An operator-asymptotic approach to periodic homogenization

(applied to equations of linearized elasticity)

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Goals

1. Explain an operator-asymptotic approach for homogenization.

2. State some extra results that we have obtained via this approach.

3. Discuss connections with other approaches.

Origins of the method

- Due to Kirill Cherednichenko and Igor Velčić.

  Applied to thin elastic plates (2022).

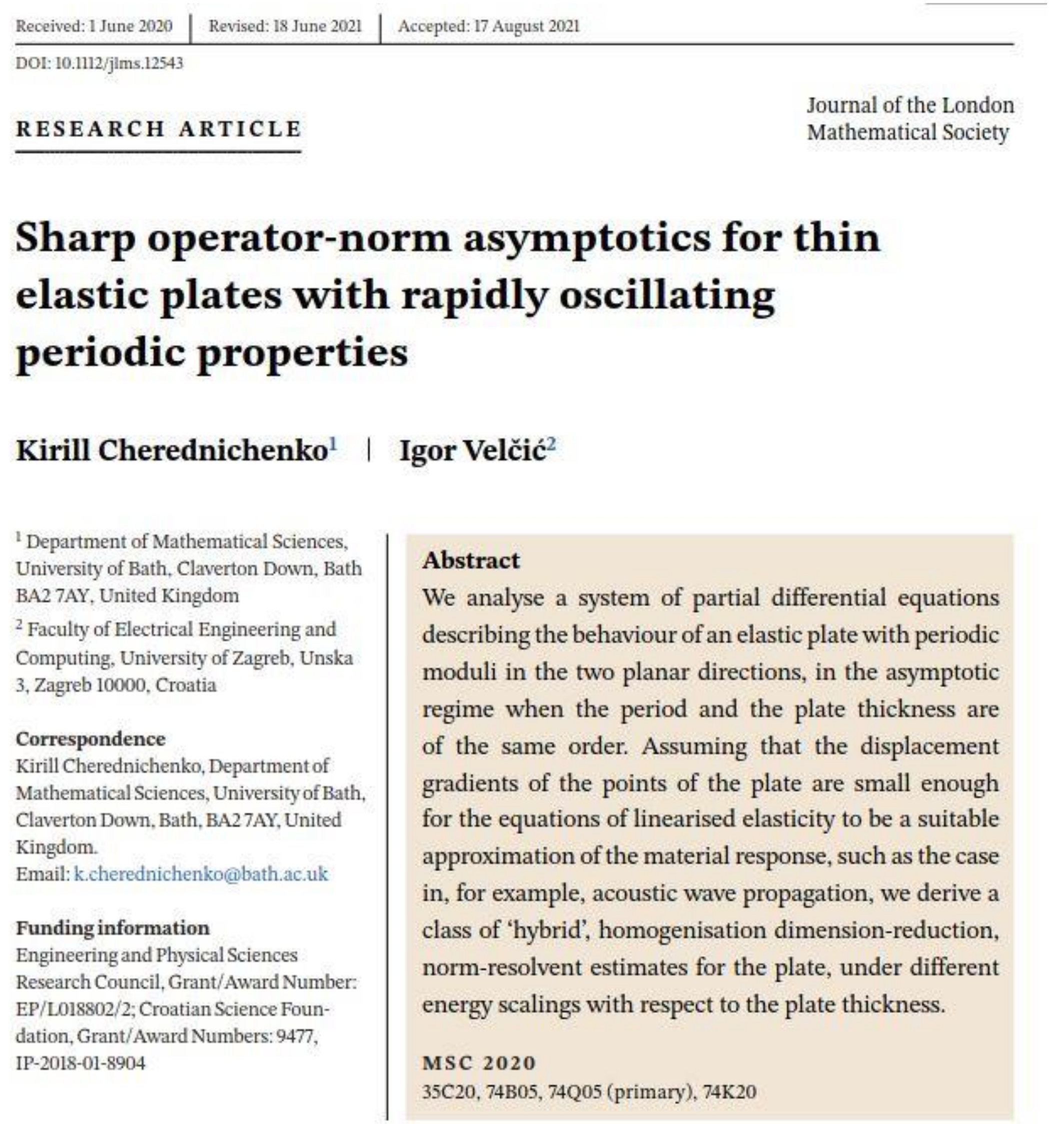
- Later applied to thin elastic rods (2023).

  Cherednichenko, Velčić, and Zubrinić

  https://arxiv.org/abs/2112.06265v3

- Dimension reduction + Homogenization.

  Let’s focus on this



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RESEARCH ARTICLE

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Sharp operator-norm asymptotics for thin elastic plates with rapidly oscillating periodic properties

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Abstract
We analyse a system of partial differential equations describing the behaviour of an elastic plate with periodic moduli in the two planar directions, in the asymptotic regime when the period and the plate thickness are of the same order. Assuming that the displacement gradients of the points of the plate are small enough for the equations of linearised elasticity to be a suitable approximation of the material response, such as the case in, for example, acoustic wave propagation, we derive a class of ‘hybrid’, homogenisation dimension-reduction, norm-resolvent estimates for the plate, under different energy scalings with respect to the plate thickness.

MSC 2020
35C20, 74B05, 74Q05 (primary), 74K20

Problem Setup

Linearized elasticity in 3D. Periodic + moderate contrast regime

Coefficient tensor

- Assumptions on the tensor $\mathbb{A} = \mathbb{A}(y)$ of material coefficients
  - (Uniformly pos. def. on $\mathbb{R}^{3\times 3}_{\mathrm{sym}}$) There exist $\nu > 0$ such that

$$\nu|\xi|^2 \leq \mathbb{A}(y)\xi : \xi \leq \frac{1}{\nu}|\xi|^2 \qquad \forall\, \xi \in \mathbb{R}^{3\times 3}_{\mathrm{sym}} \text{ and } y \in Y = [0,1)^3.$$

  - (Symmetry) For $i, j, k, l \in \{1,2,3\}$,

$$\mathbb{A}^{ik}_{jl} = \mathbb{A}^{jk}_{il} = \mathbb{A}^{ki}_{lj}.$$

  - (Boundedness)For $i, j, k, l \in \{1,2,3\}$, $\mathbb{A}^{ik}_{jl} \in L^\infty(Y; \mathbb{R}^3)$.

Key operators under study

- Let $\mathbb{A}_\varepsilon = \mathbb{A}(\frac{\cdot}{\varepsilon})$.

$\mathcal{A}_\varepsilon$ is $\varepsilon\mathbb{Z}^3$ −periodic

- Define $\mathcal{A}_\varepsilon \equiv (sym\nabla)^* \mathbb{A}_\varepsilon (sym\nabla)$ as the op. on $L^2(\mathbb{R}^3;\mathbb{C}^3)$ corresponding to the sesquilinear form:

$$a_\varepsilon(u,v) = \int_{\mathbb{R}^3} \mathbb{A}\left(\frac{x}{\varepsilon}\right) sym\nabla u(x) : \overline{sym\nabla v(x)}\, dx,$$

where $u, v \in \mathcal{D}(a_\varepsilon) = H^1(\mathbb{R}^3;\mathbb{C}^3)$.

- $\mathcal{A}_\varepsilon$ is self-adjoint, non-negative.

