Numerical analysis of the high-frequency Helmholtz eqn: a case study in applied analysis

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Goal of these 3 lectures: illustrate how (relatively simple) tools from (semiclassical) analysis can be used to answer important questions in the numerical analysis of the Helmholtz eqn.

A simple proof that the *hp*-FEM does not suffer from the pollution effect for the constant-coefficient full-space Helmholtz equation

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February 17, 2022

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Wavenumber-explicit convergence of the *hp*-FEM for the full-space heterogeneous Helmholtz equation with smooth coefficients

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Decompositions of high-frequency Helmholtz solutions via functional calculus, and application to the finite element method

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May 3, 2022

The *hp*-FEM applied to the Helmholtz equation with PML truncation does not suffer from the pollution effect

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July 13, 2022

Today

- model Helmholtz problem
- the NA question
- assemble background results

Model Helmholtz problem

given $f \in L^2_{comp}(\mathbb{R}^d)$ find $u \in H^1_{loc}(\mathbb{R}^d)$ s.t.

$$\left\{ \begin{array}{l} k^{-2}\Delta u + u = -f \\ k^{-1}\dfrac{\partial u}{\partial r} - iu = o\left(\dfrac{1}{r^{\frac{d-1}{2}}}\right) \text{ as } r := |x| \to \infty \text{ unif. in } \dfrac{x}{r} \end{array} \right.$$

$$u(x) = \frac{e^{ikr}}{r^{\frac{d-1}{2}}}\left(F_0\left(\frac{x}{r}\right) + O\left(\frac{1}{r}\right)\right) \quad r \to \infty$$

$d=3$ $\quad u(x) = k^{-2}\displaystyle\int_{\mathbb{R}^d} \frac{e^{ik|x-y|}}{4\pi|x-y|} f(y)\,dy$

$$\frac{\partial^2 U}{\partial t^2} - c^2 \Delta U = F$$

$$U(x,t) = \frac{e^{ik(r-ct)}}{r^{\frac{d-1}{2}}}\left(\cdots \right)$$

$U(x,t) = u(x)e^{-i\omega t}$, $F(x,t) = k^2 f(x) e^{-i\omega t}$

then u $\quad k = \omega/c$

Variational formulation of model Helmholtz problem

let $R>0$ s.t. $\operatorname{supp} f \subset B_R$, find $\tilde{u} \in H^1(B_R)$ s.t. $a(\hat{u},v) = F(v) \; \forall v \in H^1(B_R)$

where $a(\hat{u},v) := \int_{B_R} \left(k^{-2} \nabla \hat{u} \cdot \overline{\nabla v} - \hat{u}\,\overline{v} \right) - k^{-1} \langle \mathrm{DtN}\, \hat{u}, v \rangle_{\partial B_R}$

$$F(v) := \int_{B_R} f \, \overline{v}$$

given $g \in H^{\frac{1}{2}}(\partial B_R)$

let $w \in H^1_{loc}(\mathbb{R}^d \setminus \overline{B_R})$ be sol^n of

$$\left(k^{-2}\Delta - 1\right) w = 0 \text{ in } \mathbb{R}^d \setminus \overline{B_R}$$

$$w = g \text{ on } \partial B_R$$

$$\mathrm{DtN}\, g := k^{-1} \partial_r w$$

$$\mathrm{DtN} : H^{\frac{1}{2}}(\partial B_R) \to H^{-\frac{1}{2}}(\partial B_R)$$

Lemma $\tilde{u} = u|_{B_R}$

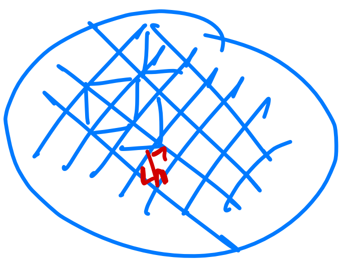
The Galerkin method

$\{\mathcal{H}_N\}_{N=0}^{\infty}$ sequence of f.d. subspaces of $H^1(B_R)$ s.t. $\forall v \in H^1(B_R)$

$$\lim_{N\to\infty}\left(\min_{v_N\in\mathcal{H}_N}\|v-v_N\|_{H^1(B_R)}\right)=0$$

find $u_N\in\mathcal{H}_N$ s.t. $a(u_N,v_N)=F(v_N)\quad\forall v_N\in\mathcal{H}_N$

we'll consider $\mathcal{H}_N$ consisting of piecewise polynomials



hFEM: $h\to 0$ as $N\to\infty$, p fixed

pFEM: $p\to\infty$ as $N\to\infty$, h fixed

hpFEM: $h\to 0$ and $p\to\infty$ as $N\to\infty$

total number of d.o.f. $\sim\left(\frac{p}{h}\right)^d$

goal: "quasi optimality" $\exists\, C_{qo}>0$ and $N_0\in\mathbb{N}$ s.t. $\forall N\geq N_0$

