Spectral and evolution analysis of thin elastic domains in high-contrast regime

Recent progress in quantitative analysis of multiscale media

Josip Žubrinić

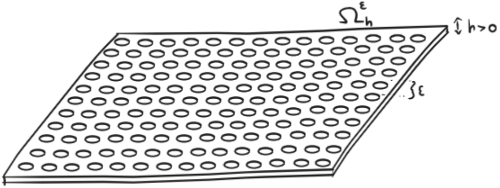
University of Zagreb
Faculty of Electrical Engineering and Computing
josip.zubrinic@fer.hr

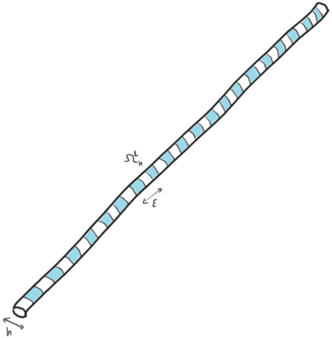
31.5.2023, Department of Mathematics, Faculty of Science, University of Split

Joint work with:
Marin Bužančić (FKIT), Kirill Cherednichenko (University of Bath), Igor Velčić (FER)

Heterogeneous thin elastic structures

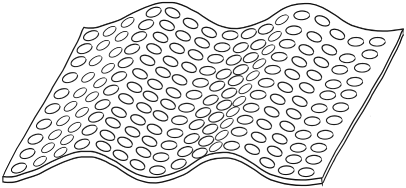
- Composite structures
- Elastic properties
- Thickness in certain directions very small

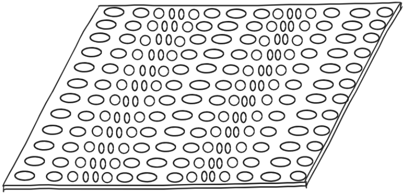


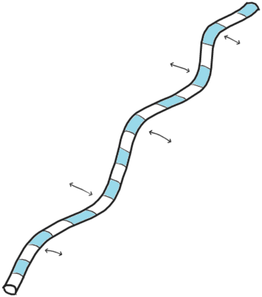


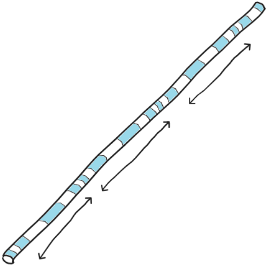
One wishes to:

- Rigorously derive lower dimensional homogeneous models of these objects
- Mathematically explain various effects associated with wave propagation
- Quantify the approximation









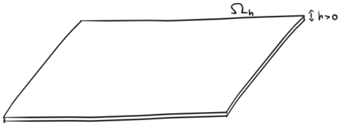
Outline of the talk

1. Elastic thin structures - setting and tools
2. Heterogeneous media in high contrast - metamaterials?
3. Elastic heterogeneous plates in high contrast

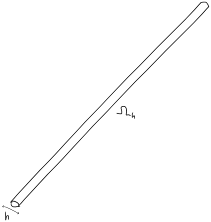
Thin (linearly) elastic structures

Parameter of thickness $h > 0$

Thin plate domain $\Omega^h := \omega \times hI$



Thin rod domain $\Omega^h := \omega_h \times I$



- Elastic properties: $\mathbb{A} \in L^\infty(\Omega^h; \mathbb{R}^{3\times3\times3\times3})$
- $\mathbb{A}$ uniformly positive definite on symmetric matrices: $\exists \alpha, \beta > 0$

$$\alpha|\xi|^2 \leqslant \mathbb{A}(x)\xi : \xi \leqslant \beta|\xi|^2, \quad \forall x \in \Omega^h, \xi \in \mathbb{R}^{3\times3}, \xi^T = \xi.$$

$$\mathbb{A}_{ijkl}(x) = \mathbb{A}_{jikl}(x) = \mathbb{A}_{klij}(x), \quad \forall x \in \Omega^h, \quad i,j,k,l \in \{1,2,3\}\,.$$

- Standard change of coordinates: $\Omega^h \to \Omega^1 := \Omega$, $\nabla \to \nabla_h$.
- The operator of linear elasticity:

$$\mathcal{A}_h \boldsymbol{u} := -\operatorname{div}_h(\mathbb{A}(x)\,\operatorname{sym}\nabla_h \boldsymbol{u}), \quad \boldsymbol{u} : \Omega \to \mathbb{R}^3, \quad \mathcal{D}(\mathcal{A}_h) \subset H^1_{\Gamma_D}(\Omega; \mathbb{R}^3).$$