Key operators under study

$Y = [0,1)^3$
$Y' = [-\pi, \pi)^3$

- Set $C^\infty_\# (Y;\mathbb{C}^3) = \{u: \mathbb{R}^3 \to \mathbb{C}^3 : u \ smooth \ and \ \mathbb{Z}^3 - periodic\}$, and define

$$H^1_\#(Y;\mathbb{C}^3) = \overline{C^\infty_\# (Y;\mathbb{C}^3)}^{\|\cdot\|_{H^1}}$$

Periodic Sobolev space

- For $\chi \in Y'$ define $X_\chi : L^2(Y;\mathbb{C}^3) \to L^2(Y;\mathbb{C}^{3\times 3})$ by

$$X_\chi u = sym(u \otimes \chi) = sym(u\chi^T)$$

$c|\chi|\|u\|_{L^2} \leq \|X_\chi u\|_{L^2} \leq C|\chi|\|u\|_{L^2}$

- For $\chi \in Y'$, define $\mathcal{A}_\chi \equiv (sym\nabla + iX_\chi)^* \mathbb{A}_\varepsilon (sym\nabla + iX_\chi)$ as the op. on $L^2(Y;\mathbb{C}^3)$ corresponding to the sesquilinear form:

$$a_\chi(u,v) = \int_{\mathbb{R}^3} \mathbb{A}(y)\,(sym\nabla + iX_\chi)u(y) : \overline{(sym\nabla + iX_\chi)v(y)}\,dy,$$

where $u, v \in \mathcal{D}(a_\chi) = H^1_\#(Y;\mathbb{C}^3)$.

Relation between $\mathcal{A}_\varepsilon$ and $\mathcal{A}_\chi$

$Y = [0,1)^3$
$Y' = [-\pi,\pi)^3$

- Define the scaled Gelfand transform $\mathcal{G}_\varepsilon$

$$\mathcal{G}_\varepsilon : L^2(\mathbb{R}^3;\mathbb{C}^3) \to L^2(Y; L^2(Y;\mathbb{C}^3)) = \int_{Y'}^{\oplus} L^2(Y;\mathbb{C}^3)\,d\chi$$

$$(\mathcal{G}_\varepsilon u)(y,\chi) := \left(\frac{\varepsilon}{2\pi}\right)^{3/2} \sum_{n\in\mathbb{Z}^3} e^{-i\chi\cdot(y+n)}\, u(\varepsilon(y+n)) \qquad y \in Y \text{ and } \chi \in Y'.$$

- **<u>Proposition</u> (Passing to the unit cell for $\mathcal{A}_\varepsilon$)**

$$\mathcal{A}_\varepsilon = \mathcal{G}_\varepsilon^* \left(\int_{Y'}^{\oplus} \frac{1}{\varepsilon^2} \mathcal{A}_\chi \, d\chi \right) \mathcal{G}_\varepsilon$$

$$(\mathcal{A}_\varepsilon - z)^{-1} = \mathcal{G}_\varepsilon^* \left(\int_{Y'}^{\oplus} \left(\frac{1}{\varepsilon^2}\mathcal{A}_\chi - z \right)^{-1} d\chi \right) \mathcal{G}_\varepsilon \qquad \text{for } z \in \rho(\mathcal{A}_\varepsilon).$$

Look to obtain uniform-in-χ estimates in the operator norm

The homogenized operator $\mathcal{A}^{\text{hom}}$

- Define the homogenized tensor $\mathbb{A}^{\text{hom}}$ through a symm bilinear form

$$a^{\text{hom}}(\xi,\zeta) = \int_Y \mathbb{A}(\xi + sym\nabla u^{\xi}) : \zeta \, dy, \qquad \forall\, \xi,\zeta \in \mathbb{R}^{3\times 3}_{\text{sym}}$$

where the corrector term $u^{\xi} \in H^1_{\#}(Y;\mathbb{R}^3)$ solves the cell-problem

$$\begin{cases} \int_Y \mathbb{A}(\xi + sym\nabla u^{\xi}) : sym\nabla v \, dy = 0, \qquad \forall v \in H^1_{\#}(Y;\mathbb{R}^3). \\ \int_Y u^{\xi} = 0. \end{cases}$$

- Define $\mathcal{A}^{\text{hom}} \equiv (sym\nabla)^* \mathbb{A}^{\text{hom}} (sym\nabla)$ as the op on $L^2(\mathbb{R}^3;\mathbb{C}^3)$ corresponding to the form

$$H^1(\mathbb{R}^3;\mathbb{C}^3) \times H^1(\mathbb{R}^3;\mathbb{C}^3) \ni (u,v) \mapsto \int_{\mathbb{R}^3} \mathbb{A}^{\text{hom}}\, sym\nabla u : \overline{sym\nabla v}\, dy$$

- $\mathbb{A}^{\text{hom}}$ satisfies the same symmetries as $\mathbb{A}$.
- $\mathbb{A}^{\text{hom}}$ is unif. pos. def. on $\mathbb{R}^{3\times 3}_{\text{sym}}$.
- $a^{\text{hom}}(\xi,\zeta) = \mathbb{A}^{\text{hom}}\xi : \zeta$

- $\mathcal{D}(\mathcal{A}^{\text{hom}}) = H^2(\mathbb{R}^3;\mathbb{C}^3)$.

Key operators under study (summary)

- $\mathcal{A}_\varepsilon \equiv (sym\nabla)^* \mathbb{A}_\varepsilon (sym\nabla)$ on $L^2(\mathbb{R}^3)$
- $\mathcal{A}_\chi \equiv \left(sym\nabla + iX_\chi\right)^* \mathbb{A}_\varepsilon \left(sym\nabla + iX_\chi\right)$ on $L^2(Y)$
- $\mathcal{A}^{\text{hom}} \equiv (sym\nabla)^* \mathbb{A}^{\text{hom}} (sym\nabla)$ on $L^2(\mathbb{R}^3)$
  - $a^{\text{hom}}(\xi,\zeta) \coloneqq \int_Y \mathbb{A}\left(\xi + sym\nabla u^\xi\right): \zeta \; dy = \mathbb{A}^{\text{hom}}\xi : \zeta,$

    where $\xi, \zeta \in \mathbb{R}^{3\times 3}_{\text{sym}}$, and u^ξ solves the cell-problem.
  - $\mathbb{A}^{\text{hom}}$ satisfies the same assumptions as $\mathbb{A}$.

Main result

Theorem There exists $C > 0$, independent of ε, such that

- $\left\| (\mathcal{A}_\varepsilon + I)^{-1} - \left(\mathcal{A}^{\text{hom}} + I\right)^{-1} \right\|_{L^2 \to L^2} \leq C\varepsilon$
- $\left\| (\mathcal{A}_\varepsilon + I)^{-1} - \left(\mathcal{A}^{\text{hom}} + I\right)^{-1} - \mathcal{R}^\varepsilon_{corr,1} \right\|_{L^2 \to H^1} \leq C\varepsilon$
- $\left\| (\mathcal{A}_\varepsilon + I)^{-1} - \left(\mathcal{A}^{\text{hom}} + I\right)^{-1} - \mathcal{R}^\varepsilon_{corr,1} - \mathcal{R}^\varepsilon_{corr,2} \right\|_{L^2 \to L^2} \leq C\varepsilon^2$

where $\mathcal{R}^\varepsilon_{corr,1}$ and $\mathcal{R}^\varepsilon_{corr,2}$ are the corrector operators defined through the asymptotic procedure.

