Asymptotic analysis of fluid-loaded elastic plates

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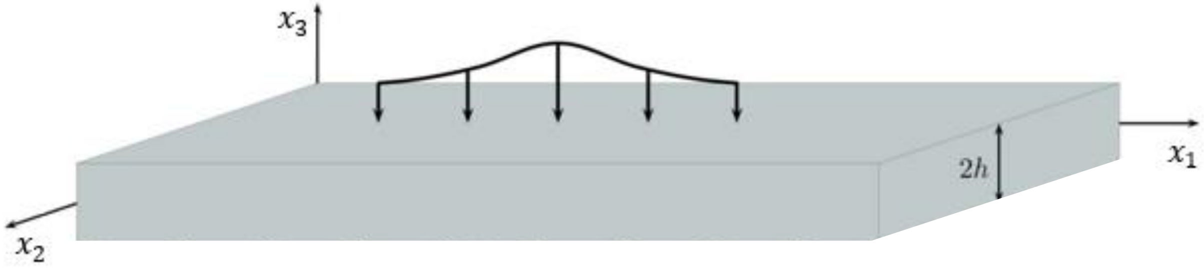
Split

3D -> 2D reduction



Dynamic equations in linear elasticity for thin elastic plates

$$\mathcal{L}\vec{u} + \lambda\vec{u} = 0.$$

Neumann boundary conditions at faces $x_3 = \pm h$

$$\ell\vec{u}\Big|_{x_3 = \pm h} = \vec{p}_{\pm}(x_1, x_2),$$

where ℓ - first-order differential operator and $\vec{p}_{\pm}(x_1, x_2)$ are **prescribed functions**.

2D equations of plate bending (l.o. in low-frequency thin plate limit)

$$D\Delta^2 w - \lambda w = q(x_1, x_2),$$

where

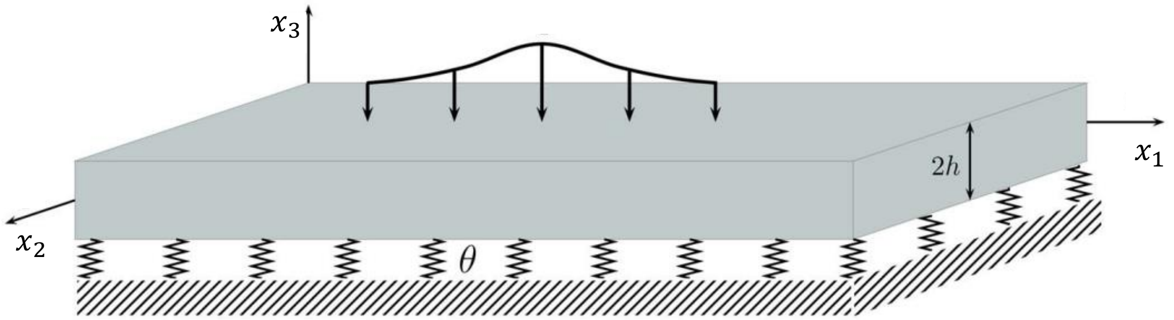
$$u_3 \Big|_{x_3 = 0} \approx w\,(x_1, x_2)$$

and

$$q = p_{3+} - p_{3-}.$$

More general boundary conditions along the faces

Winkler foundation



Mixed boundary conditions

$$\sigma_{33}\Big|_{x_3=-h} = \theta u_3\Big|_{x_3=-h}$$

Engineering approach

$$q(x_1, x_2) = -\theta w(x_1, x_2),$$

which contradicts the original assumption of Neumann boundary conditions!

Thin plate on a Winkler foundation

In this case

$$D\Delta^2 w + (\theta - \lambda) w = 0,$$

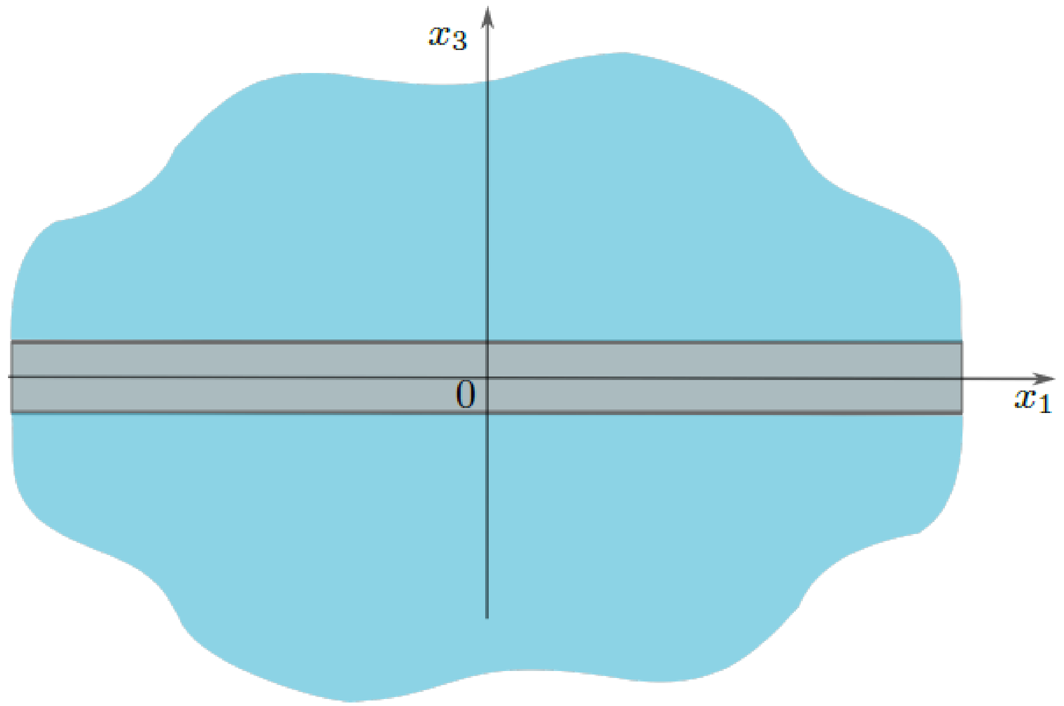
Degeneration near the cut-off frequency $\lambda = \theta$.