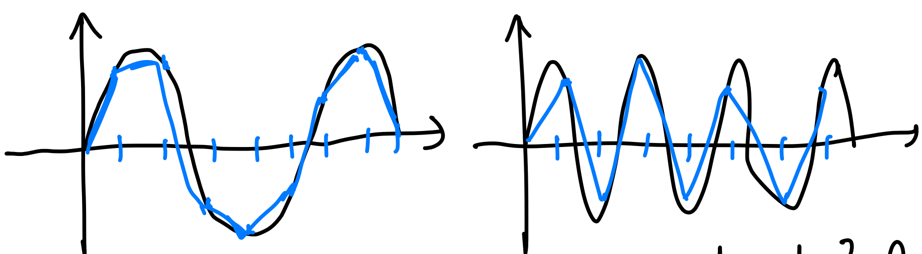
$$\|u-u_N\|_{H^1(B_R)}\leq C_{qo}\min_{v_N\in\mathcal{H}_N}\|u-v_N\|_{H^1(B_R)}$$

$$\|v\|^2_{H^m_k(B_R)}:=\sum_{0\leq|\alpha|\leq m}\|(k^{-1}\partial)^\alpha v\|^2_{L^2}$$

The question: for what $h = h(k)$, $p = p(k)$ does quasioptimality hold with $C_{qo} \sim 1$ as $k \to \infty$

Given function oscillating with freq. k, expect $\sim k^d$ d.o.f. for uniform approx as $k \to \infty$, e.g. $h \sim \frac{1}{k}$, p const.

$\left(\left(\frac{p}{h}\right)^d = \#\text{dof} \right)$



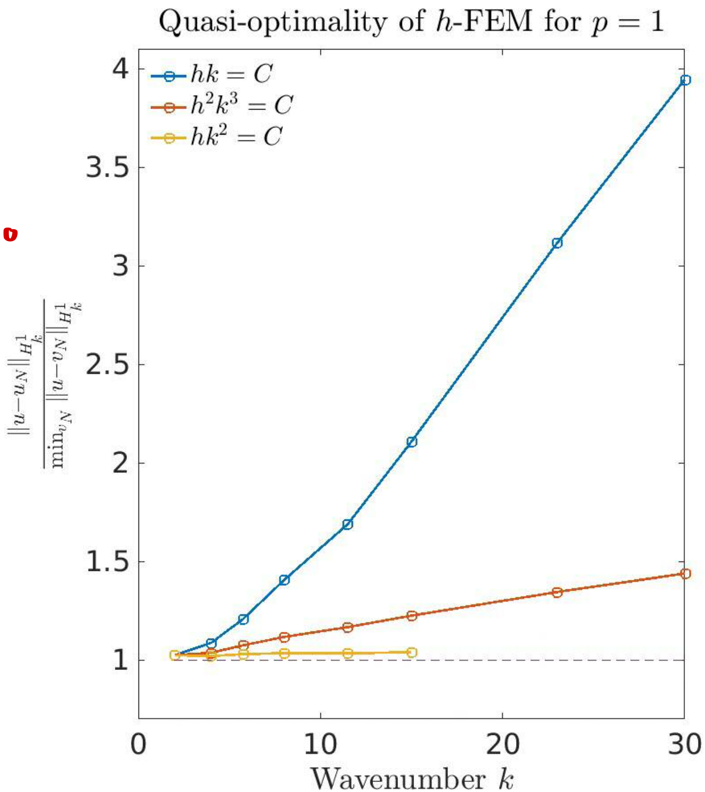
(cf Nyquist-Shannon-Whittaker)