Result extends to $(\varepsilon^{-\gamma}\mathcal{A}_\varepsilon + I)^{-1}$, $\gamma \in [-2, \infty)$

$\mathcal{R}^\varepsilon_{corr,1} f$ = first-order term of the usual 2-scale expansion

Method

$$\mathcal{A}_\varepsilon = \mathcal{G}_\varepsilon^* \left(\int_{Y'}^{\oplus} \frac{1}{\varepsilon^2} \mathcal{A}_\chi d\chi \right) \mathcal{G}_\varepsilon$$

1. **Gelfand Transform**
2. **Spectral analysis of $\mathcal{A}_\chi$**
3. **Fibrewise (= for each χ) asymptotic expansion**
4. **Back to full space via functional calculus**

Spectral analysis of $\mathcal{A}_\chi$

$$\mathcal{R}_\chi(u) = \frac{a_\chi(u,u)}{\|u\|_{L^2}}, \qquad u \in H^1_\# \setminus \{0\}$$

Proposition There exist constants $C_{rl} > c_{rl} > 0$ s.t.

$$c_{rl}|\chi|^2 \leq \mathcal{R}_\chi(u) \qquad \forall u \in H^1_\#(Y;\mathbb{C}^3)\backslash\{0\}$$

$$0 \leq \mathcal{R}_\chi(u) \leq C_{rl}|\chi|^2 \qquad \forall u \in \mathbb{C}^3\backslash\{0\}$$

$$c_{rl} \leq \mathcal{R}_\chi(u) \qquad \forall u \in (\mathbb{C}^3)^\perp \cap H^1_\#(Y;\mathbb{C}^3)\backslash\{0\}$$

- The proof follows from assumptions on $\mathbb{A}$ and

- $\|u\|_{L^2} \leq \frac{C_{\text{fourier}}}{|\chi|}\left\|(sym\nabla + iX_\chi)u\right\|_{L^2}$
- $\|\nabla u\|_{L^2} \leq C_{\text{fourier}}\left\|(sym\nabla + iX_\chi)u\right\|_{L^2}$
- $\left\|u - \int u\right\|_{L^2} \leq c_{\text{fourier}}\left\|(sym\nabla + iX_\chi)u\right\|_{L^2}$

where $u \in H^1_\# \setminus \{0\}$

Spectral analysis of $\mathcal{A}_\chi$

- **Theorem** The spectrum $\sigma(\mathcal{A}_\chi)$ contains 3 eigenvalues of order $|\chi|^2$, as $|\chi| \downarrow 0$, while the remaining eigenvalues are of order 1.
- We focus on small χ, as large χ will not contribute to the overall estimate.
- The space $\mathbb{C}^3$ is of key importance:
  - $\mathbb{C}^3 = Eig(\lambda_1^0; \mathcal{A}_0) \oplus Eig(\lambda_1^0; \mathcal{A}_0) \oplus Eig(\lambda_1^0; \mathcal{A}_0) = Eig(0; \mathcal{A}_0) = \ker(\mathcal{A}_0)$.
  - $\mathbb{C}^3 = \ker(sym\nabla_{\text{neumann}}) \cap H^1_\#(Y; \mathbb{C}^3)$

This is the set of rigid displacements $w = Ax + c$, by Korn's inequality. ($A \in \mathbb{C}^{3\times 3}, A^T = -A, c \in \mathbb{C}^3$)

Spectral analysis of $\mathcal{A}_\chi$

- The averaging operator $\mathrm{P}_{\mathbb{C}^3} = S: L^2(Y;\mathbb{C}^3) \to \mathbb{C}^3 \hookrightarrow L^2(Y;\mathbb{C}^3)$ is given by

$$Su = \int_Y u$$

- For $\varepsilon > 0$, the smoothing operator $\Xi_\varepsilon: L^2(\mathbb{R}^3;\mathbb{C}^3) \to L^2(\mathbb{R}^3;\mathbb{C}^3)$ is given by

$$\Xi_\varepsilon u = \mathcal{G}_\varepsilon^{-1}\left(\int_{Y'}^{\oplus} S\, d\chi\right)\mathcal{G}_\varepsilon = \mathcal{G}_\varepsilon^{-1}\left(\int_Y (\mathcal{G}_\varepsilon u)(y,\cdot)dy\right)$$

"Smoothing" because Ξ_ε can be written as a Fourier cutoff

viewed as a function in $y \in Y$ and $\chi \in Y'$

Definition of $\mathcal{A}_\chi^{\text{hom}}$

Assume $\chi \neq 0$.

- Instead of $\left(\frac{1}{\varepsilon^2}\mathcal{A}_\chi - z\right)^{-1}$, look at $\left(\frac{1}{|\chi|^2}\mathcal{A}_\chi - z\right)^{-1}$
- We have defined $\mathcal{A}_\varepsilon$, $\mathcal{A}_\chi$, and $\mathcal{A}^{\text{hom}}$. Now let us define $\mathcal{A}_\chi^{\text{hom}} \in \mathbb{C}^{3\times 3}$:

$$\left\langle \mathcal{A}_\chi^{\text{hom}} c, d \right\rangle_{\mathbb{C}^3} = \int_Y \mathbb{A}\left(sym\nabla u_c + iX_\chi c\right) : \overline{iX_\chi d}\,, \qquad \forall c, d \in \mathbb{C}^3.$$

Where the corrector term $u_c \in H^1_\#(Y;\mathbb{C}^3)$ solves the (χ-dependent) cell-problem

$$\begin{cases} \int_Y \mathbb{A}\left(sym\nabla u_c + iX_\chi c\right) : \overline{sym\nabla v}\, dy = 0, \qquad \forall v \in H^1_\#(Y;\mathbb{C}^3). \\ \int_Y u_c = 0. \end{cases}$$

These problems appear naturally in the asymptotic expansion.

Properties of $\mathcal{A}_\chi^{\text{hom}}$

- $\langle \mathcal{A}_\chi^{\text{hom}} c, d \rangle_{\mathbb{C}^3} = \int_Y \mathbb{A}(sym\nabla u_c + iX_\chi c) : \overline{iX_\chi d}$
- $u_c \in H^1_\#$ solves the χ-dep. cell-problem

1. $\mathcal{A}_\chi^{\text{hom}} \in \mathbb{C}^{3\times 3}$ is Hermitian.
2. $\mathcal{A}_\chi^{\text{hom}} = (iX_\chi)^* \mathbb{A}^{\text{hom}} (iX_\chi)$.
3. There exist $\nu_1 > 0$, indep of χ, such that

$$\nu_1 |\chi|^2 |c|^2 \leq \langle \mathcal{A}_\chi^{\text{hom}} c, d \rangle_{\mathbb{C}^3} \leq \frac{1}{\nu_1} |\chi|^2 |c|^2 \quad \forall c \in \mathbb{C}^3$$

4. Proposition (passing to the unit cell for $\mathcal{A}^{\text{hom}}$):

$$\mathcal{A}^{\text{hom}} \Xi_\varepsilon = \mathcal{G}_\varepsilon^* \left(\int_{Y'}^{\oplus} \frac{1}{\varepsilon^2} S^* \mathcal{A}_\chi^{\text{hom}} S \, d\chi \right) \mathcal{G}_\varepsilon.$$

Get this by comparing the definitions of $\mathcal{A}_\chi^{\text{hom}}$ and $\mathbb{A}^{\text{hom}}$.

<u>**Proof of 4**</u> go through key ingredients if time permits.

Key steps in proving $\mathcal{A}^{\text{hom}}\Xi_\varepsilon = \mathcal{G}_\varepsilon^*\left(\int_{Y'}^{\oplus}\frac{1}{\varepsilon^2}S^*\mathcal{A}_\chi^{\text{hom}}Sd\chi\right)\mathcal{G}_\varepsilon.$

- $\mathcal{A}^{\text{hom}} = (sym\nabla)^*\mathbb{A}^{\text{hom}}(sym\nabla)$ has the same form as $\mathcal{A}_\varepsilon = (sym\nabla)^*\mathbb{A}\,(sym\nabla)$, thus

$$\mathcal{A}^{hom} = \mathcal{G}_\varepsilon^*\left(\int_{Y'}^{\oplus}\frac{1}{\varepsilon^2}\mathcal{A}_\chi^{\text{hom-full}}d\chi\right)\mathcal{G}_\varepsilon,$$

where $\mathcal{A}^{\text{hom-full}} = \left(sym\nabla + iX_\chi\right)^*\mathbb{A}^{hom}(sym\nabla + iX_\chi)$, with $\mathcal{D}\left[\mathcal{A}_\chi^{\text{hom-full}}\right] = H_\#^1(Y;\mathbb{C}^3)$.

