Gauge transformations and symmetries of evolution equations

Saša Krešić-Jurić

Faculty of Science, University of Split

Recent progress in quantitative analysis of multiscale media
Split, May 29 – June 2 2023

Outline of the talk

1. General properties of integrable systems
2. Riemann–Hilbert factorization probem on loop groups
3. Gauge equivalent systems
4. Residual gauge transformations and conservation laws

Outline of the talk

1. General properties of integrable systems
2. Riemann–Hilbert factorization probem on loop groups
3. Gauge equivalent systems
4. Residual gauge transformations and conservation laws

Outline of the talk

1. General properties of integrable systems
2. Riemann–Hilbert factorization probem on loop groups
3. Gauge equivalent systems
4. Residual gauge transformations and conservation laws

Outline of the talk

1. General properties of integrable systems
2. Riemann–Hilbert factorization probem on loop groups
3. Gauge equivalent systems
4. Residual gauge transformations and conservation laws

Evolution equations in zero–curvature form

Many physically interesting evolution equations

$$\frac{\partial u}{\partial t} = K(u, u_x, \dots, u^{(n)}), \quad u = u(x,t),$$

where K is a nonlinear function can be written in zero–curvature form

$$\frac{\partial A_1}{\partial x} - \frac{\partial A_2}{\partial t} + [A_1, A_2] = 0, \quad A_1, A_2 \in \mathfrak{g},$$

where $\mathfrak{g}$ is a finite or inifinite dimensional Lie algebra.

Such equations, known as **integrable systems**, have many special properties in common:

- admit soliton solutions,
- solvable by the inverse scattering transform,
- posses Hamiltonian structure (on an infinite dimensional phase space),
- admit an infinite hierarchy of conservation laws (related to inifinitesial symmetries),
- can be solved by the Riemann–Hilbert factorization problem on Lie groups,

Evolution equations in zero–curvature form

Many physically interesting evolution equations

$$\frac{\partial u}{\partial t} = K(u, u_x, \dots, u^{(n)}), \quad u = u(x,t),$$

where K is a nonlinear function can be written in zero–curvature form

$$\frac{\partial A_1}{\partial x} - \frac{\partial A_2}{\partial t} + [A_1, A_2] = 0, \quad A_1, A_2 \in \mathfrak{g},$$

where $\mathfrak{g}$ is a finite or inifinite dimensional Lie algebra.

Such equations, known as **integrable systems**, have many special properties in common:

- admit soliton solutions,
- solvable by the inverse scattering transform,
- posses Hamiltonian structure (on an infinite dimensional phase space),
- admit an infinite hierarchy of conservation laws (related to inifinitesial symmetries),
- can be solved by the Riemann–Hilbert factorization problem on Lie groups,

Evolution equations in zero–curvature form

Many physically interesting evolution equations

$$\frac{\partial u}{\partial t} = K(u, u_x, \ldots, u^{(n)}), \quad u = u(x,t),$$

where K is a nonlinear function can be written in zero–curvature form

$$\frac{\partial A_1}{\partial x} - \frac{\partial A_2}{\partial t} + [A_1, A_2] = 0, \quad A_1, A_2 \in \mathfrak{g},$$

where $\mathfrak{g}$ is a finite or inifinite dimensional Lie algebra.

Such equations, known as **integrable systems**, have many special properties in common:

- admit soliton solutions,
- solvable by the inverse scattering transform,
- posses Hamiltonian structure (on an infinite dimensional phase space),
- admit an infinite hierarchy of conservation laws (related to inifinitesial symmetries),
- can be solved by the Riemann–Hilbert factorization problem on Lie groups,

Evolution equations in zero–curvature form

Many physically interesting evolution equations

$$\frac{\partial u}{\partial t} = K(u, u_x, \dots, u^{(n)}), \quad u = u(x,t),$$

where K is a nonlinear function can be written in zero–curvature form

$$\frac{\partial A_1}{\partial x} - \frac{\partial A_2}{\partial t} + [A_1, A_2] = 0, \quad A_1, A_2 \in \mathfrak{g},$$

where $\mathfrak{g}$ is a finite or inifinite dimensional Lie algebra.

Such equations, known as **integrable systems**, have many special properties in common:

- admit soliton solutions,
- solvable by the inverse scattering transform,
- posses Hamiltonian structure (on an infinite dimensional phase space),
- admit an infinite hierarchy of conservation laws (related to inifinitesial symmetries),
- can be solved by the Riemann–Hilbert factorization problem on Lie groups,

Evolution equations in zero–curvature form

Many physically interesting evolution equations

$$\frac{\partial u}{\partial t} = K(u, u_x, \dots, u^{(n)}), \quad u = u(x,t),$$

where K is a nonlinear function can be written in zero–curvature form

$$\frac{\partial A_1}{\partial x} - \frac{\partial A_2}{\partial t} + [A_1, A_2] = 0, \quad A_1, A_2 \in \mathfrak{g},$$

where $\mathfrak{g}$ is a finite or inifinite dimensional Lie algebra.

Such equations, known as **integrable systems**, have many special properties in common:

- admit soliton solutions,
- solvable by the inverse scattering transform,
- posses Hamiltonian structure (on an infinite dimensional phase space),
- admit an infinite hierarchy of conservation laws (related to inifinitesial symmetries),
- can be solved by the Riemann–Hilbert factorization problem on Lie groups,

Evolution equations in zero–curvature form

Many physically interesting evolution equations

$$\frac{\partial u}{\partial t} = K(u, u_x, \ldots, u^{(n)}), \quad u = u(x,t),$$

where K is a nonlinear function can be written in zero–curvature form

$$\frac{\partial A_1}{\partial x} - \frac{\partial A_2}{\partial t} + [A_1, A_2] = 0, \quad A_1, A_2 \in \mathfrak{g},$$

where $\mathfrak{g}$ is a finite or inifinite dimensional Lie algebra.

Such equations, known as **integrable systems**, have many special properties in common:

- admit soliton solutions,
- solvable by the inverse scattering transform,
- posses Hamiltonian structure (on an infinite dimensional phase space),
- admit an infinite hierarchy of conservation laws (related to inifinitesial symmetries),
- can be solved by the Riemann–Hilbert factorization problem on Lie groups,

Examples of integrable systems

1. Kortweg–de Vries (KdV) equation
$$u_t = u_{xxx} + 3uu_x$$
describes propagation of water waves in a shallow canal.
2. Nonlinear Shrödinger (NLS) equation
$$iu_t - \frac{1}{2}u_{xx} - 4u|u|^2 = 0$$
has applications in fiber optics.
3. Heisenberg magnet (HM) equation
$$S_t = S_{xx} \times S, \quad \|S\| = 1,$$
describes distribution of mangetic moments $S = (S_1, S_2, S_3)$ in a ferromangetic chain.
4. Landau–Lifshitz (LL) equation
$$S_t = S_{xx} \times S + S \times JS, \quad \|S\| = 1,$$
generalizes the HM equation to nonisotropic interaction of magnetic moments with coupling $J = \mathrm{diag}(J_1, J_2, J_3)$.

Examples of integrable systems

1. Kortweg–de Vries (KdV) equation

$$u_t = u_{xxx} + 3uu_x$$

describes propagation of water waves in a shallow canal.

2. Nonlinear Shrödinger (NLS) equation

$$iu_t - \frac{1}{2}u_{xx} - 4u|u|^2 = 0$$

has applications in fiber optics.

3. Heisenberg magnet (HM) equation

$$S_t = S_{xx} \times S, \quad \|S\| = 1,$$

describes distribution of mangetic moments $S = (S_1, S_2, S_3)$ in a ferromangetic chain.

4. Landau–Lifshitz (LL) equation

$$S_t = S_{xx} \times S + S \times JS, \quad \|S\| = 1,$$

generalizes the HM equation to nonisotropic interaction of magnetic moments with coupling $J = \text{diag}(J_1, J_2, J_3)$.

Examples of integrable systems

1. Kortweg–de Vries (KdV) equation

$$u_t = u_{xxx} + 3uu_x$$

describes propagation of water waves in a shallow canal.

2. Nonlinear Shrödinger (NLS) equation

$$iu_t - \frac{1}{2}u_{xx} - 4u|u|^2 = 0$$

has applications in fiber optics.

3. Heisenberg magnet (HM) equation

$$S_t = S_{xx} \times S, \quad \|S\| = 1,$$

describes distribution of mangetic moments $S = (S_1, S_2, S_3)$ in a ferromangetic chain.

4. Landau–Lifshitz (LL) equation

$$S_t = S_{xx} \times S + S \times JS, \quad \|S\| = 1,$$

generalizes the HM equation to nonisotropic interaction of magnetic moments with coupling $J = \mathrm{diag}(J_1, J_2, J_3)$.

Examples of integrable systems

1. Kortweg–de Vries (KdV) equation

$$u_t = u_{xxx} + 3uu_x$$

describes propagation of water waves in a shallow canal.

2. Nonlinear Shrödinger (NLS) equation

$$iu_t - \frac{1}{2}u_{xx} - 4u|u|^2 = 0$$

has applications in fiber optics.

3. Heisenberg magnet (HM) equation

$$S_t = S_{xx} \times S, \quad \|S\| = 1,$$

describes distribution of mangetic moments $S = (S_1, S_2, S_3)$ in a ferromangetic chain.

4. Landau–Lifshitz (LL) equation

$$S_t = S_{xx} \times S + S \times JS, \quad \|S\| = 1,$$

generalizes the HM equation to nonisotropic interaction of magnetic moments with coupling $J = \text{diag}(J_1, J_2, J_3)$.

Examples of integrable systems

1. Kortweg–de Vries (KdV) equation

$$u_t = u_{xxx} + 3uu_x$$

describes propagation of water waves in a shallow canal.

2. Nonlinear Shrödinger (NLS) equation

$$iu_t - \frac{1}{2}u_{xx} - 4u|u|^2 = 0$$

has applications in fiber optics.

3. Heisenberg magnet (HM) equation

$$S_t = S_{xx} \times S, \quad \|S\| = 1,$$

describes distribution of mangetic moments $S = (S_1, S_2, S_3)$ in a ferromangetic chain.

4. Landau–Lifshitz (LL) equation

$$S_t = S_{xx} \times S + S \times JS, \quad \|S\| = 1,$$

generalizes the HM equation to nonisotropic interaction of magnetic moments with coupling $J = \text{diag}(J_1, J_2, J_3)$.

Riemann–Hilbert factorization on Lie groups

Integrable systems in zero–curvature form can be constructed by the Riemann–Hilbert factorization problem on loop groups.