3D asymptotic near cut-off analysis at $\frac{\theta h}{E} \ll 1$, see [1]

$$\frac{4Eh^3}{3(1-\nu^2)}\left(1 + \frac{Eh}{(1+\nu)\theta}\Delta\right)\Delta^2 w + \left(\frac{2Eh}{1+\nu} - \frac{(1-2\nu)(3-2\nu)}{3(1-\nu)}\theta h^2\right)\Delta w +$$
$$(1-\nu)\theta\left(1 - \frac{2(1+\nu)(1-2\nu)}{3(1-\nu)}\frac{\theta h}{E}\right) w - \lambda(1-\nu)\left(1 + \frac{2Eh}{(1-\nu^2)\theta}\Delta\right) w = 0$$

Statement of the problem for an immersed elastic layer

Consider free vibrations of an isotropic elastic layer of thickness $2h$ immersed in a non-viscous compressible fluid, specifying Cartesian coordinates $-\infty < x_1, x_2 < \infty, -h \leq x_3 \leq h$, see Figure ; the axes x_2, perpendicular to the plane (x_1, x_3), is not shown in the figure. Throughout the paper we use the following notation: ρ and ρ_0 are solid and fluid densities, respectively; E is Young's modulus, ν is the Poisson's ratio, $c_2 = \sqrt{E/2\rho(1+\nu)}$ is the shear wave speed in solid and c_0 is the wave speed in fluid.



Statement of the problem for an immersed elastic layer

We restrict ourselves to a plane strain problem in the coordinates (x_1, x_3). Then, the equations of motion in linear elasticity may be written as

$$\begin{aligned}\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{31}}{\partial x_3} - \rho \frac{\partial^2 v_1}{\partial t^2} = 0,\\ \frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{33}}{\partial x_3} - \rho \frac{\partial^2 v_3}{\partial t^2} = 0.\end{aligned} \tag{1}$$

Here and below σ_{kl} $(k, l = 1, 2, 3)$ are stresses, v_m $(m = 1, 3)$ are displacements and t is time. In these equations the stresses and displacements satisfy the relations

$$\begin{aligned}\sigma_{11} &= \frac{E}{1-\nu^2}\frac{\partial v_1}{\partial x_1} + \frac{\nu}{1-\nu}\sigma_{33},\\ \sigma_{22} &= \frac{E\nu}{1-\nu^2}\frac{\partial v_1}{\partial x_1} + \frac{\nu}{1-\nu}\sigma_{33},\\ \frac{\partial v_3}{\partial x_3} &= \frac{1}{E}\left(\sigma_{33} - \nu(\sigma_{11} + \sigma_{22})\right),\\ \frac{\partial v_1}{\partial x_3} &= -\frac{\partial v_3}{\partial x_1} + \frac{2(1+\nu)}{E}\sigma_{31},\end{aligned}$$

Statement of the problem for an immersed elastic layer

In addition, we have the wave equation for the fluid displacement potential $\varphi(x_1, x_3, t)$,

$$\frac{\partial^2 \varphi}{\partial x_1^2} + \frac{\partial^2 \varphi}{\partial x_3^2} - \frac{1}{c_0^2}\frac{\partial^2 \varphi}{\partial t^2} = 0. \tag{3}$$

The interfacial conditions at $x_3 = \pm h$ are

$$\sigma_{31} = 0, \quad \sigma_{33} = \rho_0 \frac{\partial^2 \varphi}{\partial t^2}, \quad v_3 = \frac{\partial \varphi}{\partial x_3}. \tag{4}$$

The aim of the paper is to derive asymptotic models of the formulated problem over long-wave low-frequency region, where $h \ll L$ and $h/c_2 \ll T$ with L and T denoting a typical wavelength and time scale, respectively. The focus is on higher order approximations.

Scaling and dimensionless equations

Let us scale the independent variables specified in the previous section by

$$t = T\tau, \qquad x_1 = L\xi, \qquad x_3 = \begin{cases} L\eta\zeta, & \text{if } |x_3| < h, \\ L\gamma, & \text{otherwise}, \end{cases} \tag{5}$$

where $\eta = h/L \ll 1$ is a small geometric parameter. In addition, we assume that

$$T = \frac{L}{\eta^{3/2} c_2}, \tag{6}$$

which is motivated by the relation $\omega h / c_2 \sim (kh)^{5/2}$ between angular frequency ω and wave number k discovered for a fluid-borne bending wave from the asymptotic analysis of the dispersion relation for an immersed layer, see [2] .

Scaling and dimensionless equations

Now, we introduce the dimensionless stresses, displacements and fluid potential setting

$$\begin{aligned} \sigma_{11} &= E\eta\sigma_{11}^*, & \sigma_{22} &= E\eta\sigma_{22}^*, \\ \sigma_{31} &= E\eta^2\sigma_{12}^*, & \sigma_{33} &= E\eta^3\sigma_{33}^*, \\ v_1 &= L\eta v_1^*, & v_3 &= Lv_3^*, \\ \varphi &= L^2\varphi^*. \end{aligned} \tag{7}$$

Here all the starred quantities are assumed to be the same asymptotic order, i.e. of order unity. The asymptotic orders of the displacements and stresses in (7) are the same as for a bending wave on a 'dry', i.e. not contacting fluid, elastic plate, e.g. see [3] . At the same time, for the latter, a typical time scale is $T = \dfrac{L}{\eta^2 c_2}$ instead of (6).