- Apply the smoothing op Ξ_ε to both sides, on the right:

$$\begin{aligned}\mathcal{A}^{hom}\Xi_\varepsilon &= \mathcal{G}_\varepsilon^*\left(\int_{Y'}^{\oplus}\frac{1}{\varepsilon^2}\mathcal{A}_\chi^{\text{hom-full}}d\chi\right)\mathcal{G}_\varepsilon\mathcal{G}_\varepsilon^*\left(\int_{Y'}^{\oplus}Sd\chi\right)\mathcal{G}_\varepsilon \\ &= \mathcal{G}_\varepsilon^*\left(\int_{Y'}^{\oplus}\frac{1}{\varepsilon^2}\mathcal{A}_\chi^{\text{hom-full}}Sd\chi\right)\mathcal{G}_\varepsilon \\ &= \mathcal{G}_\varepsilon^*\left(\int_{Y'}^{\oplus}\frac{1}{\varepsilon^2}S\mathcal{A}_\chi^{\text{hom}}Sd\chi\right)\mathcal{G}_\varepsilon\end{aligned}$$

- $\mathbb{C}^3$ is an invariant subspace for $\mathcal{A}_\chi^{\text{hom-full}}$ (because $\mathbb{A}^{\text{hom}}$ is const. in space)
- $\mathcal{A}_\chi^{\text{hom-full}}\big|_{\mathbb{C}^3} = \left(iX_\chi\right)^*\mathbb{A}^{\text{hom}}\left(iX_\chi\right) = \mathcal{A}_\chi^{\text{hom}}$

Asymptotic expansion of $\left(\frac{1}{|\chi|^2}\mathcal{A}_\chi - z\right)^{-1}$

- Fix $\chi \neq 0$ and $z \in \rho\left(\frac{1}{|\chi|^2}\mathcal{A}_\chi\right) \cap \rho\left(\frac{1}{|\chi|^2}\mathcal{A}_\chi^{\mathrm{hom}}\right)$ and $f \in L^2(Y;\mathbb{C}^3)$.

- The resolvent equation of $\frac{1}{|\chi|^2}\mathcal{A}_\chi$, in the weak formulation is given by

$$\frac{1}{|\chi|^2}\int_Y \mathbb{A}\left(sym\nabla + iX_\chi\right)u : \overline{\left(sym\nabla + iX_\chi\right)v} - z\int_Y u\cdot\bar{v} = \int_Y f\cdot\bar{v} \qquad \forall v \in H^1_\#(Y;\mathbb{C}^3)$$

where we have a unique solution $u \in \mathcal{D}\left(\frac{1}{|\chi|^2}\mathcal{A}_\chi\right) \subset H^1_\#(Y;\mathbb{C}^3)$.

- Let us expand the solution u in the following way:

$$u = u_0 + u_1 + u_2 + u_{\mathrm{err}}, \qquad u_j, u_{\mathrm{err}} \in H^1_\#(Y;\mathbb{C}^3)$$

u_0: $\mathcal{O}(1)$ $\quad$ u_1: $\mathcal{O}(|\chi|)$ $\quad$ u_2: $\mathcal{O}(|\chi|^2)$ $\quad$ as $|\chi| \downarrow 0$. In the H^1 norm.

Asymp exp of $\left(\frac{1}{|\chi|^2}\mathcal{A}_\chi - z\right)^{-1}$ (Cycle 1)

$u = u_0 + u_1 + u_2 + u_{\text{err}}$
$u_j = \mathcal{O}(|\chi|^j)$ in the H^1 norm

- Plug the expansion for u into the resolvent eqn. We have: $\forall v \in H^1_\#(Y;\mathbb{C}^3)$,

$$\int_Y \mathbb{A}\, sym\nabla u_0 : \overline{sym\nabla v} + \int_Y \mathbb{A}\, sym\nabla u_0 : \overline{iX_\chi v} + \int_Y \mathbb{A}\, iX_\chi u_0 : \overline{sym\nabla v} + \int_Y \mathbb{A}\, iX_\chi u_0 : \overline{iX_\chi v}$$

$$+ \int_Y \mathbb{A}\, sym\nabla u_1 : \overline{sym\nabla v} + \int_Y \mathbb{A}\, sym\nabla u_1 : \overline{iX_\chi v} + \int_Y \mathbb{A}\, iX_\chi u_1 : \overline{sym\nabla v} + \int_Y \mathbb{A}\, iX_\chi u_1 : \overline{iX_\chi v}$$

$$+ \int_Y \mathbb{A}\, sym\nabla u_2 : \overline{sym\nabla v} + \int_Y \mathbb{A}\, sym\nabla u_2 : \overline{iX_\chi v} + \int_Y \mathbb{A}\, iX_\chi u_2 : \overline{sym\nabla v} + \int_Y \mathbb{A}\, iX_\chi u_2 : \overline{iX_\chi v}$$

$$+ \int_Y \mathbb{A}\, sym\nabla u_{\text{err}} : \overline{sym\nabla v} + \int_Y \mathbb{A}\, sym\nabla u_{\text{err}} : \overline{iX_\chi v} + \int_Y \mathbb{A}\, iX_\chi u_{\text{err}} : \overline{sym\nabla v} + \int_Y \mathbb{A}\, iX_\chi u_{\text{err}} : \overline{iX_\chi v}$$

$$-z|\chi|^2 \int_Y u_0 \cdot \bar{v} - z|\chi|^2 \int_Y u_1 \cdot \bar{v} - z|\chi|^2 \int_Y u_2 \cdot \bar{v} - z|\chi|^2 \int_Y u_{\text{err}} \cdot \bar{v} = |\chi|^2 \int_Y f \cdot \bar{v}$$

Legend:
$\mathcal{O}(1)$ terms
$\mathcal{O}(|\chi|)$ terms
$\mathcal{O}(|\chi|^2)$ terms
Error terms

Asymp exp of $\left(\frac{1}{|\chi|^2}\mathcal{A}_\chi - z\right)^{-1}$ (Cycle 1)

$u = u_0 + u_1 + u_2 + u_{\text{err}}$
$u_j = \mathcal{O}(|\chi|^j)$ in the H^1 norm

- $\mathcal{O}(1)$ terms gives us the problem: Seek $u_0 \in H^1_\#(Y;\mathbb{C}^3)$ that solves

$$\int_Y \mathbb{A}\, sym\nabla u_0 : \overline{sym\nabla v} = 0, \qquad \forall v \in H^1_\#$$

- By Korn's inequality (or $\mathbb{C}^3 = \ker(\mathcal{A}_0)$), $u_0 \in \mathbb{C}^3$.

- Additional constraint needed to fix this const.

Asymp exp of $\left(\frac{1}{|\chi|^2}\mathcal{A}_\chi - z\right)^{-1}$ (Cycle 1)

$u = u_0 + u_1 + u_2 + u_{\mathrm{err}}$
$u_j = \mathcal{O}(|\chi|^j)$ in the H^1 norm

- $\mathcal{O}(1)$ terms gives us: $u_0 \in \mathbb{C}^3$.
- $\mathcal{O}(|\chi|)$ terms gives us: Seek $u_1 \in \dot{H}^1_\#$ (= $H^1_\#$ with mean zero), that solves

$$\int_Y \mathbb{A}\, sym\nabla u_1 : \overline{sym\nabla v} = -\int_Y \mathbb{A}\, sym\nabla u_0 : \overline{iX_\chi v} - \int_Y \mathbb{A}\, iX_\chi u_0 : \overline{sym\nabla v}, \qquad \forall v \in H^1_\#$$

This is zero as $u_0 \in \mathbb{C}^3$.