Definition

Let G be a Banach–Lie group. We say that G admits a Riemann–Hilbert factorization if G contains closed subgroups G_- and G_+ such that

$$G_- \cap G_+ = \{e\}, \quad \text{the set } G_-G_+ \text{ is open in} G.$$

Remark

$$G_-G_+ \quad \text{is open in} \quad G \quad \Longleftrightarrow \quad \mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_+$$

where $\mathfrak{g} = T_eG$ and $\mathfrak{g}_\pm = T_eG_\pm$.

Choose $X_1, X_2, \ldots, X_n \in \mathfrak{g}_+$ such that $[X_k, X_l] = 0$ and define the **action** of $\mathbb{R}^n$ on G by

$$\mathbf{t} * g = \exp\Big(\sum_{i=1}^{n} t_i X_i\Big) g, \quad \mathbf{t} = (t_i) \in \mathbb{R}^n.$$

Riemann–Hilbert factorization on Lie groups

Integrable systems in zero–curvature form can be constructed by the Riemann–Hilbert factorization problem on loop groups.

Definition

Let G be a Banach–Lie group. We say that G admits a Riemann–Hilbert factorization if G contains closed subgroups G_- and G_+ such that

$$G_- \cap G_+ = \{e\}, \quad \text{the set } G_-G_+ \text{ is open in} G.$$

Remark

$$G_-G_+ \quad \text{is open in} \quad G \quad \Longleftrightarrow \quad \mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_+$$

where $\mathfrak{g} = T_eG$ and $\mathfrak{g}_\pm = T_eG_\pm$.

Choose $X_1, X_2, \ldots, X_n \in \mathfrak{g}_+$ such that $[X_k, X_l] = 0$ and define the **action** of $\mathbb{R}^n$ on G by

$$\mathbf{t} * g = \exp\Big(\sum_{i=1}^n t_i X_i\Big) g, \quad \mathbf{t} = (t_i) \in \mathbb{R}^n.$$

Integrable systems in zero–curvature form can be constructed by the Riemann–Hilbert factorization problem on loop groups.

Definition

Let G be a Banach–Lie group. We say that G admits a Riemann–Hilbert factorization if G contains closed subgroups G_- and G_+ such that

$$G_- \cap G_+ = \{e\}, \quad \text{the set } G_-G_+ \text{ is open in} G.$$

Remark

$$G_-G_+ \quad \text{is open in} \quad G \quad \Longleftrightarrow \quad \mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_+$$

where $\mathfrak{g} = T_eG$ and $\mathfrak{g}_\pm = T_eG_\pm$.

Choose $X_1, X_2, \ldots, X_n \in \mathfrak{g}_+$ such that $[X_k, X_l] = 0$ and define the **action** of $\mathbb{R}^n$ on G by

$$\mathbf{t} * g = \exp\Big(\sum_{i=1}^{n} t_i X_i\Big) g, \quad \mathbf{t} = (t_i) \in \mathbb{R}^n.$$

Riemann–Hilbert factorization on Lie groups

Integrable systems in zero–curvature form can be constructed by the Riemann–Hilbert factorization problem on loop groups.

Definition

Let G be a Banach–Lie group. We say that G admits a Riemann–Hilbert factorization if G contains closed subgroups G_- and G_+ such that

$$G_- \cap G_+ = \{e\}, \quad \text{the set } G_-G_+ \text{ is open in} G.$$

Remark

$$G_-G_+ \quad \text{is open in} \quad G \quad \Longleftrightarrow \quad \mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_+$$

where $\mathfrak{g} = T_eG$ and $\mathfrak{g}_\pm = T_eG_\pm$.

Choose $X_1, X_2, \ldots, X_n \in \mathfrak{g}_+$ such that $[X_k, X_l] = 0$ and define the **action** of $\mathbb{R}^n$ on G by

$$\mathbf{t} * g = \exp\big(\sum_{i=1}^{n} t_i X_i\big) g, \quad \mathbf{t} = (t_i) \in \mathbb{R}^n.$$

For $g \in G_- G_+$ we have **unique factorization**

$$\exp\Big(\sum_{i=1}^{n} t_i X_i\Big) g = g_-(\mathbf{t}) g_+(\mathbf{t}), \quad \mathbf{t} \in B_\epsilon(0) \quad \textbf{(R-H factorization)}. \qquad (1)$$

The flow $g_-(\mathbf{t}) \in G_-$ represents solutions to a hierarchy of nonlinear evolution equations written in zero–curvature form on the Lie algebra $\mathfrak{g}_+$.

$t_1 = x$ space variable, $t_k = t$ time variable of the k–th flow

$X_k \in \mathfrak{g}_+$ inifinitesimal generator of the k–th flow

By taking derivatives Eq. (1) we find

$$g_-(\mathbf{t})^{-1} X_k \, g_-(\mathbf{t}) = g_-^{-1} \frac{\partial g_-}{\partial t_k} + \frac{\partial g_+}{\partial t_k} g_+^{-1} \in \mathfrak{g}_- \oplus \mathfrak{g}_+$$

and projecting to $\mathfrak{g}_+$ we have

$$\frac{\partial g_+}{\partial t_k} = p_+\Big(g_-^{-1}(\mathbf{t}) X_k g_-(\mathbf{t})\Big) g_+, \quad p_+ : \mathfrak{g} \to \mathfrak{g}_+ \quad \text{projection.}$$

For $g \in G_-G_+$ we have **unique factorization**

$$\exp\Big(\sum_{i=1}^{n} t_i X_i\Big) g = g_-(\mathbf{t}) g_+(\mathbf{t}), \quad \mathbf{t} \in B_\epsilon(0) \quad \textbf{(R-H factorization)}. \qquad (1)$$

The flow $g_-(\mathbf{t}) \in G_-$ represents solutions to a hierarchy of nonlinear evolution equations written in zero–curvature form on the Lie algebra $\mathfrak{g}_+$.

$t_1 = x$ space variable, $t_k = t$ time variable of the k–th flow

$X_k \in \mathfrak{g}_+$ inifinitesimal generator of the k–th flow

By taking derivatives Eq. (1) we find

$$g_-(\mathbf{t})^{-1} X_k\, g_-(\mathbf{t}) = g_-^{-1} \frac{\partial g_-}{\partial t_k} + \frac{\partial g_+}{\partial t_k} g_+^{-1} \in \mathfrak{g}_- \oplus \mathfrak{g}_+$$

and projecting to $\mathfrak{g}_+$ we have

$$\frac{\partial g_+}{\partial t_k} = p_+\Big(g_-^{-1}(\mathbf{t}) X_k g_-(\mathbf{t})\Big) g_+, \quad p_+ : \mathfrak{g} \to \mathfrak{g}_+ \quad \text{projection.}$$

This yields a system of linear PDE's for g_+:

$$\frac{\partial g_+}{\partial t_k} = M_k\, g_+, \quad M_k = p_+\Big(g_-^{-1}(\mathbf{t}) X_k g_-(\mathbf{t})\Big), \quad k = 1, 2, \ldots, n.$$

Since $[X_k, X_l] = 0$ the k and l flows commute, hence

$$(g_+)_{t_k t_l} = (g_+)_{t_l t_k} \quad \Rightarrow \quad M_k \text{ and } M_l \text{ satisfy}$$

$$\frac{\partial M_k}{\partial t_l} - \frac{\partial M_l}{\partial t_k} + [M_k, M_l] = 0 \quad (\textbf{zero–curvature equation on } \mathfrak{g}_+). \qquad (2)$$

Remarks

1. Eq. (2) represents a hierarchy of evolution PDE's in the space variable $x = t_1$ and time variable $t = t_k$, $k \geq 2$ (other variables fixed).
2. Solutions of the hierarchy are determined by the group element $g_-(\mathbf{t}) \in G_-$.
3. If G is the loop group of a matrix Lie group, then Eq. (2) is equivalent with a system of nonlinear PDE's for matrix elements of M_k and M_l.

This yields a system of linear PDE's for g_+:

$$\frac{\partial g_+}{\partial t_k} = M_k\, g_+, \quad M_k = p_+\Big(g_-^{-1}(\mathbf{t}) X_k g_-(\mathbf{t})\Big), \quad k = 1, 2, \ldots, n.$$

Since $[X_k, X_l] = 0$ the k and l flows commute, hence

$$(g_+)_{t_k t_l} = (g_+)_{t_l t_k} \quad \Rightarrow \quad M_k \text{ and } M_l \text{ satisfy}$$

$$\frac{\partial M_k}{\partial t_l} - \frac{\partial M_l}{\partial t_k} + [M_k, M_l] = 0 \quad (\textbf{zero–curvature equation on } \mathfrak{g}_+). \qquad (2)$$

Remarks

1. Eq. (2) represents a hierarchy of evolution PDE's in the space variable $x = t_1$ and time variable $t = t_k$, $k \geq 2$ (other variables fixed).
2. Solutions of the hierarchy are determined by the group element $g_-(\mathbf{t}) \in G_-$.
3. If G is the loop group of a matrix Lie group, then Eq. (2) is equivalent with a system of nonlinear PDE's for matrix elements of M_k and M_l.

This yields a system of linear PDE's for g_+:

$$\frac{\partial g_+}{\partial t_k} = M_k\, g_+, \quad M_k = p_+\Big(g_-^{-1}(\mathbf{t}) X_k g_-(\mathbf{t})\Big), \quad k = 1, 2, \dots, n.$$

Since $[X_k, X_l] = 0$ the k and l flows commute, hence

$$(g_+)_{t_k t_l} = (g_+)_{t_l t_k} \quad \Rightarrow \quad M_k \text{ and } M_l \text{ satisfy}$$

$$\frac{\partial M_k}{\partial t_l} - \frac{\partial M_l}{\partial t_k} + [M_k, M_l] = 0 \quad (\textbf{zero–curvature equation on } \mathfrak{g}_+). \qquad (2)$$

Remarks

1. Eq. (2) represents a hierarchy of evolution PDE's in the space variable $x = t_1$ and time variable $t = t_k$, $k \geq 2$ (other variables fixed).
2. Solutions of the hierarchy are determined by the group element $g_-(\mathbf{t}) \in G_-$.
3. If G is the loop group of a matrix Lie group, then Eq. (2) is equivalent with a system of nonlinear PDE's for matrix elements of M_k and M_l.