Resolvent problem

$$\mathcal{A}_h \boldsymbol{u} + \alpha \boldsymbol{u} = \boldsymbol{f} \text{ on } \Omega, \quad \boldsymbol{f} \in L^2(\Omega; \mathbb{R}^3), \quad \alpha > 0.$$

Displacement approximations when $h \to 0$

Rod case:

$$\begin{bmatrix} \mathfrak{b}_1(x_3)/h \\ \mathfrak{b}_2(x_3)/h \\ (-x_1 \mathfrak{b}_1'(x_3) - x_2 \mathfrak{b}_2'(x_3)) \end{bmatrix} + \begin{bmatrix} x_2 \mathfrak{d}(x_3) \\ -x_1 \mathfrak{d}(x_3) \\ \mathfrak{a}(x_3) \end{bmatrix}$$

$\mathfrak{b}_\alpha \in H^2(I)$, $\mathfrak{d}, \mathfrak{a} \in H^1(I)$.

Plate case:

$$\begin{bmatrix} -x_3 \partial_1 \mathfrak{b}(x_1, x_2) \\ -x_3 \partial_2 \mathfrak{b}(x_1, x_2) \\ \mathfrak{b}(x_1, x_2)/h \end{bmatrix} + \begin{bmatrix} \mathfrak{a}_1(x_1, x_2) \\ \mathfrak{a}_2(x_1, x_2) \\ 0 \end{bmatrix}$$

$\mathfrak{b} \in H^2(\omega)$, $\mathfrak{a}_\alpha \in H^1(\omega)$.

The spectrum of $\mathcal{A}_h$ (The case of finite domain)

$$\sigma(\mathcal{A}_h) = \left\{ 0 < \lambda_1^h \leqslant \lambda_2^h \leqslant \dots \lambda_i^h \to +\infty, \quad i \to \infty \right\}.$$

$$\lambda_i^h \leqslant h^2 \eta_i, \quad \text{where } \eta_i \text{ does not depend on } h > 0.$$

Lemma (Korn's innequality for thin domains)

Let $\Omega \subset \mathbb{R}^3$ be a thin domain (rod or plate with regular enough boundary) and $\Gamma \subset \partial\Omega$ of positive measure. We have:

$$\|\pi_{1/h}\boldsymbol{\psi}\|^2_{H^1} \leqslant C^\gamma \left(\|\pi_{1/h}\boldsymbol{\psi}\|^2_{L^2(\Gamma;\mathbb{R}^3)} + h^{-2}\,\|\mathrm{sym}\,\nabla_h\boldsymbol{\psi}\|^2_{L^2}\right), \quad \forall\boldsymbol{\psi} \in H^1(\Omega;\mathbb{R}^3),$$

where C^γ depends only on the domain and $\pi_{1/h}$ is the appropriate scaling.

Lemma (Compactness property for plate like domains)

Let $\omega \subset \mathbb{R}^2$ (bounded with Lipschitz boundary). If the sequence $(\boldsymbol{\psi}^h)_{h>0} \subset H^1_{\gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^3)$ satisfies:

$$\limsup_{n\to\infty}\left\|\mathrm{sym}\,\nabla_h\boldsymbol{\psi}^h\right\|_{L^2} < \infty.$$

Then (on a subsequence) we have the decomposition:

$$\boldsymbol{\psi}^h = \left(\mathfrak{a}_1 - x_3\partial_1\mathfrak{b}, \mathfrak{a}_2 - x_3\partial_2\mathfrak{b}, h^{-1}\mathfrak{b}\right)^T + \tilde{\boldsymbol{\psi}}^h,$$

$$\mathrm{sym}\,\nabla_h\boldsymbol{\psi}^h = \imath\left(-x_3\nabla^2_{\hat{x}}\mathfrak{b} + \mathrm{sym}\,\nabla_{\hat{x}}\mathfrak{a}\right) + \mathrm{sym}\,\nabla_h\tilde{\boldsymbol{\psi}}^h,$$

where $\mathfrak{b} \in H^2_{\gamma_{\mathrm{D}}}(\omega)$, $\mathfrak{a} \in H^1_{\gamma_{\mathrm{D}}}(\omega;\mathbb{R}^2)$, $(\tilde{\boldsymbol{\psi}}^h)_{h>0} \subset H^1_{\gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^3)$, $h\pi_{1/h}\tilde{\boldsymbol{\psi}}^h \xrightarrow{L^2} 0$.