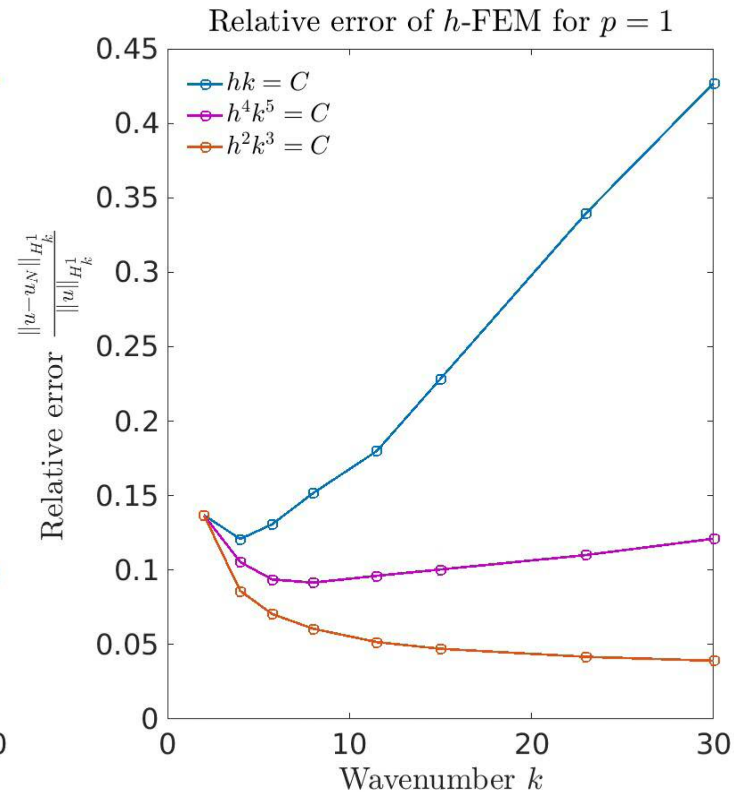
BUT hFEM with $p = 1$ need $h \sim k^{-2}$ for $C_{qo} \sim 1$

$p > 1$ need $h \sim k^{-\frac{(p+1)}{p}} \longrightarrow$ $\#\text{dof} \sim \left(k^{\frac{p+1}{p}}\right)^d \gg k^d$

"pollution effect"

C_{qo}


Quasi-optimality of h-FEM for $p = 1$
$hk = C$
$h^2k^3 = C$
$hk^2 = C$
$\frac{\|u-u_N\|_{H^1_k}}{\min_{v_N}\|u-v_N\|_{H^1_k}}$
Wavenumber k
4
3.5
3
2.5
2
1.5
1
0
10
20
30


Relative error of h-FEM for $p = 1$
$hk = C$
$h^4k^5 = C$
$h^2k^3 = C$
Relative error $\frac{\|u-u_N\|_{H^1_k}}{\|u\|_{H^1_k}}$
Wavenumber k
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0
10
20
30

Finite element solution of the Helmholtz equation with high wave number Part I: The h-version of the FEM

F Ihlenburg, I Babuška - Computers & Mathematics with Applications, 1995 - Elsevier

The paper addresses the properties of finite element solutions for the Helmholtz equation. The h-version of the finite element method with piecewise linear approximation is applied to a …

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Finite element solution of the Helmholtz equation with high wave number part II: the hp version of the FEM

F Ihlenburg, I Babuska - SIAM Journal on Numerical Analysis, 1997 - SIAM

In this paper, which is part II in a series of two, the investigation of the Galerkin finite element solution to the Helmholtz equation is continued. While part I contained results on the h …

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$h \sim k^{-\frac{(p+1)}{p}}$

Wavenumber explicit convergence analysis for Galerkin discretizations of the Helmholtz equation

JM **Melenk**, S **Sauter** - SIAM Journal on Numerical Analysis, 2011 - SIAM

We develop a stability and convergence theory for a class of highly indefinite elliptic boundary value problems (bvps) by considering the Helmholtz equation at high wavenumber k as …

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$k^{-2}\Delta u + u = -f$

Convergence analysis for finite element discretizations of the Helmholtz equation with Dirichlet-to-Neumann boundary conditions