This yields a system of linear PDE's for g_+:

$$\frac{\partial g_+}{\partial t_k} = M_k\, g_+, \quad M_k = p_+\Big(g_-^{-1}(\mathbf{t}) X_k g_-(\mathbf{t})\Big), \quad k = 1, 2, \ldots, n.$$

Since $[X_k, X_l] = 0$ the k and l flows commute, hence

$$(g_+)_{t_k t_l} = (g_+)_{t_l t_k} \quad \Rightarrow \quad M_k \text{ and } M_l \text{ satisfy}$$

$$\frac{\partial M_k}{\partial t_l} - \frac{\partial M_l}{\partial t_k} + [M_k, M_l] = 0 \quad (\textbf{zero–curvature equation on } \mathfrak{g}_+). \qquad (2)$$

Remarks

1. Eq. (2) represents a hierarchy of evolution PDE's in the space variable $x = t_1$ and time variable $t = t_k$, $k \geq 2$ (other variables fixed).
2. Solutions of the hierarchy are determined by the group element $g_-(\mathbf{t}) \in G_-$.
3. If G is the loop group of a matrix Lie group, then Eq. (2) is equivalent with a system of nonlinear PDE's for matrix elements of M_k and M_l.

This yields a system of linear PDE's for g_+:

$$\frac{\partial g_+}{\partial t_k} = M_k\, g_+, \quad M_k = p_+\Big(g_-^{-1}(\mathbf{t}) X_k g_-(\mathbf{t})\Big), \quad k = 1, 2, \ldots, n.$$

Since $[X_k, X_l] = 0$ the k and l flows commute, hence

$$(g_+)_{t_k t_l} = (g_+)_{t_l t_k} \quad \Rightarrow \quad M_k \text{ and } M_l \text{ satisfy}$$

$$\frac{\partial M_k}{\partial t_l} - \frac{\partial M_l}{\partial t_k} + [M_k, M_l] = 0 \quad (\textbf{zero–curvature equation on } \mathfrak{g}_+). \qquad (2)$$

Remarks

1. Eq. (2) represents a hierarchy of evolution PDE's in the space variable $x = t_1$ and time variable $t = t_k$, $k \geq 2$ (other variables fixed).
2. Solutions of the hierarchy are determined by the group element $g_-(\mathbf{t}) \in G_-$.
3. If G is the loop group of a matrix Lie group, then Eq. (2) is equivalent with a system of nonlinear PDE's for matrix elements of M_k and M_l.

Concrete examples of integrable systems are obtained by taking G to be a matrix group with elements in the Wienner algebra of functions

$$\mathcal{A} = \left\{ f\colon S^1 \to \mathbb{C} \mid f(z) = \sum_{n=-\infty}^{\infty} a_n z^n, \ \sum_{n=-\infty}^{\infty} |a_n| < \infty \right\}.$$

$$(\mathcal{A}, \|\cdot\|_1) \quad \text{Banach algebra with norm } \|f\|_1 = \sum_{n=-\infty}^{\infty} |a_n|$$

Introduce the algebra of $n \times n$ matrices with elements in $\mathcal{A}$, $M(n, \mathcal{A})$, and define

$$GL(n, \mathcal{A}) = \left\{ g \in M(n, \mathcal{A}) \mid \det(g(z)) \neq 0 \ \ \forall z \in S^1 \right\}.$$

Remarks

1. $M(n, \mathcal{A})$ is a Banach algebra with norm $\|g\| = \sum_{i,j=1}^{n} \|g_{ij}\|_1$.
2. By Wienner's lemma, $\det(g(z)) \neq 0 \ \ \forall z \in S^1 \quad \Rightarrow \quad g^{-1} \in M(n, \mathcal{A})$.
3. $GL(n, \mathcal{A})$ is an open subgroup of $M(n, \mathcal{A})$.
4. The Banach–Lie group G is constructed as a closed subgroup of $GL(n, \mathcal{A})$.

Concrete examples of integrable systems are obtained by taking G to be a matrix group with elements in the Wienner algebra of functions

$$\mathcal{A} = \left\{ f\colon S^1 \to \mathbb{C} \mid f(z) = \sum_{n=-\infty}^{\infty} a_n z^n, \ \sum_{n=-\infty}^{\infty} |a_n| < \infty \right\}.$$

$$(\mathcal{A}, \|\cdot\|_1) \quad \text{Banach algebra with norm } \|f\|_1 = \sum_{n=-\infty}^{\infty} |a_n|$$

Introduce the algebra of $n \times n$ matrices with elements in $\mathcal{A}$, $M(n, \mathcal{A})$, and define

$$GL(n, \mathcal{A}) = \left\{ g \in M(n, \mathcal{A}) \mid \det(g(z)) \neq 0 \ \ \forall z \in S^1 \right\}.$$

Remarks

1. $M(n, \mathcal{A})$ is a Banach algebra with norm $\|g\| = \sum_{i,j=1}^{n} \|g_{ij}\|_1$.
2. By Wienner's lemma, $\det(g(z)) \neq 0 \ \ \forall z \in S^1 \quad \Rightarrow \quad g^{-1} \in M(n, \mathcal{A})$.
3. $GL(n, \mathcal{A})$ is an open subgroup of $M(n, \mathcal{A})$.
4. The Banach–Lie group G is constructed as a closed subgroup of $GL(n, \mathcal{A})$.

The Wienner algebra splits into subalgebras

$$\mathcal{A}_+ = \left\{ f \in \mathcal{A} \mid f(z) = \sum_{n=0}^{\infty} a_n z^n \right\} \quad \text{and} \quad \mathcal{A}_- = \left\{ f \in \mathcal{A} \mid f(z) = \sum_{n=-\infty}^{0} a_n z^n \right\}.$$

The splitting allows us to

- define subgroups G_-, G_+ of G,
- and the corresponding Riemann–Hilbert factorization on G.

Nonlinear Schrödinger equation

Consider the subgroup of $GL(2,\mathcal{A})$ defined by

$$G = \left\{ g(z) \in GL(2,\mathcal{A}) \mid g(z) = \begin{bmatrix} a(z) & b(z) \\ -\bar{b}(z) & \bar{a}(z) \end{bmatrix} \right\},$$

where $a(z) = \sum_{n=-\infty}^{\infty} a_n z^n, \quad \bar{a}(z) \equiv \sum_{n=-\infty}^{\infty} \bar{a}_n z^n$, etc.

The Lie algebra of G is given by

$$\mathfrak{g} = \left\{ X(z) \in M(n,\mathcal{A}) \mid X(z) = \begin{bmatrix} a(z) & b(z) \\ -\bar{b}(z) & \bar{a}(z) \end{bmatrix} \right\}.$$

Define subgroups

$$G_- = \left\{ g(z) \in G \mid g(z) = I + \sum_{n=1}^{\infty} A_n z^{-n} \right\}, \quad G_+ = \left\{ g(z) \in G \mid g(z) = \sum_{n=0}^{\infty} B_n z^n \right\}.$$

Then

1. G_- and G_+ are closed subgroups of G,
2. $G_- \cap G_+ = \{I\}$.

Nonlinear Schrödinger equation

Consider the subgroup of $GL(2,\mathcal{A})$ defined by

$$G = \left\{ g(z) \in GL(2,\mathcal{A}) \mid g(z) = \begin{bmatrix} a(z) & b(z) \\ -\bar{b}(z) & \bar{a}(z) \end{bmatrix} \right\},$$

where $a(z) = \sum_{n=-\infty}^{\infty} a_n z^n, \quad \bar{a}(z) \equiv \sum_{n=-\infty}^{\infty} \bar{a}_n z^n$, etc.

The Lie algebra of G is given by

$$\mathfrak{g} = \left\{ X(z) \in M(n,\mathcal{A}) \mid X(z) = \begin{bmatrix} a(z) & b(z) \\ -\bar{b}(z) & \bar{a}(z) \end{bmatrix} \right\}.$$

Define subgroups

$$G_- = \left\{ g(z) \in G \mid g(z) = I + \sum_{n=1}^{\infty} A_n z^{-n} \right\}, \quad G_+ = \left\{ g(z) \in G \mid g(z) = \sum_{n=0}^{\infty} B_n z^n \right\}.$$

Then

1. G_- and G_+ are closed subgroups of G,
2. $G_- \cap G_+ = \{I\}$.

The Lie algebras of G_- and G_+ satisfy

$$\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_+ \quad \Rightarrow \quad G_-G_+ \quad \text{is open in } G.$$

Define

$$X_k(z) = \sigma z^k \in \mathfrak{g}_+, \quad \sigma = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad k = 1, 2,$$

and consider the Riemann-Hilbert factorization

$$\exp\left(xX_1(z) + tX_2(z)\right) g = g_-(x,t)g_+(x,t), \quad g \in G_-. \tag{3}$$

Theorem

If the group element

$$g_-(x,t) = I + \begin{bmatrix} * & u(x,t) \\ * & * \end{bmatrix} z^{-1} + o(z^{-2})$$

solves the Riemann-Hilbert factorization (3), then u satifes the NLS equation

$$iu_t - \frac{1}{2}u_{xx} - 4u|u|^2 = 0. \tag{4}$$

The Lie algebras of G_- and G_+ satisfy

$$\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_+ \quad \Rightarrow \quad G_-G_+ \quad \text{is open in } G.$$

Define

$$X_k(z) = \sigma z^k \in \mathfrak{g}_+, \quad \sigma = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad k = 1, 2,$$

and consider the Riemann-Hilbert factorization

$$\exp\left(xX_1(z) + tX_2(z)\right) g = g_-(x,t) g_+(x,t), \quad g \in G_-. \tag{3}$$

Theorem

If the group element

$$g_-(x,t) = I + \begin{bmatrix} * & u(x,t) \\ * & * \end{bmatrix} z^{-1} + o(z^{-2})$$

solves the Riemann-Hilbert factorization (3), then u satifes the NLS equation

$$iu_t - \frac{1}{2} u_{xx} - 4u|u|^2 = 0. \tag{4}$$

The Lie algebras of G_- and G_+ satisfy

$$\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_+ \quad \Rightarrow \quad G_-G_+ \quad \text{is open in } G.$$

Define

$$X_k(z) = \sigma z^k \in \mathfrak{g}_+, \quad \sigma = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad k = 1, 2,$$

and consider the Riemann-Hilbert factorization

$$\exp\left(xX_1(z) + tX_2(z)\right) g = g_-(x,t) g_+(x,t), \quad g \in G_-. \tag{3}$$

Theorem

If the group element

$$g_-(x,t) = I + \begin{bmatrix} * & u(x,t) \\ * & * \end{bmatrix} z^{-1} + o(z^{-2})$$

solves the Riemann-Hilbert factorization (3), then u satifes the NLS equation

$$iu_t - \frac{1}{2}u_{xx} - 4u|u|^2 = 0. \tag{4}$$

Sketch of proof. According to the general theory, the matrices

$$M_1 = p_+\big(g_-^{-1}\sigma z g_-\big), \quad M_2 = p_+\big(g_-^{-1}\sigma z^2 g_-\big)$$

satisfy the ZCE

$$\frac{\partial M_1}{\partial t} - \frac{\partial M_2}{\partial x} + [M_1, M_2] = 0. \tag{5}$$