The story of invariant subspaces

Assumption on additional material symmetries

$$\mathbb{A}_{ijk3}(x) = 0, \mathbb{A}_{i333}(x) = 0, \quad \forall x \in \Omega, \quad i,j,k \in \{1,2\}\,.$$

Bending and Stretching subspaces

$$L^2_{\mathrm{bend}} := \left\{\boldsymbol{u} \in L^2(\Omega;\mathbb{R}^3), \quad \boldsymbol{u}(S(x)) = -S\boldsymbol{u}(x)\right\},$$
$$L^2_{\mathrm{stretch}} := \left\{\boldsymbol{u} \in L^2(\Omega;\mathbb{R}^3), \quad \boldsymbol{u}(S(x)) = S\boldsymbol{u}(x)\right\},$$

where $S(x) = S_{\mathrm{rod}}(x) := (-x_1, -x_2, x_3)^T$ in the case of rod, and $S(x) = S_{\mathrm{plate}}(x) := (x_1, x_2, -x_3)^T$ in the case of plate.

Invariant subspaces

Under the assumption on additional material symmetries, the spaces L^2_{stretch} and L^2_{bend} are invariant for $\mathcal{A}_h$. The spectrum of order h^2 is contained in $\sigma(\mathcal{A}_h|_{L^2_{\mathrm{bend}}})$.

Heterogeneous media, mild contrast, scalar case

Small parameter (period of material oscillations) $\varepsilon > 0$.

Periodically oscillating heterogeneous material

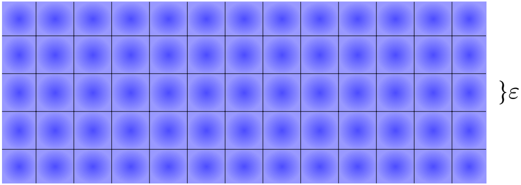
- Material properties stored in $\mathbb{A}_\varepsilon(x) \in \mathbb{R}^{d\times d}$.
- $\mathbb{A}_\varepsilon(x) = \mathbb{A}(\frac{x}{\varepsilon})$.
- $\mathbb{A}$ is Y-periodic on $\mathbb{R}^d$, $Y = [0,1]^d$.

The matrix $\mathbb{A}$ is symmetric and $\exists \alpha, \beta > 0$, such that:

$$\alpha|\xi|^2 \leqslant \mathbb{A}(x)\xi\cdot\xi \leqslant \beta|\xi|^2, \quad \forall x, \xi \in \mathbb{R}^d.$$

Elliptic operator:

$$\mathcal{A}_\varepsilon \boldsymbol{u} := -\operatorname{div}\left(\mathbb{A}_\varepsilon(x)\nabla \boldsymbol{u}\right), \quad \mathcal{D}\left(\mathcal{A}_\varepsilon\right) \subset H^1(\Omega).$$



Homogenisation in mild contrast

Heterogeneous problem

$\Omega \subset \mathbb{R}^d$ bounded, $\exists c > 0$, $c|\xi|^2 \leqslant \mathbb{A}(y)\xi \cdot \xi$, $\mathbb{A}$ Y-periodic.

$$\int_\Omega \mathbb{A}(x/\varepsilon)\nabla u_\varepsilon \cdot \nabla \varphi dx = \int_\Omega f_\varepsilon \varphi dx, \quad \forall \varphi \in H_0^1(\Omega), \quad f_\varepsilon \rightharpoonup f \text{ u } L^2(\Omega).$$

A wish of engineers and numerics people?