- Use Lax-Milgram to conclude existence + uniqueness of the prob for u_1.
- (This is the χ-dependent cell-problem with $c = u_0$)

Asymp exp of $\left(\frac{1}{|\chi|^2}\mathcal{A}_\chi - z\right)^{-1}$ (Cycle 1)

$u = u_0 + u_1 + u_2 + u_{\text{err}}$
$u_j = \mathcal{O}(|\chi|^j)$ in the H^1 norm

- $\mathcal{O}(|\chi|^2)$ terms gives us: Seek $u_2 \in \dot{H}^1_\#$ that solves

$$\int_Y \mathbb{A}\, sym\nabla u_2 : \overline{sym\nabla v} = -\int_Y \mathbb{A}\, iX_\chi u_1 : \overline{sym\nabla v} - \int_Y \mathbb{A}\, sym\nabla u_1 : \overline{iX_\chi v} - \int_Y \mathbb{A}\, iX_\chi u_0 : \overline{iX_\chi v}$$
$$+z|\chi|^2 \int_Y u_0 \cdot \bar{v} + |\chi|^2 \int_Y f \cdot \bar{v}, \qquad \forall v \in H^1_\#$$

- A necessary cond for ∃! is: The problem should hold on every test fct $v_0 \equiv v \in \mathbb{C}^3$:

  Then $sym\nabla v_0 = 0$. By how the χ-cell-problem is defined,

we get $$\frac{1}{|\chi|^2}\langle \mathcal{A}_\chi^{\text{hom}} u_0, v_0\rangle_{\mathbb{C}^3} - z\int_Y u_0 \cdot \overline{v_0} = \int_Y f \cdot \overline{v_0}, \qquad \forall v_0 \in \mathbb{C}^3.$$

i.e. $$\left(\frac{1}{|\chi|^2}\mathcal{A}_\chi^{\text{hom}} - z\right) u_0 = Sf$$

This chooses our constant $u_0 \in \mathbb{C}^3$.

With u_0 and u_1 chosen uniquely, Lax-Milgram applied to the $\mathcal{O}(|\chi|^2)$ problem gives us a unique u_2.

Summary of first cycle

$u = u_0 + u_1 + u_2 + u_{\text{err}}$
$u_j = \mathcal{O}(|\chi|^j)$ in the H^1 norm

- We write $u = u_0 + u_1 + u_2 + u_{\text{err}}$, where
  - $u_0 \in \mathbb{C}^3 \subset H^1_\#$ is given by
    $$\left(\frac{1}{|\chi|^2}\mathcal{A}_\chi^{\text{hom}} - z\right)u_0 = Sf$$
  - $u_1 \in \dot{H}^1_\#$ is the unique solution to
    $$\int_Y \mathbb{A}\, sym\nabla u_1 : \overline{sym\nabla v} = -\int_Y \mathbb{A}\, iX_\chi u_0 : \overline{sym\nabla v}, \qquad \forall v \in H^1_\#$$
  - $u_2 \in \dot{H}^1_\#$ is the unique solution to
    $$\int_Y \mathbb{A}\, sym\nabla u_2 : \overline{sym\nabla v} = -\int_Y \mathbb{A}\, iX_\chi u_1 : \overline{sym\nabla v} - \int_Y \mathbb{A}\, sym\nabla u_1 : \overline{iX_\chi v}$$
    $$-\int_Y \mathbb{A}\, iX_\chi u_0 : \overline{iX_\chi v} + z|\chi|^2 \int_Y u_0 \cdot \bar{v} + |\chi|^2 \int_Y f \cdot \bar{v}, \qquad \forall v \in H^1_\#$$

- To justify the expansion, we need estimates on u_j and u_{err} (in H^1). We iteratively prove that
  - $\|u_0\|_{H^1} \le C \|f\|_{L^2}$
  - $\|u_1\|_{H^1} \le C |\chi| \|f\|_{L^2}$
  - $\|u_2\|_{H^1} \le C |\chi|^2 \|f\|_{L^2}$
  - $\|u_{\text{err}}\|_{H^1} \le C|\chi| \|f\|_{L^2}$

- $C = C(z)$. But can be chosen independently of z, if z comes from a compact subset of both resolvents.
- It turns out that u_{err} is only $\mathcal{O}(|\chi|)$ in H^1.

Second cycle (very briefly)

- We have enough to prove the $L^2 \to L^2$ result. But we need more for $L^2 \to H^1$ and higher order $L^2 \to L^2$. How to continue the expansion?
- Thus far, we have

$$\begin{array}{lcccc} & \mathcal{O}(1) & \mathcal{O}(|\chi|) & \mathcal{O}(|\chi|^2) \\ u = & u_0 & +u_1 & +u_2 \\ & & +u_{\mathrm{err}} & \end{array}$$

Second cycle (very briefly)

- Propose a refined expansion:

$$\begin{array}{lccccc} & \mathcal{O}(1) & \mathcal{O}(|\chi|) & \mathcal{O}(|\chi|^2) & \mathcal{O}(|\chi|^3) & \\ u = & u_0 & +u_1 & +u_2 & & \\ & & +u_0^{(1)} & +u_1^{(1)} & +u_2^{(1)} & +u_{\text{err}}^{(1)} \end{array}$$

Heuristic: $u_i^{(j)}$ is $\mathcal{O}(|\chi|^{i+j})$ in H^1-norm.

- Substitute this into the resolvent equation … 7*5 + 1 = 36 terms!
  - But many terms cancel due to the problems for u_0, u_1, u_2 in Cycle 1.
  - Equate terms with same orders of $|\chi|$, something similar to Cycle 1 happens:
    - $\mathcal{O}(|\chi|)$ terms says that $u_0^{(1)} \in \mathbb{C}^3$.
    - $\mathcal{O}(|\chi|^2)$ terms gives a BVP that $u_1^{(1)} \in \dot{H}_{\#}^1$ uniquely solves.
    - $\mathcal{O}(|\chi|^3)$ terms chooses the constant $u_0^{(1)}$, and in turn provides a BVP that $u_2^{(1)} \in \dot{H}_{\#}^1$ uniquely solves.

Second cycle + Conclusion of Step 3

- Refined expansion:

	$\mathcal{O}(1)$	$\mathcal{O}(\|\chi\|)$	$\mathcal{O}(\|\chi\|^2)$	$\mathcal{O}(\|\chi\|^3)$
$u =$	u_0	$+u_1$	$+u_2$	
		$+u_0^{(1)}$	$+u_1^{(1)}$	$+u_2^{(1)}$ $+u_{\text{err}}^{(1)}$

$\longrightarrow$

- Error estimates
  - $\left\| u_0^{(1)} \right\|_{H^1} \le C \; |\chi| \; \|f\|_{L^2}$
  - $\left\| u_1^{(1)} \right\|_{H^1} \le C \; |\chi|^2 \; \|f\|_{L^2}$
  - $\left\| u_2^{(1)} \right\|_{H^1} \le C \; |\chi|^3 \; \|f\|_{L^2}$
  - $\left\| u_{\text{err}}^{(1)} \right\|_{H^1} \le C \; |\chi|^2 \; \|f\|_{L^2}$