Let $g_- = I + \sum_{n=1}^{\infty} A_n z^{-n}$. Then $g_-^{-1} = I - A_1 z^{-1} + (A_1^2 - A_2)z^{-2} + o(z^{-3})$, hence

$$M_1 = \sigma z + [\sigma, A_1], \quad M_2 = \sigma z^2 + [\sigma, A_1]z + [\sigma, A_2] - A_1[\sigma, A_1].$$

Since $A_n = \begin{bmatrix} a_n & b_n \\ -\bar{b}_n & \bar{a}_n \end{bmatrix}$ for some real valued functions a_n and b_n, we find

$$M_1 = \sigma z + 2\begin{bmatrix} 0 & ib_1 \\ -\overline{(ib_1)} & 0 \end{bmatrix}, \quad M_2 = \sigma z^2 + 2\begin{bmatrix} 0 & ib_1 \\ -\overline{(ib_1)} & 0 \end{bmatrix} z + 2\begin{bmatrix} -|b_1|^2 & v \\ -\bar{v} & i|b_1|^2 \end{bmatrix},$$

where $v = i(b_2 - a_1 b_1)$. If we denote $u = b_1$, then Eq. (5) implies

$$v = \frac{1}{2}u_x, \quad iu_t - \frac{1}{2}u_{xx} - 4u|u|^2 = 0.$$

Sketch of proof. According to the general theory, the matrices

$$M_1 = p_+\big(g_-^{-1}\sigma z g_-\big), \quad M_2 = p_+\big(g_-^{-1}\sigma z^2 g_-\big)$$

satisfy the ZCE

$$\frac{\partial M_1}{\partial t} - \frac{\partial M_2}{\partial x} + [M_1, M_2] = 0. \tag{5}$$

Let $g_- = I + \sum_{n=1}^{\infty} A_n z^{-n}$. Then $g_-^{-1} = I - A_1 z^{-1} + (A_1^2 - A_2)z^{-2} + o(z^{-3})$, hence

$$M_1 = \sigma z + [\sigma, A_1], \quad M_2 = \sigma z^2 + [\sigma, A_1]z + [\sigma, A_2] - A_1[\sigma, A_1].$$

Since $A_n = \begin{bmatrix} a_n & b_n \\ -\bar{b}_n & \bar{a}_n \end{bmatrix}$ for some real valued functions a_n and b_n, we find

$$M_1 = \sigma z + 2\begin{bmatrix} 0 & ib_1 \\ -\overline{(ib_1)} & 0 \end{bmatrix}, \quad M_2 = \sigma z^2 + 2\begin{bmatrix} 0 & ib_1 \\ -\overline{(ib_1)} & 0 \end{bmatrix} z + 2\begin{bmatrix} -|b_1|^2 & v \\ -\bar{v} & i|b_1|^2 \end{bmatrix},$$

where $v = i(b_2 - a_1 b_1)$. If we denote $u = b_1$, then Eq. (5) implies

$$v = \frac{1}{2}u_x, \quad iu_t - \frac{1}{2}u_{xx} - 4u|u|^2 = 0.$$

Sketch of proof. According to the general theory, the matrices

$$M_1 = p_+\big(g_-^{-1}\sigma z g_-\big), \quad M_2 = p_+\big(g_-^{-1}\sigma z^2 g_-\big)$$

satisfy the ZCE

$$\frac{\partial M_1}{\partial t} - \frac{\partial M_2}{\partial x} + [M_1, M_2] = 0. \tag{5}$$

Let $g_- = I + \sum_{n=1}^{\infty} A_n z^{-n}$. Then $g_-^{-1} = I - A_1 z^{-1} + (A_1^2 - A_2)z^{-2} + o(z^{-3})$, hence

$$M_1 = \sigma z + [\sigma, A_1], \quad M_2 = \sigma z^2 + [\sigma, A_1]z + [\sigma, A_2] - A_1[\sigma, A_1].$$

Since $A_n = \begin{bmatrix} a_n & b_n \\ -\bar{b}_n & \bar{a}_n \end{bmatrix}$ for some real valued functions a_n and b_n, we find

$$M_1 = \sigma z + 2\begin{bmatrix} 0 & ib_1 \\ -\overline{(ib_1)} & 0 \end{bmatrix}, \quad M_2 = \sigma z^2 + 2\begin{bmatrix} 0 & ib_1 \\ -\overline{(ib_1)} & 0 \end{bmatrix} z + 2\begin{bmatrix} -|b_1|^2 & v \\ -\bar{v} & i|b_1|^2 \end{bmatrix},$$

where $v = i(b_2 - a_1 b_1)$. If we denote $u = b_1$, then Eq. (5) implies

$$v = \frac{1}{2}u_x, \quad iu_t - \frac{1}{2}u_{xx} - 4u|u|^2 = 0.$$

One–soliton solution

Consider the initial data $g \in G_-$,

$$g = I + \begin{bmatrix} -\alpha & i\beta \\ i\beta & -\alpha \end{bmatrix} z^{-1}, \quad \alpha, \beta \in \mathbb{R}.$$

Solution of the Riemann–Hilbert factorization problem (3) yields

$$g_-(x,t) = I + \begin{bmatrix} a_1(x,t) & b_1(x,t) \\ -\bar{b}_1(x,t) & \bar{a}_1(x,t) \end{bmatrix} z^{-1}$$

where

$$a_1(x,t) = -\alpha + i\beta \tanh\big(2\beta(x+\alpha t)\big),$$
$$b_1(x,t) = i\beta \exp\big(2i(\alpha x + (\alpha^2 - \beta^2)t\big) \operatorname{sech}\big(2\beta(x+\alpha t)\big).$$

By the previous theorem, the function $u = b_1$ is a solution of the NLS equation (one–soliton solution).

Remark

If the initial data $g \in G_-$ has a pole or order N at $z = 0$, then the RH factorization problem leads to N-soliton solution of the NLS equation.

One–soliton solution

Consider the initial data $g \in G_-$,

$$g = I + \begin{bmatrix} -\alpha & i\beta \\ i\beta & -\alpha \end{bmatrix} z^{-1}, \quad \alpha, \beta \in \mathbb{R}.$$

Solution of the Riemann–Hilbert factorization problem (3) yields

$$g_-(x,t) = I + \begin{bmatrix} a_1(x,t) & b_1(x,t) \\ -\bar{b}_1(x,t) & \bar{a}_1(x,t) \end{bmatrix} z^{-1}$$

where

$$a_1(x,t) = -\alpha + i\beta \tanh\bigl(2\beta(x+\alpha t)\bigr),$$
$$b_1(x,t) = i\beta \exp\left(2i(\alpha x + (\alpha^2 - \beta^2)t\right) \operatorname{sech}\bigl(2\beta(x+\alpha t)\bigr).$$

By the previous theorem, the function $u = b_1$ is a solution of the NLS equation (one–soliton solution).

Remark

If the initial data $g \in G_-$ has a pole or order N at $z = 0$, then the RH factorization problem leads to N-soliton solution of the NLS equation.

Heisenberg magnet equation

By choosing different subgroups of G one obtains a Riemann–Hibert factorization that solves the Heisenberg magnet equation

$$S_t = S_{xx} \times S, \quad \|S\| = 1 \tag{6}$$

$S = (S_1, S_2, S_3)$ magnetic moment in a ferromagnetic material with isotropic interaction

Define closed subgroups

$$H_- = \left\{ h(z) \in G \mid h(z) = \sum_{n=0}^{\infty} A_n z^{-1} \right\}, \quad H_+ = \left\{ h(z) \in G \mid h(z) = I + \sum_{n=1}^{\infty} B_n z^n \right\}.$$

Then

1. $H_- \cap H_+ = \{I\}$,
2. $H_- H_+$ is open in G.

Consider the Riemann–Hilbert factorization

$$\exp\left(xX_1(z) + tX_2(z)\right) h = h_-(z) h_+(z), \quad h \in H_-. \tag{7}$$

By choosing different subgroups of G one obtains a Riemann–Hibert factorization that solves the Heisenberg magnet equation

$$S_t = S_{xx} \times S, \quad \|S\| = 1 \tag{6}$$

$S = (S_1, S_2, S_3)$ magnetic moment in a ferromagnetic material with isotropic interaction

Define closed subgroups

$$H_- = \Big\{ h(z) \in G \mid h(z) = \sum_{n=0}^{\infty} A_n z^{-1} \Big\}, \quad H_+ = \Big\{ h(z) \in G \mid h(z) = I + \sum_{n=1}^{\infty} B_n z^n \Big\}.$$

Then

1. $H_- \cap H_+ = \{I\}$,
2. $H_- H_+$ is open in G.

Consider the Riemann–Hilbert factorization

$$\exp\left(xX_1(z) + tX_2(z)\right) h = h_-(z) h_+(z), \quad h \in H_-. \tag{7}$$

By choosing different subgroups of G one obtains a Riemann–Hibert factorization that solves the Heisenberg magnet equation

$$S_t = S_{xx} \times S, \quad \|S\| = 1 \tag{6}$$

$S = (S_1, S_2, S_3)$ magnetic moment in a ferromagnetic material with isotropic interaction

Define closed subgroups

$$H_- = \Big\{h(z) \in G \mid h(z) = \sum_{n=0}^{\infty} A_n z^{-1}\Big\}, \quad H_+ = \Big\{h(z) \in G \mid h(z) = I + \sum_{n=1}^{\infty} B_n z^n\Big\}.$$

Then

1. $H_- \cap H_+ = \{I\}$,
2. $H_- H_+$ is open in G.

Consider the Riemann–Hilbert factorization

$$\exp\left(xX_1(z) + tX_2(z)\right) h = h_-(z) h_+(z), \quad h \in H_-. \tag{7}$$

The matrices

$$\tilde{M}_1 = \tilde{p}_+ \left(h_-^{-1} \sigma z h_- \right) = (A_0^{-1} \sigma A_0) z,$$

$$\tilde{M}_2 = \tilde{p}_+ \left(h_-^{-1} \sigma z^2 h_- \right) = (A_0^{-1} \sigma A_0) z^2 + [A_0^{-1} \sigma A_0, A_0^{-1} A_1] z$$

satisfy the ZCE

$$\frac{\tilde{M}_1}{\partial t} - \frac{\tilde{M}_2}{\partial x} + [\tilde{M}_1, \tilde{M}_2] = 0. \tag{8}$$

Define

$$S = \frac{1}{i} (A_0^{-1} \sigma A_0) = \begin{bmatrix} S_3 & S_1 - iS_2 \\ S_1 + iS_2 & -S_3 \end{bmatrix} = \sum_{i=1}^{3} S_i \sigma_i$$

where

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$ are Pauli spin matrices.