Homogenised problem

$$\int_\Omega \mathbb{A}^{\mathrm{hom}} \nabla u \cdot \nabla \varphi dx = \int_\Omega f \varphi dx, \quad \forall \varphi \in H_0^1(\Omega).$$

$$\mathbb{A}^{\mathrm{hom}} \xi \cdot \eta := \int_Y \mathbb{A}(y)[\xi + \nabla_y w_\xi(y)] \cdot \eta, \quad \int_Y \mathbb{A}(y)[\xi + \nabla_y w_\xi(y)] \cdot \nabla_y v(y) = 0.$$

Homogenised elliptic operator:

$$\mathcal{A}\boldsymbol{u} := -\operatorname{div}\left(\mathbb{A}^{\mathrm{hom}} \nabla \boldsymbol{u}\right), \quad \mathcal{D}\left(\mathcal{A}\right) = H^2(\Omega) \cap H_0^1(\Omega).$$

Qualitative method by two-scale convergence

Definition

A bounded sequence $(u_\varepsilon) \subset L^2(\Omega)$ weakly "two-scale" converges to $u(x,y) \in L^2(\Omega \times Y)$ if

$$\lim_{\varepsilon \to 0} \int_\Omega u_\varepsilon(x)\varphi(x, x/\varepsilon)dy = \int_\Omega \int_Y u(x,y)\varphi(x,y)dxdy, \quad \varphi \in C_c^\infty(\Omega; C_\#^\infty(Y)).$$

Theorem (Two-scale compactness in L^2)

Every bounded sequence in $L^2(\Omega)$ possesses a weakly two-scale convergent subsequence.

Definition

A bounded sequence $(u_\varepsilon) \subset L^2(\Omega)$ strongly "two-scale" converges to $u(x,y) \in L^2(\Omega \times Y)$ if

$$\lim_{\varepsilon \to 0} \int_\Omega u_\varepsilon(x)\varphi_\varepsilon(x)dy = \int_\Omega \int_Y u(x,y)\varphi(x,y)dxdy,$$

for all weakly "two-scale" convergent $\quad \varphi_\varepsilon \xrightarrow{2} \varphi \in L^2(\Omega \times Y).$

Homogenisation in mild contrast

Heterogeneous problem

$\Omega \subset \mathbb{R}^d$ bounded, $\exists c > 0$, $c|\xi|^2 \leqslant \mathbb{A}(y)\xi \cdot \xi$, $\mathbb{A}$ Y-periodic.

$$\int_\Omega \mathbb{A}(x/\varepsilon)\nabla u_\varepsilon \cdot \nabla \varphi dx = \int_\Omega f_\varepsilon \varphi dx, \quad \forall \varphi \in H_0^1(\Omega), \quad f_\varepsilon \rightharpoonup f \text{ u } L^2(\Omega).$$

Theorem

Every bounded sequence u_ε u $H^1(\Omega)$ possesses a subsequence such that:

$$u_\varepsilon \rightharpoonup u(x), \quad \nabla u_\varepsilon(x) \xrightharpoonup{2} \nabla u(x) + \nabla_y v(x,y), \quad \forall v \in L^2(\Omega; H^1_\#(Y)).$$

Homogenised problem

$$\int_\Omega \mathbb{A}^{\text{hom}} \nabla u \cdot \nabla \varphi dx = \int_\Omega f \varphi dx, \quad \forall \varphi \in H_0^1(\Omega).$$

$$\mathbb{A}^{\text{hom}}\xi \cdot \eta := \int_Y \mathbb{A}(y)[\xi + \nabla_y w_\xi(y)] \cdot \eta, \quad \int_Y \mathbb{A}(y)[\xi + \nabla_y w_\xi(y)] \cdot \nabla_y v(y) = 0.$$

Resolvent and spectral convergence (Zhikov, Pastukhova)

Definition

Sequence of nonneg. selfadj. $(\mathcal{A}_\varepsilon)$ operators on H_ε. $\mathcal{A}$ nonneg. selfadj. operator on a closed subspace H_0 of H. $P : H \to H_0$ orthogonal projection.
The sequence $(\mathcal{A}_\varepsilon)$ converges to $\mathcal{A}$ in the sense of strong resolvent convergence, if

$$\forall \lambda > 0, \quad (\mathcal{A}_\varepsilon + \lambda)^{-1} f_\varepsilon \xrightarrow{\varepsilon \to 0} (\mathcal{A} + \lambda)^{-1} Pf, \quad \forall f_\varepsilon, f_\varepsilon \xrightarrow{\varepsilon \to 0} f \in H.$$

Zhikov and Pastukhova: Convergence in variable Hilbert spaces - for example in the sense of two-scale convergence.