Scaling and dimensionless equations

Thus, equations (1)-(2) in the last section, taking into account relations (5) and (7), can be written in dimensionless form as

$$\frac{\partial \sigma_{31}^*}{\partial \zeta} = -\frac{\partial \sigma_{11}^*}{\partial \xi} + \frac{1}{2(1+\nu)}\eta^3 \frac{\partial^2 v_1^*}{\partial \tau^2},$$
$$\frac{\partial \sigma_{33}^*}{\partial \zeta} = -\frac{\partial \sigma_{31}^*}{\partial \xi} + \frac{1}{2(1+\nu)}\eta \frac{\partial^2 v_3^*}{\partial \tau^2}, \tag{8}$$

and

$$\sigma_{11}^* = \frac{1}{1-\nu^2}\frac{\partial v_1^*}{\partial \xi} + \frac{\nu}{1-\nu}\eta^2 \sigma_{33}^*,$$
$$\sigma_{22}^* = \frac{\nu}{1-\nu^2}\frac{\partial v_1^*}{\partial \xi} + \frac{\nu}{1-\nu}\eta^2 \sigma_{33}^*,$$
$$\frac{\partial v_3^*}{\partial \zeta} = \eta^4 \sigma_{33}^* - \nu\eta^2(\sigma_{11}^* + \sigma_{22}^*), \tag{9}$$
$$\frac{\partial v_1^*}{\partial \zeta} = -\frac{\partial v_3^*}{\partial \xi} + 2(1+\nu)\eta^2 \sigma_{31}^*.$$

Scaling and dimensionless equations

Similarly, the dimensionless form of equations (3)-(4) can be obtained using (5)-(7), which gives the following

$$\frac{\partial^2\varphi^*}{\partial\xi^2}+\frac{\partial^2\varphi^*}{\partial\gamma^2}-\eta^3\delta^2\frac{\partial^2\varphi^*}{\partial\tau^2}=0, \tag{10}$$

and

$$\begin{aligned}&\sigma^*_{31}\Big|_{\zeta=\pm1}=0, \qquad \sigma^*_{33}\Big|_{\zeta=\pm1}=\frac{r}{2(1+\nu)}\frac{\partial^2\varphi^*}{\partial\tau^2}\Big|_{\gamma=\pm\eta},\\&\frac{\partial\varphi^*}{\partial\gamma}\Big|_{\gamma=\pm\eta}=v^*_3\Big|_{\zeta=\pm1},\end{aligned} \tag{11}$$

where

$$r=\frac{\rho_0}{\rho}, \qquad \delta=\frac{c_2}{c_0}. \tag{12}$$

Scaling and dimensionless equations

Now, expand the starred stress and displacement components in an asymptotic series as

$$
\begin{aligned}
v_1^* &= v_1^{(0)} + \eta v_1^{(1)} + \eta^2 v_1^{(2)} + \ldots \\
v_3^* &= v_3^{(0)} + \eta v_3^{(1)} + \eta^2 v_3^{(2)} + \ldots \\
\sigma_{11}^* &= \sigma_{11}^{(0)} + \eta \sigma_{11}^{(1)} + \eta^2 \sigma_{11}^{(2)} + \ldots \\
\sigma_{22}^* &= \sigma_{22}^{(0)} + \eta \sigma_{22}^{(1)} + \eta^2 \sigma_{22}^{(2)} + \ldots \\
\sigma_{33}^* &= \sigma_{33}^{(0)} + \eta \sigma_{33}^{(1)} + \eta^2 \sigma_{33}^{(2)} + \ldots \\
\sigma_{31}^* &= \sigma_{31}^{(0)} + \eta \sigma_{31}^{(1)} + \eta^2 \sigma_{31}^{(2)} + \ldots \\
\varphi^* &= \varphi^{(0)} + \eta \varphi^{(1)} + \eta^2 \varphi^{(2)} + \ldots
\end{aligned}
\tag{13}
$$

Leading order approximation

We consider the leading order approximation of the problem formulated in the previous subsection, keeping only the terms with the suffix (0) in the asymptotic series (13). Firstly, integrating $(9)_3$ - $(9)_4$ along the thickness variable ζ, we obtain, respectively

$$v_3^{(0)} = V_3^{(0)}(\xi,\tau) \quad \text{and} \quad v_1^{(0)} = -\zeta \frac{\partial V_3^{(0)}}{\partial \xi}. \tag{14}$$

Substituting the displacements (14) into $(9)_1$ - $(9)_2$, we obtain

$$\sigma_{11}^{(0)} = -\zeta \frac{1}{1-\nu^2} \frac{\partial^2 V_3^{(0)}}{\partial \xi^2} \quad \text{and} \quad \sigma_{22}^{(0)} = -\zeta \frac{\nu}{1-\nu^2} \frac{\partial^2 V_3^{(0)}}{\partial \xi^2}. \tag{15}$$

Next, integrating $(8)_1$ along ζ, taking into account (14)-(15), gives

$$\sigma_{31}^{(0)} = \zeta^2 \frac{1}{2(1-\nu^2)} \frac{\partial^3 V_3^{(0)}}{\partial \xi^3} + A^{(0)}(\xi,\tau), \tag{16}$$

where $A^{(0)}$ is an arbitrary function. Finally, inserting (14)-(16) into $(8)_2$ and integrating along ζ, we obtain

$$\sigma_{33}^{(0)} = -\zeta^3 \frac{1}{6(1-\nu^2)} \frac{\partial^4 V_3^{(0)}}{\partial \xi^4} - \zeta \frac{\partial A^{(0)}}{\partial \xi}. \tag{17}$$