Note that $S^2 = I \quad \Rightarrow \quad \sum_{i=1}^{3} S_i^2 = 1.$

The matrices

$$\tilde{M}_1 = \tilde{p}_+ \left(h_-^{-1}\sigma z h_-\right) = (A_0^{-1}\sigma A_0)z,$$

$$\tilde{M}_2 = \tilde{p}_+ \left(h_-^{-1}\sigma z^2 h_-\right) = (A_0^{-1}\sigma A_0)z^2 + [A_0^{-1}\sigma A_0, A_0^{-1}A_1]z$$

satisfy the ZCE

$$\frac{\tilde{M}_1}{\partial t} - \frac{\tilde{M}_2}{\partial x} + [\tilde{M}_1, \tilde{M}_2] = 0. \tag{8}$$

Define

$$S = \frac{1}{i}(A_0^{-1}\sigma A_0) = \begin{bmatrix} S_3 & S_1 - iS_2 \\ S_1 + iS_2 & -S_3 \end{bmatrix} = \sum_{i=1}^{3} S_i\sigma_i$$

where

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$ are Pauli spin matrices.

Note that $S^2 = I \quad \Rightarrow \quad \sum_{i=1}^{3} S_i^2 = 1.$

The zero–curvature equation for $\tilde{M}_1$ and $\tilde{M}_2$ is equivalent with the matrix equation

$$\frac{\partial S}{\partial t} = \frac{1}{4i}[S, S_{xx}]. \tag{9}$$

which is the Heisenberg magnet equation (6) for the unit vector $S = (S_1, S_2, S_3)$ after rescaling $t \mapsto \frac{1}{2}t$.

Gauge transformations

Let G be a Lie group with Lie algebra $\mathfrak{g}$. Assume $A_1, A_2 \in \mathfrak{g}$ satisfy the zero–curvature equation

$$\frac{\partial A_1}{\partial x_2} - \frac{\partial A_2}{\partial x_1} + [A_1, A_2] = 0.$$

Definition

A gauge transformation $\Gamma_g : \mathfrak{g} \to \mathfrak{g}$ on a pair of matrices (A_1, A_2) by an element $g \in G$ is defined by

$$\Gamma_g(A_k) = gA_k g^{-1} + \frac{\partial g}{\partial x_k} g^{-1}, \quad k = 1, 2. \tag{10}$$

- Γ_g defines an action of the group G on the associated Lie algebra $\mathfrak{g}$.
- The zero–curvature equation is invariant under the gauge transformation:

$$\text{if } B_k = \Gamma_g(A_k), \text{ then } \mathrm{ZCE}(A_1, A_2) \quad \Leftrightarrow \quad \mathrm{ZCE}(B_1, B_2).$$

Gauge transformations

Let G be a Lie group with Lie algebra $\mathfrak{g}$. Assume $A_1, A_2 \in \mathfrak{g}$ satisfy the zero–curvature equation

$$\frac{\partial A_1}{\partial x_2} - \frac{\partial A_2}{\partial x_1} + [A_1, A_2] = 0.$$

Definition

A gauge transformation $\Gamma_g : \mathfrak{g} \to \mathfrak{g}$ on a pair of matrices (A_1, A_2) by an element $g \in G$ is defined by

$$\Gamma_g(A_k) = gA_k g^{-1} + \frac{\partial g}{\partial x_k} g^{-1}, \quad k = 1, 2. \tag{10}$$

- Γ_g defines an action of the group G on the associated Lie algebra $\mathfrak{g}$.
- The zero–curvature equation is invariant under the gauge transformation:

$$\text{if } B_k = \Gamma_g(A_k), \text{ then } \mathrm{ZCE}(A_1, A_2) \quad \Leftrightarrow \quad \mathrm{ZCE}(B_1, B_2).$$

Remarks

1. A gauge transformation generally changes the type of equation represented by ZCE. Such equations are called **gauge equivalent** (e.g. the nonlinear Schrödinger and Heisenberg magnet equation).
2. **Residual gauge transformations** leave a particular equation invariant. Such transformations preserve the particular shape of A_1 and A_2, and lead to a hierarchy of infinitesimal symmetries of the equation and associated conservation laws.

Remarks

1. A gauge transformation generally changes the type of equation represented by ZCE. Such equations are called **gauge equivalent** (e.g. the nonlinear Schrödinger and Heisenberg magnet equation).
2. **Residual gauge transformations** leave a particular equation invariant. Such transformations preserve the particular shape of A_1 and A_2, and lead to a hierarchy of infinitesimal symmetries of the equation and associated conservation laws.

Gauge transformation between the NLS and HM equations

The well known gauge equivalence between the NLS and HM equations can be interpreted in terms of different Riemann–Hilbert factorization of the group G.

Theorem

Suppose

$$\exp\left(xX_1(z)+tX_2(z)\right)=g_-(x,t)g_+(x,t),\quad g\in G_-,$$

is a solution of the RH factorization for the NLS equation, and let

$$M_k=p_+\left(g_-^{-1}\sigma z^k g_-\right),\quad k=1,2.$$

If $g_+(x,t)=\sum_{n=0}^{\infty}B_n(x,t)z^n$, then the matrices defined by the gauge transformation

$$\tilde{M}_k\equiv\Gamma_{B_0^{-1}}(M_k)=B_0^{-1}M_kB_0-B_0^{-1}\frac{\partial B_0}{\partial x},\quad k=1,2,$$

satisfy the ZCE which is equivalent with the HM equation (6).

Gauge transformation between the NLS and HM equations

The well known gauge equivalence between the NLS and HM equations can be interpreted in terms of different Riemann–Hilbert factorization of the group G.

Theorem

Suppose

$$\exp\left(xX_1(z) + tX_2(z)\right) = g_-(x,t)g_+(x,t), \quad g \in G_-,$$

is a solution of the RH factorization for the NLS equation, and let

$$M_k = p_+\left(g_-^{-1}\sigma z^k g_-\right), \quad k = 1, 2.$$

If $g_+(x,t) = \sum_{n=0}^{\infty} B_n(x,t)z^n$, then the matrices defined by the gauge transformation

$$\tilde{M}_k \equiv \Gamma_{B_0^{-1}}(M_k) = B_0^{-1}M_kB_0 - B_0^{-1}\frac{\partial B_0}{\partial x}, \quad k = 1, 2,$$

satisfy the ZCE which is equivalent with the HM equation (6).

Example

Suppose $g_-(x,t) = I + \sum_{k=1}^{N} A_k(x,t) z^{-k}$. Then $B_0(x,t)$ can be found explicitly from

$$B_0(x,t) = A_N^{-1}(x,t) A_N(0,0)$$

and the matrix

$$S = \frac{1}{i} B_0^{-1} \sigma B_0 = \begin{bmatrix} S_3 & S_1 - iS_2 \\ S_1 + iS_2 & -S_3 \end{bmatrix}$$

represents solution of the HM equation.

If we denote

$$A_N(x,t) = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}, \quad A_N(0,0) = \begin{bmatrix} a_0 & b_0 \\ -\bar{b}_0 & \bar{a}_0 \end{bmatrix},$$

then

$$S_3 = 1 + \frac{4\mathrm{Re}(ab\,\overline{a_0 b_0})}{(|a|^2 - |b|^2)(|a_0|^2 - |b_0|^2)}, \tag{11}$$

$$S_1 + iS_2 = 2\,\frac{(|a|^2 - |b|^2)\,a_0 \bar{b}_0 + ab\,\overline{b_0^2} - \overline{ab}\,a_0^2}{(|a|^2 - |b|^2)(|a_0|^2 - |b_0|^2)}. \tag{12}$$

Example

Suppose $g_-(x,t) = I + \sum_{k=1}^{N} A_k(x,t) z^{-k}$. Then $B_0(x,t)$ can be found explicitly from

$$B_0(x,t) = A_N^{-1}(x,t) A_N(0,0)$$

and the matrix

$$S = \frac{1}{i} B_0^{-1} \sigma B_0 = \begin{bmatrix} S_3 & S_1 - iS_2 \\ S_1 + iS_2 & -S_3 \end{bmatrix}$$

represents solution of the HM equation.
If we denote

$$A_N(x,t) = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}, \quad A_N(0,0) = \begin{bmatrix} a_0 & b_0 \\ -\bar{b}_0 & \bar{a}_0 \end{bmatrix},$$

then

$$S_3 = 1 + \frac{4\mathrm{Re}(ab\,\overline{a_0 b_0})}{(|a|^2 - |b|^2)(|a_0|^2 - |b_0|^2)}, \tag{11}$$

$$S_1 + iS_2 = 2\,\frac{(|a|^2 - |b|^2)\,a_0 \bar{b}_0 + ab\,\overline{b_0^2} - \overline{ab}\,a_0^2}{(|a|^2 - |b|^2)(|a_0|^2 - |b_0|^2)}. \tag{12}$$

Consider the solution of the RH problem for the NLS equation

$$g_-(x,t) = I + \begin{bmatrix} -i\alpha\tanh(2\alpha x) & -i\alpha e^{-i\alpha^2 t}\,\mathrm{sech}(2\alpha x) \\ -i\alpha e^{i2\alpha^2 t}\,\mathrm{sech}(2\alpha x) & i\alpha\tanh(2\alpha x) \end{bmatrix} z^{-1}.$$

Then Eqs. (11) and (12) yield the solution

$$S_1(x,t) = 2\cos(2\alpha^2 t)\tanh(2\alpha x)\,\mathrm{sech}(2\alpha x),$$

$$S_2(x,t) = -2\sin(2\alpha^2 t)\tanh(2\alpha x)\,\mathrm{sech}(2\alpha x),$$

$$S_3(x,t) = 2\mathrm{sech}^2(2\alpha x) - 1.$$

$S = (S_1, S_2, S_3)$ magnetization vector which rotates about the z–axis.

Residual gauge transformations

Many evolution equations can be written in zero–curvature form on finite dimensional Lie algebras.