J **Melenk**, S **Sauter** - Mathematics of Computation, 2010 - ams.org

A rigorous convergence theory for Galerkin methods for a model Helmholtz problem in ${\mathbb {R}}^{d} $, $ d\in\{1, 2, 3\} $ is presented. General conditions on the approximation …

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! hpFEM does not suffer from the pollution effect !

if $\frac{hk}{p} \lesssim C_1$, $p \geq C_2 \log k$ then $C_{qo} \sim 1$, $\#\mathrm{dof} \sim \left(\frac{p}{h}\right)^d \sim k^d$

What info do we need about Helmholtz v.f.?

find $u \in H^1(B_R)$ s.t. $a(u,v) = F(v) \quad \forall v \in H^1(B_R)$

- sol$^{\underline{n}}$ exists and is unique

- <u>continuity of $a(\cdot,\cdot)$</u>

  given $k_0>0$, $R_0>0$ $\exists$ $C_{cont}>0$ s.t. $\forall k \geq k_0$, $\forall R \geq R_0$

$$|a(u,v)| \leq C_{cont} \|u\|_{H^1_k(B_R)} \|v\|_{H^1_k(B_R)} \quad \forall u,v \in H^1(B_R)$$

- <u>Gårding inequality</u>

$$\operatorname{Re} a(v,v) \geq \|v\|^2_{H^1_k(B_R)} - 2\|v\|^2_{L^2(B_R)} \quad \forall v \in H^1(B_R)$$

What info do we need about Helmholtz solution operator?

$$\|u\|_{H^1_k(B_R)} \lesssim C_{sol} \|f\|_{L^2(B_R)}$$

<u>Thm</u> (Morawetz 1968, 1975)

$$C_{sol} \leq 2kR\sqrt{1+\left(\frac{d-1}{2kR}\right)^2}$$

multiply $k^{-2}\Delta u + u = -f$ by $x\cdot\nabla u - ik\beta u + \alpha u$ and integrate

<u>Cor</u> given $k_0, R_0 > 0$ $\exists C > 0$ st

$$\|u\|_{H^2_k(B_R)} \leq C\, kR \|f\|_{L^2(B_R)} \quad \forall k \geq k_0,\ R \geq R_0$$

Recap

$(k^{-2}\Delta - 1)u = f$ in $\mathbb{R}^d$ + radiation condition

reformulated as: find $u_N \in \cancel{H^1(B_R)}\ \mathcal{H}_N$ s.t. $a(u_N, v_N) = F(v_N)$ $\forall v_N \in \cancel{H^1(B_R)}\ \mathcal{H}_N$

$$a(u,v) = \int_{B_R} \left(k^{-2}\nabla u \cdot \nabla \bar{v} - u\bar{v}\right) - k^{-1}\langle \mathrm{DtN}\, u, v\rangle_{\partial B_R}$$

Galerkin method: given $\mathcal{H}_N \subset H^1(B_R)$

Goal: find conditions on h and p s.t. $\|u - u_N\|_{H^1_k} \le C_{qo} \min_{v_N \in \mathcal{H}_N} \|u - v_N\|_{H^1_k}$ "quasi-optimality"

indep of k

- $p = 1$, $hk^2 \le C_0$
- $\frac{hk}{p} \le C_1$, $p \ge C_2 \log k$ [Melenk + Sauter 2010]

$\#\text{dof} \sim \left(\frac{p}{h}\right)^d$

$\#\text{dof} \sim k^d$

$|a(u,v)| \le C_{cont} \|u\|_{H^1_k} \|v\|_{H^1_k}$ $\forall u, v$

Gårding: $\Re\, a(v,v) \ge \|v\|^2_{H^1_k} - 2\|v\|^2_{L^2}$ $\forall v$

$\|u\|_{H^2_k} \le C kR \|f\|_{L^2}$

Warm up: Q.O. when $a(\cdot,\cdot)$ is cts and coercive

$$|a(v,v)| \geq C_{coer} \|v\|^2_{H^1_k} \quad \forall v$$

Galerkin o/s:

$$a(u, v_N) = F(v_N) \quad \forall v_N \in \cancel{H^1}\, \mathcal{H}_N$$
$$a(u_N, v_N) = F(v_N) \quad \forall v_N \in \mathcal{H}_N$$