Leading order approximation

Now, the fluid potential at leading order can be determined by inserting $(13)_7$ into (10), giving the Laplace equation

$$\frac{\partial^2 \varphi^{(0)}}{\partial \xi^2} + \frac{\partial^2 \varphi^{(0)}}{\partial \gamma^2} = 0, \tag{18}$$

governing incompressible fluid. Further, the boundary conditions, obtained by inserting (13) into (11), are

$$\sigma_{31}^{(0)}\Big|_{\zeta=1} = 0, \quad \sigma_{33}^{(0)}\Big|_{\zeta=1} = \frac{r}{2(1+\nu)} \frac{\partial^2 \varphi^{(0)}}{\partial \tau^2}\Big|_{\gamma=\eta}, \quad \frac{\partial \varphi^{(0)}}{\partial \gamma}\Big|_{\gamma=\eta} = v_3^{(0)}. \tag{19}$$

By applying $(19)_1$ and $(19)_2$ to (16) and (17), respectively, we obtain

$$A^{(0)} = -\frac{1}{2(1-\nu^2)} \frac{\partial^3 V_3^{(0)}}{\partial \xi^3}, \tag{20}$$

and

$$\frac{2}{3(1-\nu)} \frac{\partial^4 V_3^{(0)}}{\partial \xi^4} - r \frac{\partial^2 \varphi^{(0)}}{\partial \tau^2}\Big|_{\gamma=\eta} = 0. \tag{21}$$

First order approximation

Now we have

$$\frac{\partial^2 \varphi^{(1)}}{\partial \xi^2} + \frac{\partial^2 \varphi^{(1)}}{\partial \gamma^2} = 0, \tag{22}$$

and

$$\begin{aligned}
v_3^{(1)} &= V_3^{(1)}(\xi, \tau), \\
v_1^{(1)} &= -\zeta \frac{\partial V_3^{(1)}}{\partial \xi}, \\
\sigma_{11}^{(1)} &= -\zeta \frac{1}{1-\nu^2} \frac{\partial^2 V_3^{(1)}}{\partial \xi^2}, \\
\sigma_{22}^{(1)} &= -\zeta \frac{\nu}{1-\nu^2} \frac{\partial^2 V_3^{(1)}}{\partial \xi^2}, \\
\sigma_{31}^{(1)} &= \zeta^2 \frac{1}{2(1-\nu^2)} \frac{\partial^3 V_3^{(1)}}{\partial \xi^3} + A^{(1)}(\xi, \tau), \\
\sigma_{33}^{(1)} &= -\zeta^3 \frac{1}{6(1-\nu^2)} \frac{\partial^4 V_3^{(1)}}{\partial \xi^4} - \zeta \frac{\partial A^{(1)}}{\partial \xi} + \zeta \frac{1}{2(1+\nu)} \frac{\partial^2 V_3^{(0)}}{\partial \tau^2},
\end{aligned} \tag{23}$$

First order approximation

where $A^{(1)}$ is an arbitrary function, and the boundary conditions are

$$\sigma_{31}^{(1)}\Big|_{\zeta=1} = 0, \quad \sigma_{33}^{(1)}\Big|_{\zeta=1} = \frac{r}{2(1+\nu)}\frac{\partial^2\varphi^{(1)}}{\partial\tau^2}\Big|_{\gamma=\eta}, \quad \frac{\partial\varphi^{(1)}}{\partial\gamma}\Big|_{\gamma=\eta} = v_3^{(1)}. \qquad (24)$$

Note that the last term in the expression for $\sigma_{33}^{(1)}$ in (23) corresponds to elastic plate inertia which does not affect the leading order approximation.

Next, applying $(24)_1$ and $(24)_2$ to $(23)_6$ and $(23)_7$, respectively, we obtain

$$A^{(1)} = -\frac{1}{2(1-\nu^2)}\frac{\partial^3 V_3^{(1)}}{\partial\xi^3}, \qquad (25)$$

and

$$\frac{2}{3(1-\nu)}\frac{\partial^4 V_3^{(1)}}{\partial\xi^4} + \frac{\partial^2 V_3^{(0)}}{\partial\tau^2} - r\frac{\partial^2\varphi^{(1)}}{\partial\tau^2}\Big|_{\gamma=\eta} = 0. \qquad (26)$$

The derived equation corresponds to a fluid-loaded Kirchhoff plate.

Second order approximation

In this case we have as above

$$\frac{\partial^2 \varphi^{(2)}}{\partial \xi^2} + \frac{\partial^2 \varphi^{(2)}}{\partial \gamma^2} = 0, \tag{27}$$

and also

$$\begin{aligned}
v_3^{(2)} &= \zeta^2 \frac{\nu}{2(1-\nu)} \frac{\partial^2 V_3^{(0)}}{\partial \xi^2} + V_3^{(2)}(\xi,\tau),\\
v_1^{(2)} &= \frac{1}{6(1-\nu)}((2-\nu)\zeta^3 - 6\zeta)\frac{\partial^3 V_3^{(0)}}{\partial \xi^3} - \zeta \frac{\partial V_3^{(2)}}{\partial \xi},\\
\sigma_{11}^{(2)} &= \frac{1}{6(1-\nu)(1-\nu^2)}(2(1-\nu)\zeta^3 + 3(\nu-2)\zeta)\frac{\partial^4 V_3^{(0)}}{\partial \xi^4}\\
&\quad - \zeta \frac{1}{1-\nu^2}\frac{\partial^2 V_3^{(2)}}{\partial \xi^2},\\
\sigma_{22}^{(2)} &= \frac{\nu}{6(1-\nu)(1-\nu^2)}((1-\nu)\zeta^3 - 3\zeta)\frac{\partial^4 V_3^{(0)}}{\partial \xi^4} - \zeta \frac{\nu}{1-\nu^2}\frac{\partial^2 V_3^{(2)}}{\partial \xi^2},\\
\sigma_{31}^{(2)} &= -\frac{1}{12(1-\nu)(1-\nu^2)}((1-\nu)\zeta^4 + 3(\nu-2)\zeta^2)\frac{\partial^5 V_3^{(0)}}{\partial \xi^5}\\
&\quad + \zeta^2 \frac{1}{2(1-\nu^2)}\frac{\partial^3 V_3^{(2)}}{\partial \xi^3} + A^{(2)}(\xi,\tau),\\
\sigma_{33}^{(2)} &= \frac{1}{60(1-\nu)(1-\nu^2)}((1-\nu)\zeta^5 + 5(\nu-2)\zeta^3)\frac{\partial^6 V_3^{(0)}}{\partial \xi^6}\\
&\quad - \zeta^3 \frac{1}{6(1-\nu^2)}\frac{\partial^4 V_3^{(2)}}{\partial \xi^4} - \zeta \frac{\partial A^{(2)}}{\partial \xi} + \zeta \frac{1}{2(1+\nu)}\frac{\partial^2 V_3^{(1)}}{\partial \tau^2},
\end{aligned} \tag{28}$$