Examples

(1) The Kortweg–de Vries (KdV) equation

$$u_t = u_{xxx} + 3uu_x$$

is equivalent with the ZCE for $A_1, A_2 \in sl(2,\mathbb{R})$,

$$A_1 = \begin{bmatrix} 0 & 1 \\ -\frac{1}{2}u & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -\frac{1}{2}u_x & u \\ -\frac{1}{2}u_{xx} - \frac{1}{2}u^2 & \frac{1}{2}u_x \end{bmatrix}.$$

(2) The Harry Dym equation

$$u_t = -\frac{1}{4}u^3 u_{xxx}$$

is equivalent the ZCE for $A_1, A_2 \in sl(2,\mathbb{R})$ given by

$$A_1 = \begin{bmatrix} 0 & 1 \\ -\frac{1}{u^2} & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -\frac{1}{2}u_x & u \\ -\frac{1}{2}u_{xx} - \frac{1}{u} & \frac{1}{2}u_x \end{bmatrix}.$$

(3) The focusing nonlinear Schrödinger equation for a complex valued function u,

$$iu_t - u_{xx} - 2|u|^2 u = 0,$$

is equivalent with the ZCE for $A_1, A_2 \in su(2,\mathbb{C})$ defined by

$$A_1 = \begin{bmatrix} 0 & u \\ -\bar{u} & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -i|u|^2 & -iu_x \\ -i\bar{u}_x & i|u|^2 \end{bmatrix}.$$

(2) The Harry Dym equation

$$u_t = -\frac{1}{4}u^3 u_{xxx}$$

is equivalent the ZCE for $A_1, A_2 \in sl(2,\mathbb{R})$ given by

$$A_1 = \begin{bmatrix} 0 & 1 \\ -\frac{1}{u^2} & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -\frac{1}{2}u_x & u \\ -\frac{1}{2}u_{xx} - \frac{1}{u} & \frac{1}{2}u_x \end{bmatrix}.$$

(3) The focusing nonlinear Schrödinger equation for a complex valued function u,

$$iu_t - u_{xx} - 2|u|^2 u = 0,$$

is equivalent with the ZCE for $A_1, A_2 \in su(2,\mathbb{C})$ defined by

$$A_1 = \begin{bmatrix} 0 & u \\ -\bar{u} & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -i|u|^2 & -iu_x \\ -i\bar{u}_x & i|u|^2 \end{bmatrix}.$$

Residual gauge transformations for the KdV equation

Suppose that $A_1, A_2 \in sl(2,\mathbb{R})$, $A_i = A_i(x,t)$,

$$A_1 = \begin{bmatrix} R & S \\ T & -R \end{bmatrix}, \quad A_2 = \begin{bmatrix} p & u \\ q & -p \end{bmatrix}$$

satisfy the ZCE

$$\frac{\partial A_1}{\partial t} + \frac{\partial A_2}{\partial x} + [A_1, A_2] = 0. \tag{13}$$

We fix a particular gauge of A_1 as

$$A_1 = \begin{bmatrix} 0 & 1 \\ T & 0 \end{bmatrix} \qquad \text{(Drinfeld–Sokolov gauge).} \tag{14}$$

If $T = -\frac{1}{2}u$, then A_2 is *completely* determined by u since $p = -\frac{1}{2}u_x$ and $q = p_x - \frac{1}{2}u^2$, hence

$$A_2 = \begin{bmatrix} -\frac{1}{2}u_x & u \\ -\frac{1}{2}u_{xx} - \frac{1}{2}u^2 & \frac{1}{2}u_x \end{bmatrix}. \tag{15}$$

In this case, the ZCE equation (13) is equivalent with the KdV equation.

Residual gauge transformations for the KdV equation

Suppose that $A_1, A_2 \in sl(2,\mathbb{R})$, $A_i = A_i(x,t)$,

$$A_1 = \begin{bmatrix} R & S \\ T & -R \end{bmatrix}, \quad A_2 = \begin{bmatrix} p & u \\ q & -p \end{bmatrix}$$

satisfy the ZCE

$$\frac{\partial A_1}{\partial t} + \frac{\partial A_2}{\partial x} + [A_1, A_2] = 0. \qquad (13)$$

We fix a particular gauge of A_1 as

$$A_1 = \begin{bmatrix} 0 & 1 \\ T & 0 \end{bmatrix} \qquad \text{(Drinfeld–Sokolov gauge).} \qquad (14)$$

If $T = -\frac{1}{2}u$, then A_2 is *completely* determined by u since $p = -\frac{1}{2}u_x$ and $q = p_x - \frac{1}{2}u^2$, hence

$$A_2 = \begin{bmatrix} -\frac{1}{2}u_x & u \\ -\frac{1}{2}u_{xx} - \frac{1}{2}u^2 & \frac{1}{2}u_x \end{bmatrix}. \qquad (15)$$

In this case, the ZCE equation (13) is equivalent with the KdV equation.

Goal: determine residual gauge transformations that leave the DS gauge of A_1 invariant. Bu successive application of such transformations we can find new solutions of the KdV equation.

Idea: represent the $sl(2,\mathbb{R})$ matrices in DS gauge as the level set of a function and find the gauge symmetry group of the set.

If G is a local Lie group of transformations acting on the manifold M, then to every $X \in T_eG$ we associate the vector field $\hat{X}\colon M \to TM$ by

$$\hat{X}(a) = \frac{d}{d\tau}\Big|_{\tau=0} \exp(\tau X)\cdot a.$$

Theorem

Let G be a connected local Lie group of transformations acting on the m–dimensional manifold M. Suppose that $F\colon M \to \mathbb{R}^l$, $l \leq m$, is of maximal rank on the level set

$$\mathcal{S} = \{x \in M \mid F(x) = 0\}.$$

Then G is a symmetry group of $\mathcal{S}$ if and only if the Lie derivative

$$\mathcal{L}_{\hat{X}} F(x) = 0 \qquad \forall x \in \mathcal{S}, \quad k = 1, 2, \ldots, l,$$

for every infinitesimal generator X of G.

Goal: determine residual gauge transformations that leave the DS gauge of A_1 invariant. Bu successive application of such transformations we can find new solutions of the KdV equation.

Idea: represent the $sl(2,\mathbb{R})$ matrices in DS gauge as the level set of a function and find the gauge symmetry group of the set.

If G is a local Lie group of transformations acting on the manifold M, then to every $X \in T_eG$ we associate the vector field $\hat{X} \colon M \to TM$ by

$$\hat{X}(a) = \frac{d}{d\tau}\Big|_{\tau=0} \exp(\tau X) \cdot a.$$

Theorem

Let G be a connected local Lie group of transformations acting on the m–dimensional manifold M. Suppose that $F \colon M \to \mathbb{R}^l$, $l \le m$, is of maximal rank on the level set

$$\mathcal{S} = \{x \in M \mid F(x) = 0\}.$$

Then G is a symmetry group of $\mathcal{S}$ if and only if the Lie derivative

$$\mathcal{L}_{\hat{X}} F(x) = 0 \qquad \forall x \in \mathcal{S}, \quad k = 1, 2, \ldots, l,$$

for every infinitesimal generator X of G.

Goal: determine residual gauge transformations that leave the DS gauge of A_1 invariant. Bu successive application of such transformations we can find new solutions of the KdV equation.

Idea: represent the $sl(2,\mathbb{R})$ matrices in DS gauge as the level set of a function and find the gauge symmetry group of the set.

If G is a local Lie group of transformations acting on the manifold M, then to every $X \in T_eG$ we associate the vector field $\hat{X}\colon M \to TM$ by

$$\hat{X}(a) = \frac{d}{d\tau}\Big|_{\tau=0} \exp(\tau X)\cdot a.$$

Theorem

Let G be a connected local Lie group of transformations acting on the m–dimensional manifold M. Suppose that $F\colon M \to \mathbb{R}^l$, $l \leq m$, is of maximal rank on the level set

$$\mathcal{S} = \{x \in M \mid F(x) = 0\}.$$

Then G is a symmetry group of $\mathcal{S}$ if and only if the Lie derivative

$$\mathcal{L}_{\hat{X}} F(x) = 0 \qquad \forall x \in \mathcal{S}, \quad k = 1, 2, \ldots, l,$$

for every infinitesimal generator X of G.

Goal: determine residual gauge transformations that leave the DS gauge of A_1 invariant. Bu successive application of such transformations we can find new solutions of the KdV equation.

Idea: represent the $sl(2,\mathbb{R})$ matrices in DS gauge as the level set of a function and find the gauge symmetry group of the set.

If G is a local Lie group of transformations acting on the manifold M, then to every $X \in T_eG$ we associate the vector field $\hat{X}\colon M \to TM$ by

$$\hat{X}(a) = \frac{d}{d\tau}\Big|_{\tau=0} \exp(\tau X)\cdot a.$$

Theorem

Let G be a connected local Lie group of transformations acting on the m–dimensional manifold M. Suppose that $F\colon M \to \mathbb{R}^l$, $l \leq m$, is of maximal rank on the level set

$$\mathcal{S} = \{x \in M \mid F(x) = 0\}.$$

Then G is a symmetry group of $\mathcal{S}$ if and only if the Lie derivative

$$\mathcal{L}_{\hat{X}} F(x) = 0 \qquad \forall x \in \mathcal{S}, \quad k = 1, 2, \ldots, l,$$

for every infinitesimal generator X of G.

Define $F\colon sl(2,\mathbb{R}) \to \mathbb{R}^2$ by

$$F\left(\begin{bmatrix} a & b \\ c & -a \end{bmatrix}\right) = (a, b-1).$$

Then

$$\mathcal{S} = \left\{A \in sl(2,\mathbb{R}) \mid F(A) = 0\right\}$$

consists of matrices in DS gauge. Using $sl(2,\mathbb{R}) \simeq \mathbb{R}^3$, we find

$$DF(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \forall\, A \in sl(2,\mathbb{R}) \quad \Rightarrow \quad \mathrm{rank}(DF) = \max.\ \text{on } \mathcal{S}.$$

Consider a one–parameter groups of gauge transformations

$$G = \left\{\Gamma_{g(\tau)} \mid g(\tau) = \exp(\tau L(x,t)),\ \tau \in \mathbb{R}\right\}, \quad L \in sl(2,\mathbb{R}).$$

We want to determine conditions on L such that $\Gamma_{g(\tau)}$ is a residual transformation.