Definition

The sequence of spectra $\sigma(\mathcal{A}_\varepsilon)$ converges to the spectrum of $\mathcal{A}$ in the sense of Hausdorff, if:

- $\forall \lambda \in \sigma(\mathcal{A})$ exists a sequence $\lambda_\varepsilon \in \sigma(\mathcal{A}_\varepsilon)$ such that $\lambda_\varepsilon \to \lambda$.
- If $\lambda_\varepsilon \in \sigma(\mathcal{A}_\varepsilon)$ and $\lambda_\varepsilon \to \lambda$, then $\lambda \in \sigma(\mathcal{A})$.

The first one is the consequence of strong resolvent convergence. The second needs a bit more work.

Quantitative results in the homogenisation of elliptic PDE

Quantitative results are given with the norm-resolvent estimates (Birman, Suslina 2001., 2005., 2006., ...):

Quantitative result

$$\|(\mathcal{A}_\varepsilon + I)^{-1} - (\mathcal{A} + I)^{-1}\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} \leqslant C\varepsilon,$$

$$\|(\mathcal{A}_\varepsilon + I)^{-1} - (\mathcal{A} + I)^{-1} - \varepsilon\mathcal{R}_{\mathrm{corr}}(\varepsilon)\|_{L^2(\mathbb{R}^d)\to H^1(\mathbb{R}^d)} \leqslant C\varepsilon,$$

$$\|(\mathcal{A}_\varepsilon + I)^{-1} - (\mathcal{A} + I)^{-1} - \varepsilon\widehat{\mathcal{R}}_{\mathrm{corr}}(\varepsilon)\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} \leqslant C\varepsilon^2,$$

Cherednichenko, Velčić (2021. thin heterogeneous plates in mild contrast)
These quantitative results immediately yield spectral convergence.

High-contrast materials

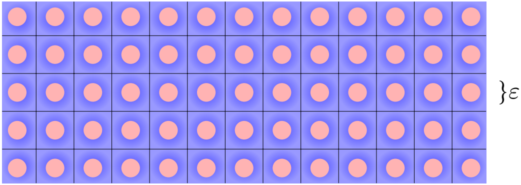
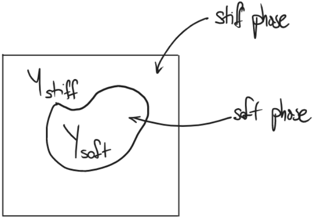


Figure: Depiction of a material with high-contrast inclusions



- Tensor of material coefficients:

$$\mathbb{A}(y) = \begin{cases} \mathbb{A}_{\mathrm{stiff}}(y), & y \in Y_{\mathrm{stiff}}, \\ \varepsilon^2\, \mathbb{A}_{\mathrm{soft}}(y), & y \in Y_{\mathrm{soft}}. \end{cases}$$

- $\mathbb{A}_{\mathrm{stiff}}, \mathbb{A}_{\mathrm{soft}}$ uniformly positive definite.

Homogenisation in high contrast

Heterogeneous problem

$\Omega \subset \mathbb{R}^d$ bounded, $\mathbb{A}_\varepsilon(y) = \varepsilon^2 \chi_{\text{soft}}(y)\mathbb{A}_{\text{soft}}(y) + \chi_{\text{stiff}}(y)\mathbb{A}_{\text{stiff}}(y)$

$$\int_\Omega \mathbb{A}_\varepsilon(x/\varepsilon)\nabla u_\varepsilon \cdot \nabla\varphi dx = \int_\Omega f_\varepsilon \varphi dx, \quad \forall \varphi \in H_0^1(\Omega), \quad f_\varepsilon \xrightarrow{2} f \text{ u } L^2(\Omega \times Y).$$

Theorem

Let $u_\varepsilon \subset H^1(\Omega)$ *such that both* u_ε *and* $\varepsilon\nabla u_\varepsilon$ *are bounded in* $L^2(\Omega)$. *Then (on a subsequence):*

$$u_\varepsilon \xrightarrow{2} u(x,y), \quad \varepsilon\nabla u_\varepsilon(x) \xrightarrow{2} \nabla_y u(x,y), \quad u \in L^2(\Omega; H^1_\#(Y)).$$