Second order approximation

$$\sigma_{31}^{(2)}\Big|_{\zeta=1} = 0, \quad \sigma_{33}^{(2)}\Big|_{\zeta=1} = \frac{r}{2(1+\nu)}\frac{\partial^2\varphi^{(2)}}{\partial\tau^2}\Big|_{\gamma=\eta}, \quad \frac{\partial\varphi^{(2)}}{\partial\gamma}\Big|_{\gamma=\eta} = v_3^{(2)}\Big|_{\zeta=1}. \quad (29)$$

Applying $(29)_1$ and $(29)_2$ to $(28)_6$ and $(28)_7$, respectively, we derive

$$A^{(2)} = \frac{2\nu-5}{12(1-\nu)(1-\nu^2)}\frac{\partial^5 V_3^{(0)}}{\partial\xi^5} - \frac{1}{2(1-\nu^2)}\frac{\partial^3 V_3^{(2)}}{\partial\xi^3}, \quad (30)$$

and

$$\frac{8-3\nu}{15(1-\nu)^2}\frac{\partial^6 V_3^{(0)}}{\partial\xi^6} + \frac{2}{3(1-\nu)}\frac{\partial^4 V_3^{(2)}}{\partial\xi^4} + \frac{\partial^2 V_3^{(1)}}{\partial\tau^2} - r\frac{\partial^2\varphi^{(2)}}{\partial\tau^2}\Big|_{\gamma=\eta} = 0. \quad (31)$$

Eliminating the 6th order derivative in (31) using (21), see [3] and [4] for more detail, we arrive at

$$\frac{2}{3(1-\nu)}\frac{\partial^4 V_3^{(2)}}{\partial\xi^4} + \frac{\partial^2 V_3^{(1)}}{\partial\tau^2} + r\frac{\partial^2}{\partial\tau^2}\left(\frac{(8-3\nu)}{10(1-\nu)}\frac{\partial^2\varphi^{(0)}}{\partial\xi^2} - \varphi^{(2)}\right)\Big|_{\gamma=\eta} = 0. \quad (32)$$

Third order approximation

$$\frac{\partial^3\varphi^{(3)}}{\partial\xi^2}+\frac{\partial^2\varphi^{(3)}}{\partial\gamma^2}-\delta^2\frac{\partial^2\varphi^{(0)}}{\partial\tau^2}=0, \tag{33}$$

and

$$\begin{aligned}
v_3^{(3)} &= \zeta^2\frac{\nu}{2(1-\nu)}\frac{\partial^2 V_3^{(1)}}{\partial\xi^2}+V_3^{(3)}(\xi,\tau),\\
v_1^{(3)} &= \frac{1}{6(1-\nu)}((2-\nu)\zeta^3-6\zeta)\frac{\partial^3 V_3^{(1)}}{\partial\xi^3}-\zeta\frac{\partial V_3^{(3)}}{\partial\xi},\\
\sigma_{11}^{(3)} &= \frac{1}{6(1-\nu)(1-\nu^2)}(2(1-\nu)\zeta^3+3(\nu-2)\zeta)\frac{\partial^4 V_3^{(1)}}{\partial\xi^4}\\
&\quad-\zeta\frac{1}{1-\nu^2}\frac{\partial^2 V_3^{(3)}}{\partial\xi^2}+\zeta\frac{\nu}{2(1-\nu^2)}\frac{\partial^2 V_3^{(0)}}{\partial\tau^2},\\
\sigma_{22}^{(3)} &= \frac{\nu}{6(1-\nu)(1-\nu^2)}((1-\nu)\zeta^3-3\zeta)\frac{\partial^4 V_3^{(1)}}{\partial\xi^4}\\
&\quad-\zeta\frac{\nu}{1-\nu^2}\frac{\partial^2 V_3^{(3)}}{\partial\xi^2}+\zeta\frac{\nu}{2(1-\nu^2)}\frac{\partial^2 V_3^{(0)}}{\partial\tau^2},\\
\sigma_{31}^{(3)} &= -\frac{1}{12(1-\nu)(1-\nu^2)}((1-\nu)\zeta^4+3(\nu-2)\zeta^2)\frac{\partial^5 V_3^{(1)}}{\partial\xi^5}\\
&\quad+\zeta^2\frac{1}{2(1-\nu^2)}\frac{\partial^3 V_3^{(3)}}{\partial\xi^3}-\zeta^2\frac{1}{4(1-\nu^2)}\frac{\partial^3 V_3^{(0)}}{\partial\xi\partial\tau^2}+A^{(3)}(\xi,\tau),\\
\sigma_{33}^{(3)} &= \frac{1}{60(1-\nu)(1-\nu^2)}((1-\nu)\zeta^5+5(\nu-2)\zeta^3)\frac{\partial^6 V_3^{(0)}}{\partial\xi^6}\\
&\quad-\zeta^3\frac{1}{6(1-\nu^2)}\frac{\partial^4 V_3^{(3)}}{\partial\xi^4}\\
&\quad+\zeta^3\frac{1}{12(1-\nu)}\frac{\partial^4 V_3^{(0)}}{\partial\xi^2\partial\tau^2}-\zeta\frac{\partial A^{(3)}}{\partial\xi}+\zeta\frac{1}{2(1+\nu)}\frac{\partial^2 V_3^{(2)}}{\partial\tau^2},
\end{aligned} \tag{34}$$