The group G acts on $sl(2,\mathbb{R})$ by

$$\Gamma_{g(\tau)}(A) = g(\tau)\, A\, g(\tau)^{-1} + \frac{\partial g(\tau)}{\partial x} g(\tau)^{-1}$$

Define $F\colon sl(2,\mathbb{R}) \to \mathbb{R}^2$ by

$$F\left(\begin{bmatrix} a & b \\ c & -a \end{bmatrix}\right) = (a, b-1).$$

Then

$$\mathcal{S} = \{A \in sl(2,\mathbb{R}) \mid F(A) = 0\}$$

consists of matrices in DS gauge. Using $sl(2,\mathbb{R}) \simeq \mathbb{R}^3$, we find

$$DF(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \forall\, A \in sl(2,\mathbb{R}) \quad \Rightarrow \quad \mathrm{rank}(DF) = \max.\ \text{on } \mathcal{S}.$$

Consider a one–parameter groups of gauge transformations

$$G = \left\{\Gamma_{g(\tau)} \mid g(\tau) = \exp(\tau L(x,t)),\ \tau \in \mathbb{R}\right\}, \quad L \in sl(2,\mathbb{R}).$$

We want to determine conditions on L such that $\Gamma_{g(\tau)}$ is a residual transformation.

The group G acts on $sl(2,\mathbb{R})$ by

$$\Gamma_{g(\tau)}(A) = g(\tau)\, A\, g(\tau)^{-1} + \frac{\partial g(\tau)}{\partial x} g(\tau)^{-1}$$

Hence, G is a symmetry group of the level set $\mathcal{S}$ iff

$$\mathcal{L}_{\hat{L}} F(A) = 0 \quad \forall\, A \in \mathcal{S}$$

where $\hat{L}$ is the vector field on $sl(2,\mathbb{R})$ defined by

$$\hat{L}(A) = \frac{d}{d\tau}\Big|_{\tau=0} \Gamma_{g(\tau)}(A) = [L, A] + \frac{\partial L}{\partial x}.$$

Hence $L \in sl(2,\mathbb{R})$ is an infinitesimal generator of the gauge symmetry group of $\mathcal{S}$ iff

$$\mathcal{L}_{\hat{L}} F(A) = DF(A)\Big([L, A] + \frac{\partial L}{\partial x}\Big) = 0 \quad \forall\, A \in \mathcal{S}. \tag{16}$$

To evaluate the condition (16), denote

$$L = \begin{bmatrix} w & y \\ v & -w \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ T & 0 \end{bmatrix} \in \mathcal{S}.$$

Then

$$[L, A] + \frac{\partial L}{\partial x} = \begin{bmatrix} Ty - v + w_x & 2w + y_x \\ -2wT + v_x & -Ty + v - w_x \end{bmatrix}.$$

Hence, G is a symmetry group of the level set $\mathcal{S}$ iff

$$\mathcal{L}_{\hat{L}} F(A) = 0 \quad \forall\, A \in \mathcal{S}$$

where $\hat{L}$ is the vector field on $sl(2,\mathbb{R})$ defined by

$$\hat{L}(A) = \frac{d}{d\tau}\Big|_{\tau=0} \Gamma_{g(\tau)}(A) = [L, A] + \frac{\partial L}{\partial x}.$$

Hence $L \in sl(2,\mathbb{R})$ is an infinitesimal generator of the gauge symmetry group of $\mathcal{S}$ iff

$$\mathcal{L}_{\hat{L}} F(A) = DF(A)\Big([L, A] + \frac{\partial L}{\partial x}\Big) = 0 \quad \forall\, A \in \mathcal{S}. \tag{16}$$

To evaluate the condition (16), denote

$$L = \begin{bmatrix} w & y \\ v & -w \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ T & 0 \end{bmatrix} \in \mathcal{S}.$$

Then

$$[L, A] + \frac{\partial L}{\partial x} = \begin{bmatrix} Ty - v + w_x & 2w + y_x \\ -2wT + v_x & -Ty + v - w_x \end{bmatrix}.$$

Hence, G is a symmetry group of the level set $\mathcal{S}$ iff

$$\mathcal{L}_{\hat{L}} F(A) = 0 \quad \forall\, A \in \mathcal{S}$$

where $\hat{L}$ is the vector field on $sl(2,\mathbb{R})$ defined by

$$\hat{L}(A) = \frac{d}{d\tau}\Big|_{\tau=0} \Gamma_{g(\tau)}(A) = [L, A] + \frac{\partial L}{\partial x}.$$

Hence $L \in sl(2,\mathbb{R})$ is an infinitesimal generator of the gauge symmetry group of $\mathcal{S}$ iff

$$\mathcal{L}_{\hat{L}} F(A) = DF(A)\Big([L, A] + \frac{\partial L}{\partial x}\Big) = 0 \quad \forall\, A \in \mathcal{S}. \tag{16}$$

To evaluate the condition (16), denote

$$L = \begin{bmatrix} w & y \\ v & -w \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ T & 0 \end{bmatrix} \in \mathcal{S}.$$

Then

$$[L, A] + \frac{\partial L}{\partial x} = \begin{bmatrix} Ty - v + w_x & 2w + y_x \\ -2wT + v_x & -Ty + v - w_x \end{bmatrix}.$$

Using $DF(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ we find

$$DF(A)\left([L, A] + \frac{\partial L}{\partial x}\right) = \begin{bmatrix} Ty - v + w_x \\ 2w + y_x \end{bmatrix},$$

hence L must satisfy the condition

$$Ty - v + w_x = 0, \quad 2w + y_x = 0.$$

The choice $T = -\frac{1}{2}u$ yields the KdV equation, thus

$$L = \begin{bmatrix} -\frac{1}{2}y_x & y \\ -\frac{1}{2}uy - \frac{1}{2}y_{xx} & \frac{1}{2}y_x \end{bmatrix} \tag{17}$$

is an infinitesimal generator of the gauge symmetry group for the KdV equation parametrized by the function $y(x, t)$.

- The function y satisfies a linear PDE depending on the solution u of the KdV equation.
- The equation is found by expanding the gauge transformations of the KdV matrices into powers of τ.

Using $DF(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ we find

$$DF(A)\left([L, A] + \frac{\partial L}{\partial x}\right) = \begin{bmatrix} Ty - v + w_x \\ 2w + y_x \end{bmatrix},$$

hence L must satisfy the condition

$$Ty - v + w_x = 0, \quad 2w + y_x = 0.$$

The choice $T = -\frac{1}{2}u$ yields the KdV equation, thus

$$L = \begin{bmatrix} -\frac{1}{2}y_x & y \\ -\frac{1}{2}uy - \frac{1}{2}y_{xx} & \frac{1}{2}y_x \end{bmatrix} \tag{17}$$

is an infinitesimal generator of the gauge symmetry group for the KdV equation parametrized by the function $y(x, t)$.

- The function y satisfies a linear PDE depending on the solution u of the KdV equation.
- The equation is found by expanding the gauge transformations of the KdV matrices into powers of τ.

Using $DF(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ we find

$$DF(A)\Big([L, A] + \frac{\partial L}{\partial x}\Big) = \begin{bmatrix} Ty - v + w_x \\ 2w + y_x \end{bmatrix},$$

hence L must satisfy the condition

$$Ty - v + w_x = 0, \quad 2w + y_x = 0.$$

The choice $T = -\frac{1}{2}u$ yields the KdV equation, thus

$$L = \begin{bmatrix} -\frac{1}{2}y_x & y \\ -\frac{1}{2}uy - \frac{1}{2}y_{xx} & \frac{1}{2}y_x \end{bmatrix} \tag{17}$$

is an infinitesimal generator of the gauge symmetry group for the KdV equation parametrized by the function $y(x, t)$.

- The function y satisfies a linear PDE depending on the solution u of the KdV equation.
- The equation is found by expanding the gauge transformations of the KdV matrices into powers of τ.

For L given by Eq. (17), consider the gauge transformation of

$$A_1 = \begin{bmatrix} 0 & 1 \\ -\frac{1}{2}u & \end{bmatrix}, \quad A_2 = \begin{bmatrix} -\frac{1}{2}u_x & u \\ -\frac{1}{2}u_{xx} - \frac{1}{2}u^2 & \frac{1}{2}u_x \end{bmatrix}.$$

$$\Gamma_{g(\tau)}(A_1) = \begin{bmatrix} 0 & 1 \\ -\frac{1}{2}u & 0 \end{bmatrix} + \tau \begin{bmatrix} 0 & 0 \\ -\frac{1}{2}y_{xxx} - uy_x - \frac{1}{2}u_x y & 0 \end{bmatrix} + o(\tau^2), \tag{18}$$

$$\Gamma_{g(\tau)}(A_2) = \begin{bmatrix} -\frac{1}{2}u_x & u \\ -\frac{1}{2}u_{xx} - \frac{1}{2}u^2 & \frac{1}{2}u_x \end{bmatrix} + \tau \begin{bmatrix} * & y_t - uy_x + u_x y \\ * & * \end{bmatrix} + o(\tau^2). \tag{19}$$

Equations (18) and (19) give deformations of u to first order in τ:

$$\delta u(y) = y_{xxx} + 2y_x + u_x y,$$
$$\delta u(y) = y_t - uy_x + u_x y.$$

which imply that

$$y_t = y_{xxx} + 3uy_x. \tag{20}$$

Remark

Eq. (20) is the linear equation associated to KdV found by Gardner by the Inverse scattering transform.

For L given by Eq. (17), consider the gauge transformation of

$$A_1 = \begin{bmatrix} 0 & 1 \\ -\frac{1}{2}u & \end{bmatrix}, \quad A_2 = \begin{bmatrix} -\frac{1}{2}u_x & u \\ -\frac{1}{2}u_{xx} - \frac{1}{2}u^2 & \frac{1}{2}u_x \end{bmatrix}.$$

$$\Gamma_{g(\tau)}(A_1) = \begin{bmatrix} 0 & 1 \\ -\frac{1}{2}u & 0 \end{bmatrix} + \tau \begin{bmatrix} 0 & 0 \\ -\frac{1}{2}y_{xxx} - uy_x - \frac{1}{2}u_x y & 0 \end{bmatrix} + o(\tau^2), \tag{18}$$

$$\Gamma_{g(\tau)}(A_2) = \begin{bmatrix} -\frac{1}{2}u_x & u \\ -\frac{1}{2}u_{xx} - \frac{1}{2}u^2 & \frac{1}{2}u_x \end{bmatrix} + \tau \begin{bmatrix} * & y_t - uy_x + u_x y \\ * & * \end{bmatrix} + o(\tau^2). \tag{19}$$

Equations (18) and (19) give deformations of u to first order in τ:

$$\delta u(y) = y_{xxx} + 2y_x + u_x y,$$

$$\delta u(y) = y_t - uy_x + u_x y.$$

which imply that

$$y_t = y_{xxx} + 3uy_x. \tag{20}$$

Remark

Eq. (20) is the linear equation associated to KdV found by Gardner by the Inverse scattering transform.