Homogenised problem

$$\begin{cases} \displaystyle\int_\Omega \mathbb{A}^{\text{hom}}\nabla u \cdot \nabla\varphi dx = \int_\Omega \int_{Y_{\text{stiff}}} f(x,y)\varphi(x) dx, \quad \forall \varphi \in H_0^1(\Omega). \\ \displaystyle\int_\Omega \int_{Y_{\text{soft}}} \mathbb{A}(y)\nabla_y u_0(x,y) \cdot \nabla_y \xi(x,y) dydx = \int_\Omega \int_{Y_{\text{soft}}} f(x,y)\xi(x,y) dydx, \quad \forall \xi. \end{cases}$$

$$\mathbb{A}^{\text{hom}}\xi \cdot \eta := \int_{Y_{\text{stiff}}} \mathbb{A}(y)(\xi + \nabla_y w_\xi(y)) \cdot \eta, \quad \int_{Y_{\text{stiff}}} \mathbb{A}(y)[\xi + \nabla_y w_\xi(y)] \cdot \nabla_y v(y) = 0.$$

Two-scale limit operator and spectral characterisation

Micro and Macro operators

$$\mathcal{A}_{\text{macro}} \longleftrightarrow \int_\Omega \mathbb{A}^{\text{hom}} \nabla u \cdot \nabla \varphi dx, \quad \mathcal{D}(\mathcal{A}_{\text{macro}}) \subset H_0^1(\Omega)$$

$$\mathcal{A}_{\text{micro}} \longleftrightarrow \int_\Omega \int_{Y_{\text{soft}}} \mathbb{A}(y) \nabla_y u_0(x,y) \cdot \nabla_y \xi(x,y) dy dx, \quad \mathcal{D}(\mathcal{A}_{\text{micro}}) \subset L^2(\Omega; X),$$

$$X = \left\{\varphi \in H^1_\#(Y), \quad \varphi = 0 \text{ on } Y_{\text{stiff}}\right\}$$

Self-adjoint, nonnegative operator $\mathcal{A}$ defined through bilinear form:

$$\int_\Omega \mathbb{A}^{\text{hom}} \nabla u \cdot \nabla \varphi dx + \int_\Omega \int_{Y_{\text{soft}}} \mathbb{A}(y) \nabla_y u_0(x,y) \cdot \nabla_y \xi(x,y) dy dx,$$

$$\mathcal{D}(\mathcal{A}) \subset H_0^1(\Omega) + L^2(\Omega; X) =: V$$

Theorem

$$\sigma(\mathcal{A}) = \sigma(\mathcal{A}_{\text{micro}}) \cup \{\lambda > 0, \quad \beta(\lambda) \in \sigma(\mathcal{A}_{\text{macro}})\},$$

$$\beta(\lambda) = \lambda + \sum_{m=1}^{\infty} \frac{\lambda^2 c_m^2}{\omega_m - \lambda}, \quad \omega_m \in \sigma(\mathcal{A}_{\text{micro}}).$$

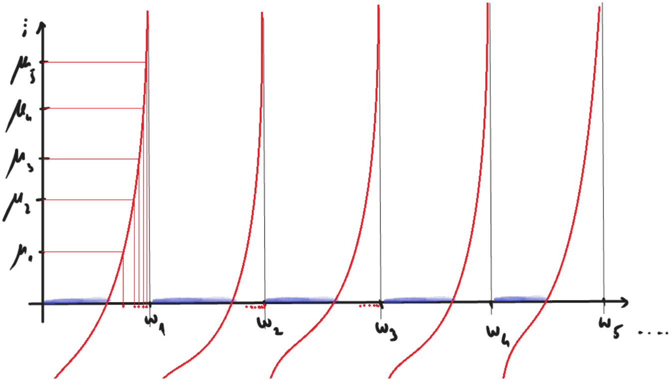
Limit operator

Equivalent formulations of $\mathcal{A}u = \lambda u$

$$\begin{cases} \mathcal{A}_{\text{micro}} u_{\text{micro}} = \lambda(u_{\text{macro}} + u_{\text{micro}}) \\ \mathcal{A}_{\text{macro}} u_{\text{macro}} = \lambda(u_{\text{macro}} + \langle u_{\text{micro}} \rangle) \\ u = u_{\text{micro}} + u_{\text{macro}} \end{cases} \iff \begin{cases} \mathcal{A}_{\text{macro}} u_{\text{macro}} = \beta(\lambda) u_{\text{macro}} \\ u_{\text{macro}} \neq 0 \end{cases}$$

or

$$\begin{cases} \langle u_{\text{micro}} \rangle = u_{\text{macro}} = 0 \\ \mathcal{A}_{\text{micro}} u_{\text{micro}} = \lambda u_{\text{micro}} \end{cases}$$