Third order approximation

$$\sigma_{31}^{(3)}\Big|_{\zeta=1} = 0, \quad \sigma_{33}^{(3)}\Big|_{\zeta=1} = \frac{r}{2(1+\nu)} \frac{\partial^2 \varphi^{(3)}}{\partial \tau^2}\Big|_{\gamma=\eta}, \quad \frac{\partial \varphi^{(3)}}{\partial \gamma}\Big|_{\gamma=\eta} = v_3^{(3)}\Big|_{\zeta=1}. \quad (35)$$

Applying $(35)_1$ and $(35)_2$ to $(34)_6$ and $(34)_7$, respectively, we have

$$A^{(3)} = \frac{2\nu-5}{12(1-\nu)(1-\nu^2)} \frac{\partial^5 V_3^{(1)}}{\partial \xi^5} - \frac{1}{2(1-\nu^2)} \frac{\partial^3 V_3^{(3)}}{\partial \xi^3} + \frac{1}{4(1-\nu^2)} \frac{\partial^3 V_3^{(0)}}{\partial \xi \partial \tau^2}. \quad (36)$$

Also, after eliminating the term with a 6th order derivative in the same manner as in the previous subsection, we arrive at

$$\begin{aligned} &\frac{2}{3(1-\nu)} \frac{\partial^4 V_3^{(3)}}{\partial \xi^4} + \frac{\partial^2}{\partial \tau^2}\left(V_3^{(2)} + \frac{7\nu-17}{15(1-\nu)} \frac{\partial^2 V_3^{(0)}}{\partial \xi^2} \right) \\ &+ r\frac{\partial^2}{\partial \tau^2}\left(\frac{(8-3\nu)}{10(1-\nu)} \frac{\partial^2 \varphi^{(1)}}{\partial \xi^2} - \varphi^{(3)} \right)\Bigg|_{\gamma=\eta} = 0. \end{aligned} \quad (37)$$

Hierarchy of asymptotic models

At leading order, we have from (18), $(19)_3$ and (21)

$$\left.\frac{1}{3(1-\nu^2)}\frac{\partial^4 v_3^*}{\partial \xi^4} - \frac{r}{2(1+\nu)}\frac{\partial^2 \varphi^*}{\partial \tau^2}\right|_{\gamma=\eta} = 0, \tag{38}$$

$$\frac{\partial^2 \varphi^*}{\partial \xi^2} + \frac{\partial^2 \varphi^*}{\partial \gamma^2} = 0, \tag{39}$$

and

$$\left.\frac{\partial \varphi^*}{\partial \gamma}\right|_{\gamma=\eta} = v_3^*, \tag{40}$$

where $v_3^*(\xi, 0, \tau) = V_3^{(0)}(\xi, \tau)$ and $\varphi^*(\xi, \gamma, \tau) = \varphi^{(0)}(\xi, \gamma, \tau)$. In terms of original

Hierarchy of asymptotic models

In terms of original variables

$$\frac{Eh^3}{3(1-\nu^2)}\frac{\partial^4 v_3}{\partial x_1^4} - \rho_0 \frac{\partial^2 \varphi}{\partial t^2}\bigg|_{x_3=h} = 0, \tag{41}$$

$$\frac{\partial^2 \varphi}{\partial x_1^2} + \frac{\partial^2 \varphi}{\partial x_3^2} = 0, \tag{42}$$

and

$$\frac{\partial \varphi}{\partial x_3}\bigg|_{x_3=h} = v_3. \tag{43}$$

Here and below in this section the transverse displacement v_3 is taken at the mid plane $x_3 = 0$, i.e. $v_3 = v_3(x_1, 0, t)$.

Hierarchy of asymptotic models

First order

Now, we consider the sum of equation (21) and equation (26), multiplied by the small parameter η, to obtain

$$\frac{2}{3(1-\nu)}\frac{\partial^4}{\partial\xi^4}\left(V_3^{(0)}+\eta V_3^{(1)}\right)-r\frac{\partial^2}{\partial\tau^2}\left(\varphi^{(0)}+\eta\varphi^{(1)}\right)\Big|_{\gamma=\eta} \qquad (44)$$

$$+\eta\frac{\partial^2}{\partial\tau^2}\left(V_3^{(0)}+\eta V_3^{(1)}\right)-\eta^2\frac{\partial^2 V_3^{(1)}}{\partial\tau^2}=0.$$

Below, η^2 term in the last equation is neglected. Similarly, we have from (18) , (22) and $(19)_3$, (24), respectively

$$\frac{\partial^2}{\partial\xi^2}\left(\varphi^{(0)}+\eta\varphi^{(1)}\right)+\frac{\partial^2}{\partial\gamma^2}\left(\varphi^{(0)}+\eta\varphi^{(1)}\right)=0, \qquad (45)$$

and

$$\frac{\partial}{\partial\gamma}\left(\varphi^{(0)}+\eta\varphi^{(1)}\right)\Big|_{\gamma=\eta}=V_3^{(0)}+\eta V_3^{(1)}. \qquad (46)$$