For L given by Eq. (17), consider the gauge transformation of

$$A_1 = \begin{bmatrix} 0 & 1 \\ -\frac{1}{2}u & \end{bmatrix}, \quad A_2 = \begin{bmatrix} -\frac{1}{2}u_x & u \\ -\frac{1}{2}u_{xx} - \frac{1}{2}u^2 & \frac{1}{2}u_x \end{bmatrix}.$$

$$\Gamma_{g(\tau)}(A_1) = \begin{bmatrix} 0 & 1 \\ -\frac{1}{2}u & 0 \end{bmatrix} + \tau \begin{bmatrix} 0 & 0 \\ -\frac{1}{2}y_{xxx} - uy_x - \frac{1}{2}u_x y & 0 \end{bmatrix} + o(\tau^2), \tag{18}$$

$$\Gamma_{g(\tau)}(A_2) = \begin{bmatrix} -\frac{1}{2}u_x & u \\ -\frac{1}{2}u_{xx} - \frac{1}{2}u^2 & \frac{1}{2}u_x \end{bmatrix} + \tau \begin{bmatrix} * & y_t - uy_x + u_x y \\ * & * \end{bmatrix} + o(\tau^2). \tag{19}$$

Equations (18) and (19) give deformations of u to first order in τ:

$$\delta u(y) = y_{xxx} + 2y_x + u_x y,$$

$$\delta u(y) = y_t - uy_x + u_x y.$$

which imply that

$$y_t = y_{xxx} + 3uy_x. \tag{20}$$

Remark

Eq. (20) is the linear equation associated to KdV found by Gardner by the Inverse scattering transform.

Theorem (Infinitesimal transformations of KdV)

If u satisfies the KdV equation and y is a solution of the associated linear equation (20), then

$$L = \begin{bmatrix} -\frac{1}{2}y_x & y \\ -\frac{1}{2}uy - \frac{1}{2}y_{xx} & \frac{1}{2}y_x \end{bmatrix}$$

is an infinitesimal generator of the gauge symmetry group for the KdV equation and

$$\tilde{u} = u + \tau(y_{xxx} + 2y_x + u_x y)$$

satisfies the KdV equation to first order in τ.

Hierarchy of residual gauge transformations for KdV and associated conservation laws

To each solution u of the KdV equation one can associate a hierarchy of residual gauge transformations $L^{(1)}, L^{(1)}, L^{(2)}, \dots$ and associated local conservation laws.

Suppose $y^{(1)}$ is a solution of the associated linear equation (20) and define

$$G^{(1)} = \delta u(y^{(1)}) = y^{(1)}_{xxx} + 2uy^{(1)}_x + u_x y^{(1)}.$$

Then $G^{(1)}$ satisfies the evolution equation

$$G^{(1)}_t = G^{(1)}_{xxx} + 3u_x G^{(1)} + 3uG^{(1)}_x$$

which can be written as the local conservation law

$$\frac{\partial G^{(1)}}{\partial t} = \frac{\partial F^{(1)}}{\partial x}, \quad F^{(1)} = G^{(1)}_{xx} + 3uG^{(1)}$$

for density $G^{(1)}$ and flux $F^{(1)}$.

Hierarchy of residual gauge transformations for KdV and associated conservation laws

To each solution u of the KdV equation one can associate a hierarchy of residual gauge transformations $L^{(1)}, L^{(1)}, L^{(2)}, \ldots$ and associated local conservation laws.

Suppose $y^{(1)}$ is a solution of the associated linear equation (20) and define

$$G^{(1)} = \delta u(y^{(1)}) = y^{(1)}_{xxx} + 2uy^{(1)}_x + u_x y^{(1)}.$$

Then $G^{(1)}$ satisfies the evolution equation

$$G^{(1)}_t = G^{(1)}_{xxx} + 3u_x G^{(1)} + 3uG^{(1)}_x$$

which can be written as the local conservation law

$$\frac{\partial G^{(1)}}{\partial t} = \frac{\partial F^{(1)}}{\partial x}, \quad F^{(1)} = G^{(1)}_{xx} + 3uG^{(1)}$$

for density $G^{(1)}$ and flux $F^{(1)}$.

Hierarchy of residual gauge transformations for KdV and associated conservation laws

To each solution u of the KdV equation one can associate a hierarchy of residual gauge transformations $L^{(1)}, L^{(1)}, L^{(2)}, \dots$ and associated local conservation laws.

Suppose $y^{(1)}$ is a solution of the associated linear equation (20) and define

$$G^{(1)} = \delta u(y^{(1)}) = y^{(1)}_{xxx} + 2uy^{(1)}_x + u_x y^{(1)}.$$

Then $G^{(1)}$ satisfies the evolution equation

$$G^{(1)}_t = G^{(1)}_{xxx} + 3u_x G^{(1)} + 3uG^{(1)}_x$$

which can be written as the local conservation law

$$\frac{\partial G^{(1)}}{\partial t} = \frac{\partial F^{(1)}}{\partial x}, \quad F^{(1)} = G^{(1)}_{xx} + 3uG^{(1)}$$

for density $G^{(1)}$ and flux $F^{(1)}$.

Define $y^{(2)}$ by the condition $y_x^{(2)} = G^{(1)}$. Then

$$y_t^{(2)} = \int \frac{\partial G^{(1)}}{\partial t} dx = F^{(1)} = y_{xxx}^{(2)} + 3u y_x^{(2)},$$

hence $y^{(2)}$ also satisfies Eq. (20).

By iterating the above procedure we obtain an infinite hierarchy of

1. solutions of Eq. (20) defined by

$$y_x^{(n+1)} = G^{(n)}, \quad G^{(n)} = \delta u(y^{(n)}),$$

2. local conservation laws

$$\frac{\partial G^{(n)}}{\partial t} = \frac{\partial F^{(n)}}{\partial x}, \quad F^{(n)} = G_{xx}^{(n)} + 3uG^{(n)}, \quad n \geq 1.$$

By starting with the trivial solution $y^{(1)} = 1$, we find

$$y^{(2)} = u,$$

$$y^{(3)} = \frac{3}{2}u^2 + u_{xx},$$

$$y^{(4)} = \frac{15}{6}u^3 + \frac{5}{2}u_x^2 + 5uu_{xx} + u_{xxx} \qquad \text{etc.}$$

Define $y^{(2)}$ by the condition $y_x^{(2)} = G^{(1)}$. Then

$$y_t^{(2)} = \int \frac{\partial G^{(1)}}{\partial t}\, dx = F^{(1)} = y_{xxx}^{(2)} + 3u y_x^{(2)},$$

hence $y^{(2)}$ also satisfies Eq. (20).

By iterating the above procedure we obtain an infinite hierarchy of

1. solutions of Eq. (20) defined by

$$y_x^{(n+1)} = G^{(n)}, \quad G^{(n)} = \delta u(y^{(n)}),$$

2. local conservation laws

$$\frac{\partial G^{(n)}}{\partial t} = \frac{\partial F^{(n)}}{\partial x}, \quad F^{(n)} = G_{xx}^{(n)} + 3uG^{(n)}, \quad n \geq 1.$$

By starting with the trivial solution $y^{(1)} = 1$, we find

$$y^{(2)} = u,$$
$$y^{(3)} = \frac{3}{2}u^2 + u_{xx},$$
$$y^{(4)} = \frac{15}{6}u^3 + \frac{5}{2}u_x^2 + 5uu_{xx} + u_{xxx} \qquad \text{etc.}$$

Define $y^{(2)}$ by the condition $y_x^{(2)} = G^{(1)}$. Then

$$y_t^{(2)} = \int \frac{\partial G^{(1)}}{\partial t} dx = F^{(1)} = y_{xxx}^{(2)} + 3uy_x^{(2)},$$

hence $y^{(2)}$ also satisfies Eq. (20).

By iterating the above procedure we obtain an infinite hierarchy of

1. solutions of Eq. (20) defined by

$$y_x^{(n+1)} = G^{(n)}, \quad G^{(n)} = \delta u(y^{(n)}),$$

2. local conservation laws

$$\frac{\partial G^{(n)}}{\partial t} = \frac{\partial F^{(n)}}{\partial x}, \quad F^{(n)} = G_{xx}^{(n)} + 3uG^{(n)}, \quad n \geq 1.$$

By starting with the trivial solution $y^{(1)} = 1$, we find

$$y^{(2)} = u,$$

$$y^{(3)} = \frac{3}{2}u^2 + u_{xx},$$

$$y^{(4)} = \frac{15}{6}u^3 + \frac{5}{2}u_x^2 + 5uu_{xx} + u_{xxx} \qquad \text{etc.}$$

Define $y^{(2)}$ by the condition $y_x^{(2)} = G^{(1)}$. Then

$$y_t^{(2)} = \int \frac{\partial G^{(1)}}{\partial t} dx = F^{(1)} = y_{xxx}^{(2)} + 3u y_x^{(2)},$$

hence $y^{(2)}$ also satisfies Eq. (20).

By iterating the above procedure we obtain an infinite hierarchy of

1. solutions of Eq. (20) defined by

$$y_x^{(n+1)} = G^{(n)}, \quad G^{(n)} = \delta u(y^{(n)}),$$

2. local conservation laws

$$\frac{\partial G^{(n)}}{\partial t} = \frac{\partial F^{(n)}}{\partial x}, \quad F^{(n)} = G_{xx}^{(n)} + 3uG^{(n)}, \quad n \geq 1.$$

By starting with the trivial solution $y^{(1)} = 1$, we find

$$\begin{aligned} y^{(2)} &= u, \\ y^{(3)} &= \frac{3}{2}u^2 + u_{xx}, \\ y^{(4)} &= \frac{15}{6}u^3 + \frac{5}{2}u_x^2 + 5uu_{xx} + u_{xxx} \qquad \text{etc.} \end{aligned}$$

The corresponding infinitesimal generators of the gauge symmetry group are given by

$$L^{(1)} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{2}u & 0 \end{bmatrix},$$

$$L^{(2)} = \begin{bmatrix} -\frac{1}{2}u_x & u \\ -\frac{1}{2}u_{xx} - \frac{1}{2}u^2 & \frac{1}{2}u_x \end{bmatrix},$$

$$L^{(3)} = \begin{bmatrix} -\frac{1}{2}u_{xxx} - \frac{3}{2}uu_x & u_{xx} + \frac{3}{2}u^2 \\ -\frac{1}{2}u_{xxxx} - 2uu_{xx} - \frac{3}{2}u_x^2 - \frac{3}{4}u^3 & \frac{1}{2}u_{xxx} + \frac{3}{2}uu_x \end{bmatrix},$$

$$\vdots$$