subtract $\quad a(u-u_N, v_N) = 0 \quad \forall v_N \in \mathcal{H}_N$

need $\|u-u_N\|_{L^2} \leq \frac{1}{2}\|u-u_N\|_{H^1_k}$

u_N exists by Lax-Milgram

$$\cancel{C_{coer}}\, \|u-u_N\|^2_{H^1_k} \leq |a(u-u_N, u-u_N)| + 2\|u-u_N\|^2_{L^2}$$
$$= |a(u-u_N, u-v_N)| + 2\|u-u_N\|^2_{L^2} \quad \forall v_N \in \mathcal{H}_N \quad (\text{since } u_N - v_N \in \mathcal{H}_N)$$
$$\leq C_{cont} \|u-u_N\|_{H^1_k} \|u-v_N\|_{H^1_k} + 2\|u-u_N\|^2_{L^2}$$

$$\Rightarrow \boxed{\|u-u_N\|_{H^1_k} \leq \frac{C_{cont}}{\cancel{C_{coer}}} \|u-v_N\|_{H^1_k} \quad \forall v_N \in \mathcal{H}_N}$$

[Céa's lemma]

How to get suff. condition for $\|u-u_N\|_{L^2} \leq \frac{1}{2}\|u-u_N\|_{H^1_k}$

("Aubin-Nitsche trick" "Schatz argument")

<u>Defn</u> given $f \in L^2$ let $S^* f \in H^1$ be sol^n of $a(v, S^* f) = (v, f)_{L^2}$ $\forall v \in H^1$

<u>Lemma</u> $a(\overline{S^* f}, v) = (\bar{f}, v)_{L^2}$ $\forall v \in H^1$

<u>Lemma</u> if $\eta(\mathcal{H}_N) := \sup\limits_{f \in L^2} \min\limits_{v_N \in \mathcal{H}_N} \dfrac{\|S^* f - v_N\|_{H^1_k}}{\|f\|_{L^2}} \leq \dfrac{1}{2 C_{cont}}$

then $\|u-u_N\|_{L^2} \leq \frac{1}{2}\|u-u_N\|_{H^1_k}$ and $\|u-u_N\|_{H^1_k} \leq 2 C_{cont} \min\limits_{v_N \in \mathcal{H}_N} \|u-v_N\|_{H^1_k}$

how well sol^{ns} of | adjoint eq^n | are approximated in $\mathcal{H}_N$
| Helmholtz eq^n |

$$\|u-u_N\|_{L^2}^2 = a(u-u_N, \mathcal{S}^*(u-u_N)) \quad \text{by def}^{\text{n}} \text{ of } \mathcal{S}^*$$

$$\Big(a(\underset{u-u_N}{\overset{u-u_N}{v}}, \mathcal{S}^* \underset{u-u_N}{f}) = (\overset{u-u_N}{v}, \underset{u-u_N}{f}) \quad \forall v \in H^1 \Big)$$

$$= a(u-u_N, \mathcal{S}^*(u-u_N) - v_N) \quad \forall v_N \in \mathcal{H}_N$$

by Galerkin o/g

$$\le C_{cont} \|u-u_N\|_{H^1_k} \underbrace{\|\mathcal{S}^*(u-u_N) - v_N\|_{H^1_k}}$$

by def$^{\text{n}}$ of $\eta(\mathcal{H}_N)$ $\exists\, v_N \in \mathcal{H}_N$ st

$$\le \eta(\mathcal{H}_N)\, \|u-u_N\|_{L^2}$$

$$\eta(\mathcal{H}_N) := \sup_{f \in L^2} \min_{v_N \in \mathcal{H}_N} \frac{\|\mathcal{S}^* f - v_N\|_{H^1_k}}{\|f\|_{L^2}}$$

$$\Rightarrow \|u-u_N\|_{L^2} \le C_{cont}\, \eta(\mathcal{H}_N)\, \|u-u_N\|_{H^1_k}$$

$$\therefore \text{ need } \eta(\mathcal{H}_N) \le \frac{1}{2 C_{cont}}$$

Piecewise polynomial approx. theory

$$\|v - I_h v\|_{H^m(D)} \le C \underbrace{h^{s-m} |v|_{H^s(D)}} \qquad \text{if } p \ge s-1$$