- **<u>Theorem</u>** Let $\chi \in Y' \setminus \{0\}$ and $z \in \rho\left(\frac{1}{|\chi|^2}\mathcal{A}_\chi\right) \cap \rho\left(\frac{1}{|\chi|^2}\mathcal{A}_\chi^{\text{hom}}\right)$. There exist a constant $C > 0$, which does not depend on χ, (and z if z is taken from a compact subset of) such that
  - $\left\| \left(\frac{1}{|\chi|^2}\mathcal{A}_\chi - z\right)^{-1} - \left(\frac{1}{|\chi|^2}\mathcal{A}_\chi^{\text{hom}} - z\right)^{-1} S \right\|_{L^2 \to H^1} \le C\,|\chi| \|f\|_{L^2}.$
  - $\left\| \left(\frac{1}{|\chi|^2}\mathcal{A}_\chi - z\right)^{-1} - \left(\frac{1}{|\chi|^2}\mathcal{A}_\chi^{\text{hom}} - z\right)^{-1} S - \mathcal{R}_{corr,1,\chi}(z) - \mathcal{R}_{corr,2,\chi}(z) \right\|_{L^2 \to H^1} \le C\,|\chi|^2 \|f\|_{L^2}.$

u u_0 u_1 $u_0^{(1)}$

Step 4 (back to the full space)

Putting everything together…

The contour Γ

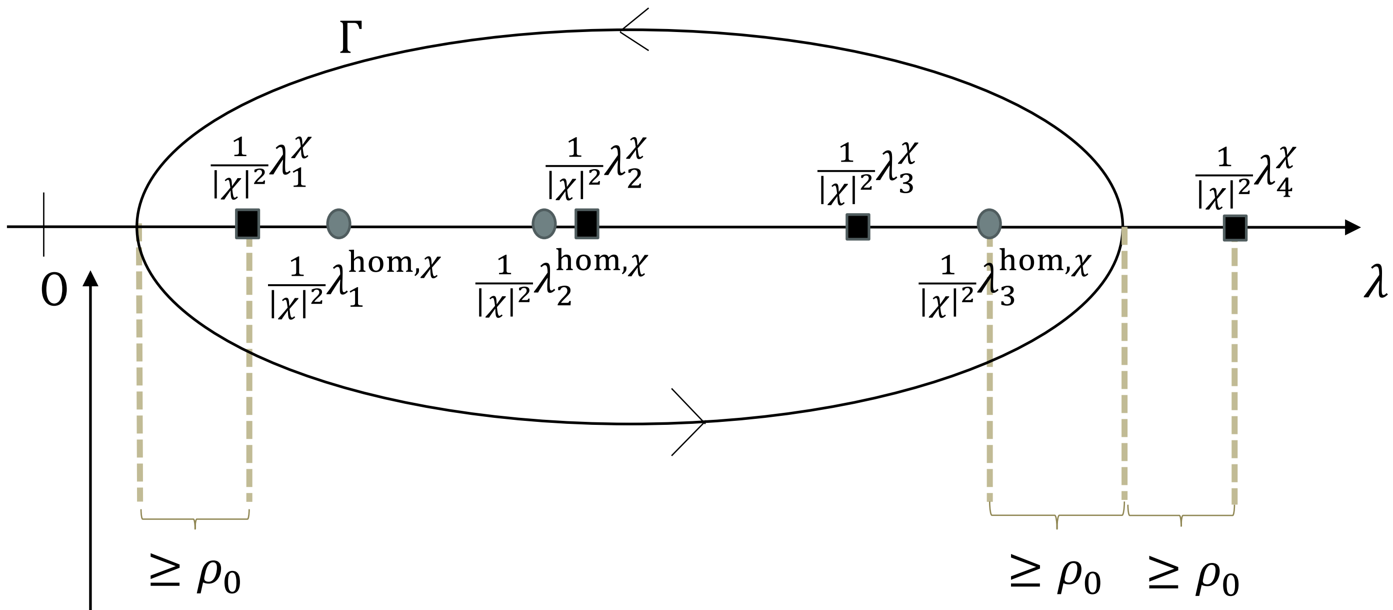
For $\chi \in [-\mu, \mu]^3 \setminus \{0\}$

- Focus on small χ. How small do we need χ to be?
- **Definition** Let $\Gamma \subset \{z \in \mathbb{C} : Re(z) > 0\}$ be a closed contour, oriented anti-clockwise, s.t.
  - (**Separation of spectrum**) There exist $\mu > 0$, s.t. Γ encloses the three smallest evalues of $\frac{1}{|\chi|^2}\mathcal{A}_\chi$ and $\frac{1}{|\chi|^2}\mathcal{A}_\chi^{\text{hom}}$, and nothing else.
  - (**Buffer between contour and spectra**) There exist some $\rho_0 > 0$ s.t.

$$\inf_{\substack{z\in\Gamma \\ \chi\in[\mu,\mu]^3\setminus\{0\} \\ i\in\{1,2,3,4\}}} \left| z - \frac{1}{|\chi|^2}\lambda_i^\chi \right| \geq \rho_0$$

and

$$\inf_{\substack{z\in\Gamma \\ \chi\in[\mu,\mu]^3\setminus\{0\} \\ i\in\{1,2,3\}}} \left| z - \frac{1}{|\chi|^2}\lambda_i^{\text{hom},\chi} \right| \geq \rho_0$$

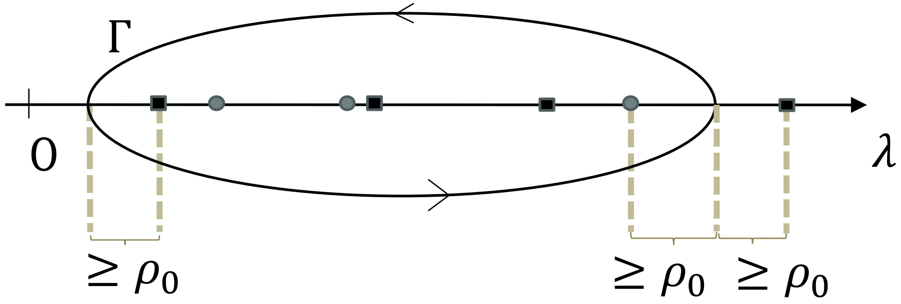


This gap implies that the fct $g_{\varepsilon,\chi} : \Gamma \to \mathbb{C}$ with $g_{\varepsilon,\chi}(z) = \left(\frac{|\chi|^2}{\varepsilon^2} + 1\right)^{-1}$ satisfies

$$\left|g_{\varepsilon,\chi}(z)\right| \leq C \max\left\{\frac{|\chi|^2}{\varepsilon^2}, 1\right\}^{-1}.$$

($g_{\varepsilon,\chi}$ connects $\frac{1}{|\chi|^2}\mathcal{A}_\chi$ back to $\frac{1}{\varepsilon^2}\mathcal{A}_\chi$)

Proof of $L^2 \to L^2$



- To show: $\left\| (\mathcal{A}_\varepsilon + I)^{-1} - \left(\mathcal{A}^{\text{hom}} + I\right)^{-1} \right\|_{L^2 \to L^2} \leq C\varepsilon.$

- **Step A.** Look at estimates on $L^2(Y)$ first. If $\chi \in [-\mu, \mu]^3 \setminus \{0\}$, then

$$P_\chi \left(\frac{1}{\varepsilon^2} \mathcal{A}_\chi + I \right)^{-1} P_\chi = g_{\varepsilon,\chi}\left(\frac{1}{|\chi|^2} \mathcal{A}_\chi \right) P_{\Gamma, \frac{1}{|\chi|^2}\mathcal{A}_\chi} = -\frac{1}{2\pi i} \oint_\Gamma g_{\varepsilon,\chi}(z) \left(\frac{1}{|\chi|^2} \mathcal{A}_\chi - z \right)^{-1} dz$$

$$\left(\frac{1}{\varepsilon^2} \mathcal{A}_\chi^{\text{hom}} + I_{\mathbb{C}^3} \right)^{-1} S = g_{\varepsilon,\chi}\left(\frac{1}{|\chi|^2} \mathcal{A}_\chi^{\text{hom}} \right) P_{\Gamma, \frac{1}{|\chi|^2}\mathcal{A}_\chi^{\text{hom}}} = -\frac{1}{2\pi i} \oint_\Gamma g_{\varepsilon,\chi}(z) \left(\frac{1}{|\chi|^2} \mathcal{A}_\chi^{\text{hom}} - z \right)^{-1} dz$$

Proj onto espace of the first 3 evalues for $\mathcal{A}_\chi$

Proj onto espace of the evalues enclosed by Γ, for the operator $\frac{1}{|\chi|^2}\mathcal{A}_\chi^{\text{hom}}$.