Hierarchy of asymptotic models

Formulae (44)-(46) can be re-written as

$$\left.\frac{2}{3(1-\nu)}\frac{\partial^4 v_3^*}{\partial \xi^4}+\eta\frac{\partial^2 v_3^*}{\partial \tau^2}-r\frac{\partial^2 \varphi^*}{\partial \tau^2}\right|_{\gamma=\eta}=0, \tag{47}$$

where $v_3^* = V_3^{(0)} + \eta V_3^{(1)}$ and $\varphi^* = \varphi^{(0)} + \eta\varphi^{(1)}$. In original variables (47) becomes

$$\left.\frac{Eh^3}{3(1-\nu^2)}\frac{\partial^4 v_3}{\partial x_1^4}+\rho h\frac{\partial^2 v_3}{\partial t^2}-\rho_0\frac{\partial^2 \varphi}{\partial t^2}\right|_{x_3=h}=0. \tag{48}$$

Hierarchy of asymptotic models

Second order

Next, similarly to the derivation above, we have from (21), (26) and (32) neglecting $O(\eta^3)$ terms

$$\frac{2}{3(1-\nu)}\frac{\partial^4 v_3^*}{\partial \xi^4} - r\left(1-\eta^2\frac{(8-3\nu)}{10(1-\nu)}\frac{\partial^2}{\partial \xi^2}\right)\frac{\partial^2 \varphi^*}{\partial \tau^2}\bigg|_{\gamma=\eta} + \eta\frac{\partial^2 v_3^*}{\partial \tau^2} = 0, \tag{49}$$

where $v_3^* = V_3^{(0)} + \eta V_3^{(1)} + \eta^2 V_3^{(2)}$ and $\varphi^* = \varphi^{(0)} + \eta\varphi^{(1)} + \eta^2\varphi^{(2)}$. In this case the impenetrability condition to within the same truncation error becomes, see (13), (23) and (28),

$$\frac{\partial \varphi^*}{\partial \gamma}\bigg|_{\gamma=\eta} = \left(1+\eta^2\frac{\nu}{2(1-\nu)}\frac{\partial^2}{\partial \xi^2}\right)v_3^*. \tag{50}$$

In terms of original variables, (49) and (50) are given by

$$\frac{Eh^3}{3(1-\nu^2)}\frac{\partial^4 v_3}{\partial x_1^4} + \rho h\frac{\partial^2 v_3}{\partial t^2} - \rho_0\left(1-h^2\frac{8-3\nu}{10(1-\nu)}\frac{\partial^2}{\partial x_1^2}\right)\frac{\partial^2 \varphi}{\partial t^2}\bigg|_{x_3=h} = 0, \tag{51}$$

and

$$\frac{\partial \varphi}{\partial x_3}\bigg|_{x_3=h} = \left(1+\frac{\nu h^2}{2(1-\nu)}\frac{\partial^2}{\partial x_1^2}\right)v_3. \tag{52}$$

Hierarchy of asymptotic models

Third order

Finally, we deduce from (21), (26), (32) and (37), neglecting $O(\eta^4)$ terms

$$\frac{2}{3(1-\nu)}\frac{\partial^4 v_3^*}{\partial\xi^4} - r\left(1-\eta^2\frac{(8-3\nu)}{10(1-\nu)}\frac{\partial^2}{\partial\xi^2}\right)\frac{\partial^2\varphi^*}{\partial\tau^2}\Big|_{\gamma=\eta} + \eta\left(1+\eta^2\frac{7\nu-17}{15(1-\nu)}\frac{\partial^2}{\partial\xi^2}\right)\frac{\partial^2 v_3^*}{\partial\tau^2} = 0, \tag{53}$$

where $v_3^* = V_3^{(0)} + \eta V_3^{(1)} + \eta^2 V_3^{(2)} + \eta^3 V_3^{(3)}$ and $\varphi^* = \varphi^{(0)} + \eta\varphi^{(1)} + \eta^2\varphi^{(2)} + \eta^3\varphi^{(3)}$. Now, the equation governing fluid motion, taking into account formulae (18), (22), (27) and (33) and neglecting $O(\eta^4)$ terms, can be written as

$$\frac{\partial^2\varphi^*}{\partial\xi^2} + \frac{\partial^2\varphi^*}{\partial\gamma^2} - \eta^3\delta^2\frac{\partial^2\varphi^*}{\partial\tau^2} = 0. \tag{54}$$

In original variables, (53) takes the form

$$\frac{Eh^3}{3(1-\nu^2)}\frac{\partial^4 v_3}{\partial x_1^4} + \rho h\left(1+h^2\frac{7\nu-17}{15(1-\nu)}\frac{\partial^2}{\partial x_1^2}\right)\frac{\partial^2 v_3}{\partial t^2} - \rho_0\left(1-h^2\frac{8-3\nu}{10(1-\nu)}\frac{\partial^2}{\partial x_1^2}\right)\frac{\partial^2\varphi}{\partial t^2}\Big|_{x_3=h} = 0, \tag{55}$$