To sum up

Theorem

Let operators $(\mathcal{A}_\varepsilon)$, $\mathcal{A}$ be nonneg. selfadj. operators related to mild-contrast homogenisation/high-contrast homogenisation. Then we have both the strong resolvent convergence of the operators and Hausdorff convergence of spectra.

Remarks

Vector version of Zhikov's beta function

$$\beta(\lambda) = \lambda\mathcal{I} + \sum_{n=1}^{\infty} \frac{\lambda^2}{\omega_n - \lambda} \langle \overline{\varphi_n} \rangle \otimes \langle \overline{\varphi_n} \rangle$$

- $\exists C > 0$ s.t. $\forall \lambda > 0$

$$\langle \beta'(\lambda)\xi, \xi \rangle > C|\xi|^2, \quad \forall \xi \in \mathbb{R}^2.$$

- The set of all $\lambda > 0$ for which the generalised eigenvalue problem is solvable is countable.

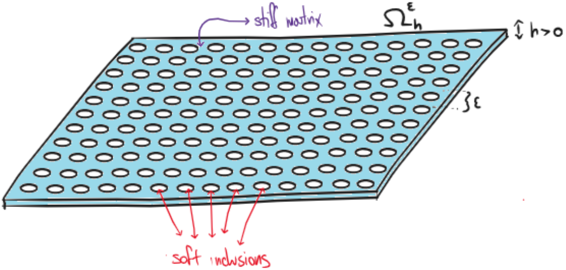
Scaling of frequencies

$$\text{Studying } (\mathcal{A} + \alpha I)^{-1} \iff \text{Studying } \partial_{tt}\boldsymbol{u} + \mathcal{A}\boldsymbol{u} = \boldsymbol{f}$$

$$\text{Studying } \left(\tfrac{1}{\eta}\mathcal{A} + \alpha I\right)^{-1} \iff \text{Studying } \eta\partial_{tt}\boldsymbol{u} + \mathcal{A}\boldsymbol{u} = \boldsymbol{f}$$

The inertia term $\eta\partial_{tt}\boldsymbol{u}$ is equivalent to $\partial_{\tilde{t}\tilde{t}}\boldsymbol{u}$ with $\tilde{t} = \sqrt{\eta}t$.

Heterogeneous thin elastic plate model



- $$\mathcal{A}_{\varepsilon_h} \cdot := -\operatorname{div}_h \left(\mathbb{A}^{\varepsilon_h}\left(\tfrac{x_1}{\varepsilon_h}, \tfrac{x_2}{\varepsilon_h}\right) \operatorname{sym} \nabla_h \cdot \right), \quad \mathcal{D}(\mathcal{A}_{\varepsilon_h}) \subset H^1_{\gamma_D}(\Omega),$$
- $$\mathbb{A}^{\varepsilon_h}(y) = \begin{cases} \mathbb{A}_{\text{stiff}}(y), & y \in Y_{\text{stiff}}, \\ \varepsilon^2 \mathbb{A}_{\text{soft}}(y), & y \in Y_{\text{soft}}. \end{cases}$$
- Simultaneous homogenisation and dimension reduction (both parameters $\varepsilon, h \to 0$)

Goals

- To plug the dimension reduction and homogenisation of thin plates in the abstract setting of operator theory (as much as possible)
- To identify phenomena that come from dimension reduction, phenomena that come from high-contrast homogenisation, and phenomena that come from the interaction of these two settings
- To give a systematic overview of models of heterogeneous thin plates in various interesting regimes
- To answer some important questions (What happens with the spectrum? What are the properties of evolution equations?)