Recall previous slide:

$$g_{\varepsilon,\chi}(z) = \left(\frac{|\chi|^2}{\varepsilon^2} + 1 \right)^{-1}$$

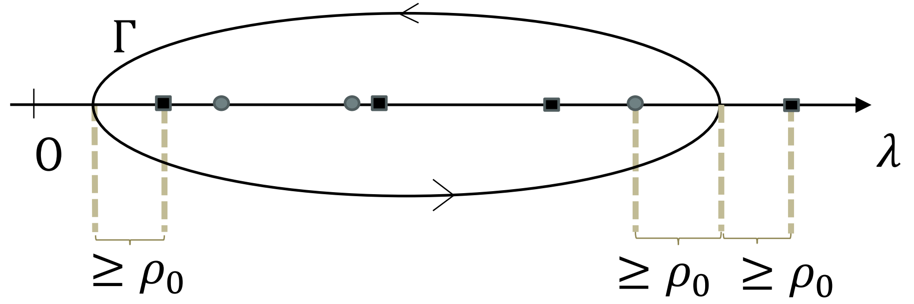
$$\left| g_{\varepsilon,\chi}(z) \right| \leq C \max\left\{ \frac{|\chi|^2}{\varepsilon^2}, 1 \right\}^{-1}$$

- By Step 3 (resolvent expansion)

  $\| \quad - \quad \|_{L^2 \to L^2} \leq C|\chi|$

- (Important!) C does not depend on z and χ, by the properties of the contour Γ.

Proof of $L^2 \to L^2$



- To show: $\left\| (\mathcal{A}_\varepsilon + I)^{-1} - \left(\mathcal{A}^{\text{hom}} + I\right)^{-1} \right\|_{L^2 \to L^2} \leq C\varepsilon.$

- **Step A.** Estimates on $L^2(Y)$.

- **Step A-I.** If $\chi \in [-\mu, \mu]^3 \setminus \{0\}$, then $\left\| P_\chi \left(\frac{1}{\varepsilon^2} \mathcal{A}_\chi + I \right)^{-1} P_\chi - \left(\frac{1}{\varepsilon^2} \mathcal{A}_\chi^{\text{hom}} + I_{\mathbb{C}^3} \right)^{-1} S \right\|_{L^2 \to L^2} \leq C \max\left\{ \frac{|\chi|^2}{\varepsilon^2}, 1 \right\}^{-1} |\chi| \leq C\varepsilon.$

- **Step A-II.** If $\chi \in Y' \setminus [-\mu, \mu]^3$, then by Step 2 (spec analysis of $\mathcal{A}_\chi$)

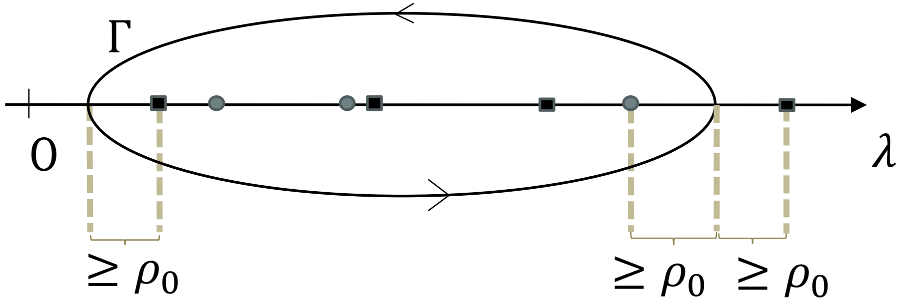
$$\left\| P_\chi \left(\tfrac{1}{\varepsilon^2} \mathcal{A}_\chi + I \right)^{-1} P_\chi \right\|_{L^2 \to L^2} \leq C\varepsilon^2 \quad \text{and} \quad \left\| \left(\tfrac{1}{\varepsilon^2} \mathcal{A}_\chi^{\text{hom}} + I_{\mathbb{C}^3} \right)^{-1} S \right\|_{L^2 \to L^2} \leq C\varepsilon^2$$

- **Step A-III.** The spec analysis of $\mathcal{A}_\chi$ also tells us that for all χ,

$$\left\| (I - P_\chi) \left(\tfrac{1}{\varepsilon^2} \mathcal{A}_\chi + I \right)^{-1} (I - P_\chi) \right\|_{L^2 \to L^2} \leq C\varepsilon^2$$

Overall estimate: $\mathcal{O}(\varepsilon)$ in the $L^2(Y; \mathbb{C}^3) \to L^2(Y; \mathbb{C}^3)$ norm

Proof of $L^2 \to L^2$



- To show: $\left\|(\mathcal{A}_\varepsilon + I)^{-1} - \left(\mathcal{A}^{\mathrm{hom}} + I\right)^{-1}\right\|_{L^2\to L^2} \le C\varepsilon.$

- **Step B.** Back to Estimates on $L^2(\mathbb{R}^3)$. Recall the "passing to the unit cell" formulas

$$(\mathcal{A}_\varepsilon + I)^{-1} = \mathcal{G}_\varepsilon^*\left(\int_{Y'}^{\oplus}\left(\frac{1}{\varepsilon^2}\mathcal{A}_\chi + I\right)^{-1} d\chi\right)\mathcal{G}_\varepsilon$$

$$\left(\mathcal{A}^{\mathrm{hom}} + I\right)^{-1}\Xi_\varepsilon = \mathcal{G}_\varepsilon^*\left(\int_{Y'}^{\oplus}\left(\frac{1}{\varepsilon^2}\mathcal{A}_\chi^{\mathrm{hom}} + I_{\mathbb{C}^3}\right)^{-1} S d\chi\right)\mathcal{G}_\varepsilon$$

Step A →

$$\left\|(\mathcal{A}_\varepsilon + I)^{-1} - \left(\mathcal{A}^{\mathrm{hom}} + I\right)^{-1}\Xi_\varepsilon\right\|_{L^2\to L^2} \le C\varepsilon$$

- **Step C.** Show that you can drop Ξ_ε without affecting the estimates.

$$\left\|\left(\mathcal{A}^{\mathrm{hom}} + I\right)^{-1}(I - \Xi_\varepsilon)\right\|_{L^2\to L^2} \le C\varepsilon^2$$

(Prove this via Fourier transform)

Additional results

Omitted from the talk

1. Extend the results to arbitrary spectral scaling $\gamma \in [-2, \infty)$, e.g.

$$\left\| \left(\frac{1}{\varepsilon^{\gamma}} \mathcal{A}_{\varepsilon} + I \right)^{-1} - \left(\frac{1}{\varepsilon^{\gamma}} \mathcal{A}^{\text{hom}} + I \right)^{-1} \right\|_{L^2 \to L^2} \leq C \varepsilon^{\frac{\gamma+2}{2}}.$$