Dispersion relations for derived asymptotic models

Let us begin with the travelling wave solution of the problem (41)-(43), setting

$$
\begin{aligned}
v_3 &= e^{i(kx_1 - \omega t)}, \\
\varphi &= -\frac{1}{k} e^{-k(x_3 - h) + i(kx_1 - \omega t)},
\end{aligned}
\tag{56}
$$

where k and ω are wavenumber and angular frequency, respectively. On substituting (56) into the aforementioned formulae, we arrive at the dispersion relation

$$\Omega^2 = \frac{2}{3r(1-\nu)} K^5, \tag{57}$$

where K and Ω are dimensionless wavenumber and frequency, respectively, with

$$K = kh, \quad \Omega = \frac{\omega h}{c_2}. \tag{58}$$

Dispersion relations for derived asymptotic models

Next, substituting (56) into (42), (43) and (48), we arrive at the dispersion relation, corresponding to the first order asymptotic model

$$\Omega^2 = \frac{2}{3(1-\nu)} \frac{K^5}{r+K}. \tag{59}$$

Now, we set

$$\begin{aligned} v_3(x_1, 0) &= e^{i(kx_1 - \omega t)}, \\ \varphi &= -\frac{1}{k}\left(1 - \frac{\nu h^2 k^2}{2(1-\nu)}\right) e^{-k(x_3 - h) + i(kx_1 - \omega t)}, \end{aligned} \tag{60}$$

in the equations (42), (51) and (52), resulting in the dispersion relation for the second order model, given by

$$\Omega^2 = \frac{40(1-\nu)K^5}{60r(1-\nu)^2 + 12(4rK^2 + 5K)(1-\nu)^2 + 3r\nu(3\nu - 8)K^4}. \tag{61}$$

Dispersion relations for derived asymptotic models

Similarly, insert

$$v_3(x_1, 0) = e^{i(kx_1 - \omega t)},$$
$$\varphi = -\left(k^2 - \frac{\omega^2}{c_0^2}\right)^{-\frac{1}{2}} \left(1 - \frac{\nu h^2 k^2}{2(1-\nu)}\right) e^{-k(x_3 - h) + i(kx_1 - \omega t)}. \tag{62}$$

into the equations of the third order approximation, given by (3), (52) and (55). As a result, we have

$$\Omega^2 \Big(60(1-\nu)^2 H + 3r(3\nu - 8)(2(\nu-1) + \nu K^2)K^2 + 4(1-\nu)(17 - 7\nu)K^2 H$$
$$- 30r(1-\nu)(2(\nu-1) + \nu K^2) \Big) = 40(1-\nu)K^4 H, \tag{63}$$

where

$$H = \sqrt{K^2 - \Omega^2 \delta^2}. \tag{64}$$

Exact dispersion relation

$$(2K^2 - \Omega^2)^2 \tanh A - 4K^2 AB \tanh B + \frac{\Omega^4 Ar}{H} = 0, \tag{65}$$

where

$$A = \sqrt{K^2 - \Omega^2\kappa^2}, \quad B = \sqrt{K^2 - \Omega^2}, \quad H = \sqrt{K^2 - \Omega^2\delta^2}, \tag{66}$$

with

$$\kappa = \sqrt{\frac{1 - 2\nu}{2 - 2\nu}}. \tag{67}$$

Numerical results

The problem parameters are $c_0 = 1480ms^{-1}$, $\nu = 0.2$, $c_2 = 3156ms^{-1}$, $\rho = 2710kgm^{-3}$ and $\rho_0 = 1000kgm^{-3}$.

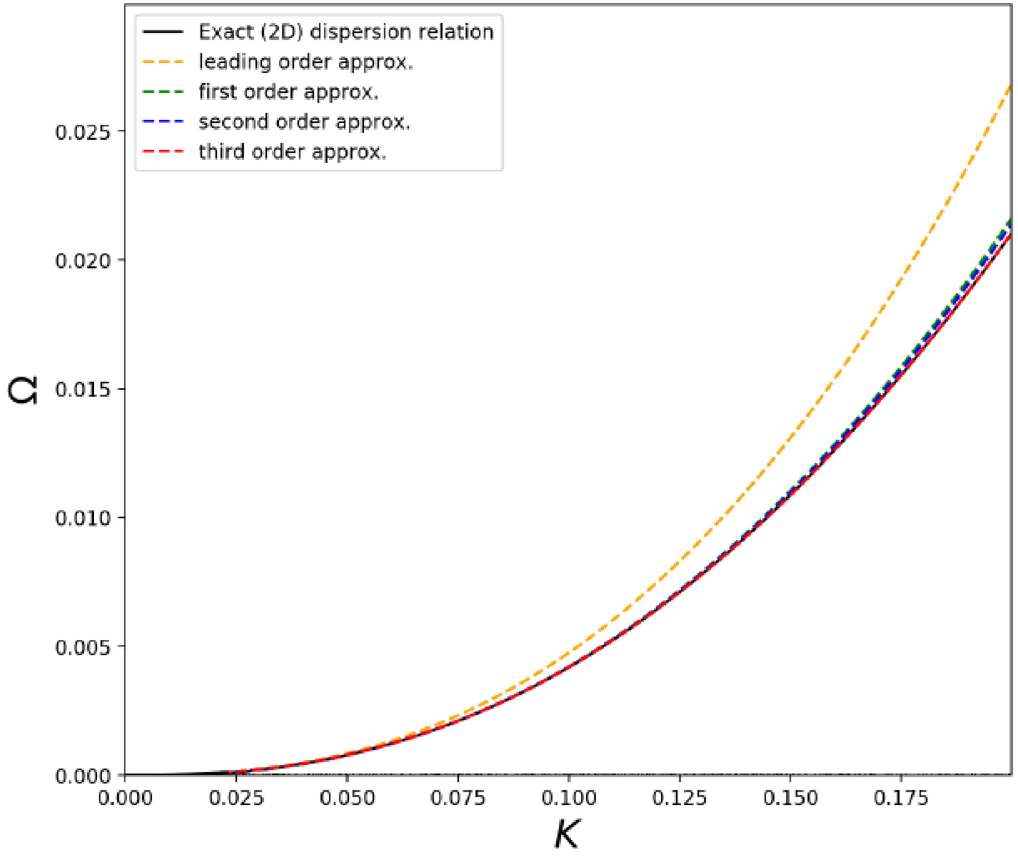


Figure 2: Comparison of the full dispersion relation (65) (solid black line) with the leading (57) (dashed orange line), first (59) (dashed green line), second (61) (dashed blue line) and third order (63) (dashed red line) shortened dispersion relations.

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