$\uparrow$ interpolation operator $I_h v \in \mathcal{V}_N$

e.g. $m=0$ $\quad p = s-1$

Taylor series $(s-1)$ terms $\quad$ remainder $\underline{h^s (\partial^s v)}$

given $v \in H^s$

$$\min_{w_N \in \mathcal{V}_N} \|v - w_N\|_{H^1_k} \le C_{\text{approx}} \left(\frac{hk}{p}\right)^{s-1} \left(1 + \frac{hk}{p}\right) \underbrace{k^{-s} |v|_{H^s}}_{|v|_{H^s_k}} \qquad \text{if } p \ge s-1$$

$\uparrow$ depends on s

---

bound on $\eta(\mathcal{V}_N)$

$\uparrow$ use $p=1$, $s=2$

$$\eta(\mathcal{V}_N) := \sup_{f \in L^2} \frac{1}{\|f\|_{L^2}} \min_{v_N \in \mathcal{V}_N} \|\mathcal{S}^* f - v_N\|_{H^1_k} \le \sup_{f \in L^2} \frac{1}{\|f\|_{L^2}} C_{\text{approx}}\, hk\,(1+hk)\, \underbrace{k^{-2} |\mathcal{S}^* f|_{H^2}}_{\le C kR \|f\|_{L^2}}$$

$$\le \underbrace{C_{\text{approx}} C\, hk . kR (1+hk)}$$ "hk^2 sufft. small"

Theorem (Melenk + Sauter 2010)

choose $k_0 > 0$

$$u\big|_{B_R} = u_{H^2} + u_{\mathcal{A}}$$

where $\|u_{H^2}\|_{H^2_k(B_R)} \leq C_1 \|f\|_{L^2}$ $\forall k \geq k_0$

one power of kR better

and $\|(k^{-1}\partial)^\alpha u_{\mathcal{A}}\|_{L^2(B_R)} \leq C_2\, kR\, (C_3)^{|\alpha|} \|f\|_{L^2(B_R)}$ $\forall k \geq k_0$ $\forall \alpha$

same kR dependence

(recall $\|u\|_{H^2_k(B_R)} \leq CkR \|f\|_{L^2(B_R)}$)

use splitting to bound $\eta(\mathcal{M}_N)$

$$\eta(\mathcal{M}_N) \leq \sup_f \left(\min_{v_N^{(1)} \in \mathcal{M}_N} \frac{\|u_{H^2} - v_N^{(1)}\|_{H^1_k}}{\|f\|_{L^2}} + \min_{v_N^{(2)} \in \mathcal{M}_N} \frac{\|u_{\mathcal{A}} - v_N^{(2)}\|_{H^1_k}}{\|f\|_{L^2}} \right)$$

$\leq C \frac{hk}{p}\left(1 + \frac{hk}{p}\right)$

can show $\leq CkR\left(\frac{hk}{\sigma p}\right)^p$

suff small if

$\frac{hk}{p} \leq C_1$, $p \geq C_2 \log k$

Recap

$$(-k^{-2}\Delta - 1)u = f \text{ in } \mathbb{R}^d \quad + \text{ r.c.}$$

have $\|u\|_{H^2_k(B_R)} \le C\, kR\, \|f\|_{L^2(B_R)}$

Thm (Melenk + Sauter 2010)

components of u with freq $\geq \lambda k$, $\lambda > 1$

$\ldots \ldots \le \lambda k$

$$u|_{B_R} = u_{H^2} + u_{\mathcal{A}}$$

where

$$\|u_{H^2}\|_{H^2_k(B_R)} \le C_1 \|f\|_{L^2(B_R)} \quad \forall k \geq k_0$$

$$\|(k^{-1}\partial)^{\alpha} u_{\mathcal{A}}\|_{L^2(B_R)} \le C_2\, kR\, (C_3)^{|\alpha|} \|f\|_{L^2(B_R)} \quad \forall k \geq k_0, \ \forall \alpha$$