2. Defining the full-space corrector ops $\mathcal{R}^{\varepsilon}_{corr,j}$ using $\mathcal{R}_{corr,j,\chi}(z)$, e.g.

$$\begin{aligned} \mathcal{R}^{\varepsilon}_{corr,1} &= \mathcal{G}^{*}_{\varepsilon} \left(\int_{Y'}^{\oplus} \mathcal{R}^{\varepsilon}_{corr,1,\chi} d\chi \right) \mathcal{G}_{\varepsilon} \\ &= \mathcal{G}^{*}_{\varepsilon} \left(\int_{Y'}^{\oplus} \mathcal{B}_{corr,1,\chi} \left(\frac{1}{\varepsilon^{\gamma+2}} \mathcal{A}^{\text{hom}}_{\chi} + I_{\mathbb{C}^3} \right)^{-1} S d\chi \right) \mathcal{G}_{\varepsilon}. \end{aligned}$$

$\mathcal{B}_{corr,1,\chi}$ takes $c \in \mathbb{C}^3$ to the solution of the χ-dependent cell-problem $u_1 \in H^1_{\#}$ (recall defn of $\mathcal{A}^{\text{hom}}_{\chi}$)

For $\chi \in [-\mu, \mu] \setminus \{0\}$,

$$\mathcal{R}^{\varepsilon}_{corr,1,\chi} = -\frac{1}{2\pi i} \oint_{\Gamma} g_{\varepsilon,\chi}(z) \mathcal{R}_{corr,1,\chi}(z) dz$$

u_1

Omitted from the talk

3. $L^2 \to H^1$ and higher order $L^2 \to L^2$

$$\left\| \left(\frac{1}{\varepsilon^\gamma} \mathcal{A}_\varepsilon + I \right)^{-1} - \left(\frac{1}{\varepsilon^\gamma} \mathcal{A}^{\text{hom}} + I \right)^{-1} - \mathcal{R}^\varepsilon_{corr,1} \right\|_{L^2 \to H^1} \leq C \max\left\{ \varepsilon^{\gamma+1}, \varepsilon^{\frac{\gamma+2}{2}} \right\}.$$

This is somewhat tedious, many cases to enumerate. Some care needed when passing from $L^2(Y)$ back to $L^2(\mathbb{R}^3)$!
(see next slide)

$$\left\| \left(\frac{1}{\varepsilon^\gamma} \mathcal{A}_\varepsilon + I \right)^{-1} - \left(\frac{1}{\varepsilon^\gamma} \mathcal{A}^{\text{hom}} + I \right)^{-1} - \mathcal{R}^\varepsilon_{corr,1} - \mathcal{R}^\varepsilon_{corr,2} \right\|_{L^2 \to L^2} \leq C \varepsilon^{\gamma+2}.$$

This is easy.

4. Connection between $\mathcal{R}^\varepsilon_{corr,1} f$ and the $\mathcal{O}(\varepsilon)$ term in classical two-scale expansion.

We show that they are the same!

(Thanks Igor for the hint)

Proof structure of $L^2 \to H^1$

	On $L^2(Y;\mathbb{C}^3)$	
	$\chi \in [-\mu,\mu]^3 \backslash \{0\}$	$\chi \in Y' \backslash [-\mu,\mu]^3$
L^2 norm	$\varepsilon^{\frac{\gamma+2}{2}}$	$\varepsilon^{\gamma+2}$
L^2 norm of the gradient	$\varepsilon^{\gamma+2}$	$\varepsilon^{\gamma+2}$

Proof via contour integral | Deal with the terms individually

$\mathcal{G}_\varepsilon^*$

On $L^2(\mathbb{R}^3;\mathbb{C}^3)$
$\varepsilon^{\frac{\gamma+2}{2}}$
$\varepsilon^{\gamma+1}$

Because $\mathcal{G}_\varepsilon$ is unitary

Treat the two terms ∇_y and $\_\otimes\chi$ separately:

- ∇_y part, use [shaded box] to get $\mathcal{O}(\varepsilon^{\gamma+1})$
- $\_\otimes\chi$ part, modify L^2-norm's argument.

  Separately get $\mathcal{O}(\varepsilon^{\gamma+1})$ for small and large χ.

$$\underbrace{\left(\frac{1}{\varepsilon^{\gamma+2}}\mathcal{A}_\chi + I\right)^{-1}}_{\mathbb{X}} - \underbrace{\left(\frac{1}{\varepsilon^{\gamma+2}}\mathcal{A}_\chi^{\text{hom}} + I_{\mathbb{C}^3}\right)^{-1}}_{\mathbb{Y}} - \underbrace{\mathcal{R}_{\text{corr},1,\chi}^{\varepsilon}}_{\mathbb{Z}}$$

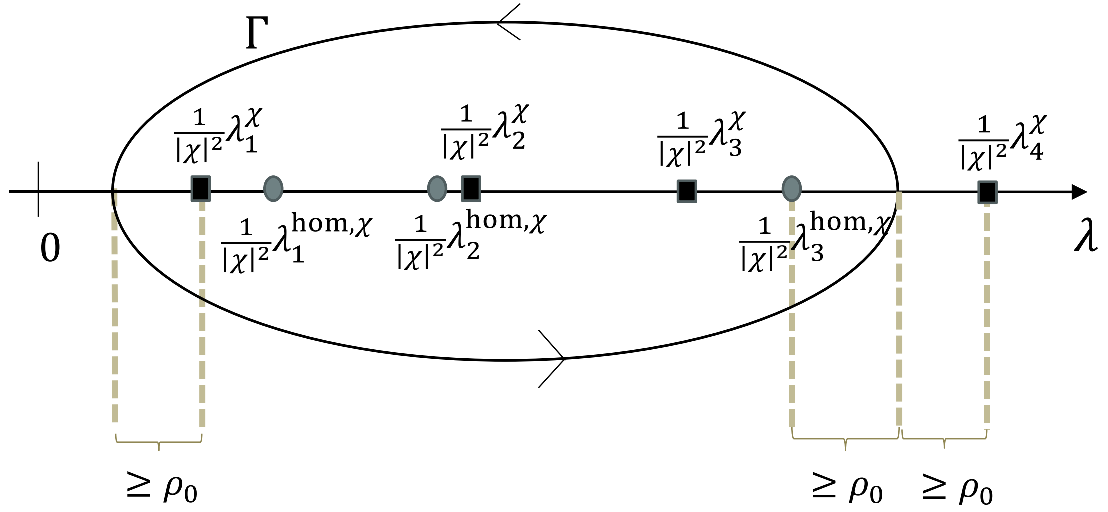
$$\underbrace{\left(\frac{1}{\varepsilon^{\gamma}}\mathcal{A}_\varepsilon + I\right)^{-1}}_{\mathcal{X}} - \underbrace{\left(\frac{1}{\varepsilon^{\gamma}}\mathcal{A}^{\text{hom}} + I_{\mathbb{C}^3}\right)^{-1}\Xi_\varepsilon}_{\mathcal{Y}} - \underbrace{\mathcal{R}_{\text{corr},1}^{\varepsilon}}_{\mathcal{Z}}$$

Connections to existing approaches

Birman-Suslina (2004) spectral germ approach.

Zhikov (1989) spectral approach.

Cooper-Waurick (2019) fibre-homogenisation.


Γ
$\frac{1}{|\chi|^2}\lambda_1^\chi$
$\frac{1}{|\chi|^2}\lambda_2^\chi$
$\frac{1}{|\chi|^2}\lambda_3^\chi$
$\frac{1}{|\chi|^2}\lambda_4^\chi$
0
$\frac{1}{|\chi|^2}\lambda_1^{\mathrm{hom},\chi}$
$\frac{1}{|\chi|^2}\lambda_2^{\mathrm{hom},\chi}$
$\frac{1}{|\chi|^2}\lambda_3^{\mathrm{hom},\chi}$
λ
$\geq \rho_0$
$\geq \rho_0$
$\geq \rho_0$

Thank you!