Fourier transform

$$\mathcal{F}_k \phi(\xi) := \int_{\mathbb{R}^d} e^{-ikx\cdot\xi} \phi(x)\, dx$$

$$\mathcal{F}_h^{-1} \psi(x) := \left(\frac{k}{2\pi}\right)^d \int_{\mathbb{R}^d} e^{ikx\cdot\xi} \psi(\xi)\, d\xi$$

$$\mathcal{F}_k\Big(\big(\underbrace{-ik^{-1}\partial}_{k^{-1}D}\big)^\alpha \phi\Big)(\xi) = \xi^\alpha \mathcal{F}_h \phi(\xi) \qquad -i\partial = D$$

$$\|\phi\|_{L^2(\mathbb{R}^d)} = \left(\frac{k}{2\pi}\right)^d \|\mathcal{F}_h \phi\|_{L^2(\mathbb{R}^d)}$$

$$\|\phi\|^2_{H^s_h(\mathbb{R}^d)} = \left(\frac{k}{2\pi}\right)^d \int_{\mathbb{R}^d} \langle\xi\rangle^{2s} |\mathcal{F}_h \phi(\xi)|^2\, d\xi \quad \text{where } \langle\xi\rangle := (1+|\xi|^2)^{\frac{1}{2}}$$

Fourier multipliers

$$\boxed{(a(k^{-1}D)v)(x) := \underset{\xi \mapsto x}{\mathcal{F}_h^{-1}}\big(a(\xi)(\mathcal{F}_h v)(\xi)\big)(x)}$$

motivation: $-k^{-2}\Delta - 1 = p(k^{-1}D)$ when $p(\xi) = |\xi|^2 - 1$ e.g. $p \in (FS)^2$

Def^n (non standard) a is a Fourier symbol of order $m \in \mathbb{R}$ if $\exists\, C > 0$ st

$$|a(\xi)| \leq C \langle \xi \rangle^m \quad \forall \xi \qquad , \text{ write } a \in (FS)^m$$

e.g. 2 $a(\xi) = 1$ then $a \in (FS)^0$ $\quad a(k^{-1}D)v(x) = v(x)$

e.g. 3 supp a compact then $a \in (FS)^{-N}$ $\forall N > 1$, $1 - a \in (FS)^0$

Lemma $a \in (FS)^{m_a}$, $b \in (FS)^{m_b}$ $\Rightarrow$ $ab \in (FS)^{m_a + m_b}$

$$a(k^{-1}D)b(k^{-1}D) = (ab)(k^{-1}D) = b(k^{-1}D)a(k^{-1}D)$$

$$\|a(k^{-1}D)v\|^2_{H_h^{s-m_a}} = \int \langle \xi \rangle^{2s-2m_a} |a(\xi)\mathcal{F}_h v(\xi)|^2 \leq C \int \langle \xi \rangle^{2s} |\mathcal{F}_h v(\xi)|^2 d\xi$$

$\rightarrow$ $a(k^{-1}D) : H_k^s \to H_k^{s-m_a}$ with $\|a(k^{-1}D)\|_{H_h^s \to H_h^{s-m_a}} \leq C$

pf of thm

let $\chi_\lambda(\xi) := \mathbb{1}_{|\xi| \leq \lambda}(\xi)$

$$\Pi_L := \chi_\lambda(h^{-1}D) \quad \text{i.e.} \quad \Pi_L v = \mathcal{F}_h^{-1}\Big(\chi_\lambda(\cdot)(\mathcal{F}_h v)(\cdot)\Big)$$

low freq. cutoff

returns freq. $\leq \lambda$

$$\Pi_H := I - \Pi_L = (1-\chi_\lambda)(h^{-1}D)$$

returns freq. $\geq \lambda$

$\varphi \in C^\infty_{comp}(\mathbb{R}^d; [0,1])$ $\quad \varphi \equiv 1$ on B_R, $\operatorname{supp} \varphi \subset B_{2R}$

$$u_{\mathcal{L}} := \Pi_L(\varphi u)\Big|_{B_R}, \quad u_{H^2} := \Pi_H(\varphi u)\Big|_{B_R}$$

on B_R $\quad u_{\mathcal{L}} + u_{H^2} = \varphi u = u$

pf of bound on u_A

function with compactly supported Fourier transform is analytic

$$\|(k^{-1}\partial)^\alpha u_A\|_{L^2(B_R)} = \|(k^{-1}\partial)^\alpha \Pi_L(\varphi u)\|_{L^2(B_R)} \le \|(k^{-1}\partial)^\alpha \Pi_L(\varphi u)\|_{L^2(\mathbb{R}^d)}$$

$$= \left(\frac{k}{2\pi}\right)^d \|\underbrace{\xi^\alpha \chi_\lambda(\xi)}_{\lesssim \lambda^{|\alpha|}} \mathcal{F}_h(\varphi u)(\xi)\|_{L^2(\mathbb{R}^d)}$$

$$\lesssim \left(\frac{k}{2\pi}\right)^d \lambda^{|\alpha|} \|\mathcal{F}_h(\varphi u)\|_{L^2(\mathbb{R}^d)}$$

$$= \lambda^{|\alpha|} \|\varphi u\|_{L^2(\mathbb{R}^d)}$$

$$\lesssim \lambda^{|\alpha|} \|u\|_{L^2(B_{2R})} \lesssim \lambda^\alpha C kR \|f\|_{L^2}$$

a priori bound

$$\|(k^{-1}\partial)^\alpha u_A\|_{L^2(B_R)} \le C_2 kR (C_3)^{|\alpha|} \|f\|_{L^2(B_R)}$$

$C_3 = \lambda$

$C_2 = C$

pf of bound on $u_{\mathcal{H}^2}$

$$\|u_{\mathcal{H}^2}\|_{H^2_h(B_R)} \lesssim C_1 \|f\|_{L^2(B_R)}$$

$$\|u_{\mathcal{H}^2}\|_{H^2_h(B_R)} = \|\Pi_{\mathcal{H}}(\varphi u)\|_{H^2_h(B_R)} \leq \|\Pi_{\mathcal{H}}(\varphi u)\|_{H^2_h(\mathbb{R}^d)}$$

$$= \|(1-\chi_\mu)(h^{-1}D)\varphi u\|_{H^2_h(\mathbb{R}^d)}$$

if $\lambda \geq \lambda_0 > 1$ then $\exists C > 0$ s.t.

$$\underbrace{\big|\,|\xi|^2 - 1\,\big|}_{p(\xi)} \geq C \langle \xi \rangle^2 \text{ for } |\xi| \geq \lambda$$

$(1+|\xi|^2)$

$$= \left\| \frac{(1-\chi_\mu)(h^{-1}D)}{p(h^{-1}D)} p(h^{-1}D)\varphi u \right\|_{H^2_h(\mathbb{R}^d)}$$

restrict to $|\xi| \geq \lambda$

$$\underbrace{\frac{(1-\chi_\mu)(\xi)}{p(\xi)}} \in (FS)^{-2}$$

$(-h^{-2}\Delta - 1)$

$$\leq C \|p(h^{-1}D)\varphi u\|_{L^2(\mathbb{R}^d)}$$

$$= C \|\varphi f + [-h^{-2}\Delta - 1, \varphi] u\|_{L^2(\mathbb{R}^d)} \lesssim C \|f\|_{L^2(\mathbb{R}^d)} + \frac{1}{hR}\|u\|_{H^1_h(B_{2R})}$$

$(-h^{-2}\Delta - 1)\varphi u = \varphi(-h^{-2}\Delta - 1)u + [-h^{-2}\Delta - 1, \varphi]u$

$$\lesssim \tilde{C} \|f\|_{L^2(\mathbb{R}^d)}$$

Thm suppose $a \in (FS)^{m_a}$, $b \in (FS)^{m_b}$ and $\exists\, c > 0$

s.t. $|b(\xi)| \geq c \langle \xi \rangle^{m_b}$ for $\xi \in \operatorname{supp} a$

then $a(k^{-1}D) = q(k^{-1}D)\, b(k^{-1}D)$

$\in (FS)^{m_a - m_b}$ s.t. $q(\xi) = \dfrac{a(\xi)}{b(\xi)}$

means q makes sense

$a(\xi) = (1-\chi_\Lambda)(\xi)$

$b(\xi) = p(\xi) = |\xi|^2 - 1$

$\geq C \langle \xi \rangle^2$

on $|\xi| \geq \Lambda > 1$

$$\boxed{-k^{-2}\Delta - 1 \text{ is semiclassically elliptic for } |\xi| > 1}$$

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variable coeff. Helmholtz operator $-k^{-2}\nabla\cdot(A\nabla) - n$ is not a Fourier multiplier

if is a (semiclassical) pseudo differential operator

generalisation of this thm is the (semiclassical) "elliptic parametrix"