Systematic overview of properties

Time scale The level of contrast	Standard time scale: $t \in [0, T]$ The resolvent of unscaled operator $(\mathcal{A}_{\varepsilon_h} + \alpha I)^{-1}$ Unscaled spectrum $\sigma(\mathcal{A}_{\varepsilon_h})$	Long time scale: $\tilde{t} = \frac{t}{h^2} \in [0, \frac{T}{h^2}]$ The resolvent of scaled operator $\left(\frac{1}{h^2}\mathcal{A}_{\varepsilon_h} + \alpha I\right)^{-1}$ Scaled spectrum $\sigma(\frac{1}{h^2}\mathcal{A}_{\varepsilon_h})$
High contrast $\mu_{\varepsilon_h} = {\varepsilon_h}^2$		
Very high contrast $\mu_{\varepsilon_h} = {\varepsilon_h}^2 h^2$		

First theorem on resolvent convergence - real time scale

Theorem

Let $\mathcal{A}_{\varepsilon_h}$ be an operator of linear elasticity on the thin plate with oscillating material coefficients in high contrast. Then we have strong resolvent convergence:

$$\mathcal{A}_{\varepsilon_h} \to \mathcal{A},$$

where $\mathcal{A}$ is the operator associated with the problem:
Find $(\mathfrak{a}, \mathfrak{b})^\top \in H^1_{\gamma_D}(\omega; \mathbb{R}^2) \times L^2(\omega)$, $\mathring{\boldsymbol{u}} \in L^2(\omega; H^1_{00}(I \times Y_{\mathrm{soft}}; \mathbb{R}^3))$, such that:

$$\mathcal{A}_{\mathrm{macro}}(\mathfrak{a}, \mathfrak{b}) + \lambda(\mathfrak{a}, \mathfrak{b}) + \lambda\langle\mathring{\boldsymbol{u}}\rangle = \langle\boldsymbol{f}\rangle,$$
$$\mathcal{A}_{\mathrm{micro}}(\mathring{\boldsymbol{u}}) + \lambda(\mathfrak{a}, \mathfrak{b}) + \lambda\mathring{\boldsymbol{u}} = \boldsymbol{f},$$

and we have $\mathcal{A}_{\mathrm{macro}}(\mathfrak{a}, \mathfrak{b}) = \mathcal{A}_{\mathrm{macro}}(\mathfrak{a}, 0)$.

Second theorem on resolvent convergence - Long time scale

Theorem

Let $\mathcal{A}_{\varepsilon_h}$ be an operator of linear elasticity on the thin plate with oscillating material coefficients in high contrast. Then we have strong resolvent convergence:

$$\frac{1}{h^2}\mathcal{A}_{\varepsilon_h} \to \mathcal{A},$$

where $\mathcal{A}$ is the operator associated with the problem:
Find $\mathfrak{a} \in H^1_{\gamma_D}(\omega;\mathbb{R}^2)$, $\mathfrak{b} \in H^2_{\gamma_D}(\omega)$, $\mathring{\boldsymbol{u}} \in L^2(\omega; H^1_{00}(I \times Y_{\mathrm{soft}};\mathbb{R}^3))$, such that:

$$\begin{aligned} &\mathcal{A}^{\mathfrak{b}}_{\mathrm{macro}}\mathfrak{b} + \lambda\mathfrak{b} = \mathcal{M}_{\mathrm{bend}}\langle \boldsymbol{f} \rangle, \\ &\mathfrak{a} = \mathfrak{a}^{\mathfrak{b}} + \mathfrak{a}^{\boldsymbol{f}}, \\ &\mathcal{A}_{\mathrm{micro}}\mathring{\boldsymbol{u}} = \boldsymbol{f}. \end{aligned}$$

Third theorem on resolvent convergence - Long time scale II.

Theorem

*Let $\mathcal{A}_{\varepsilon_h}$ be an operator of linear elasticity on the thin plate with oscillating material coefficients in **VERY** high contrast. Then we have strong resolvent convergence:*

$$\frac{1}{h^2}\mathcal{A}_{\varepsilon_h} \to \mathcal{A},$$

where $\mathcal{A}$ is the operator associated with the problem:
Find $\mathfrak{a} \in H^1_{\gamma_{\mathrm{D}}}(\omega;\mathbb{R}^2)$, $\mathfrak{b} \in H^2_{\gamma_{\mathrm{D}}}(\omega)$, $\mathring{\boldsymbol{u}} \in L^2(\omega; H^1_{00}(I \times Y_{\mathrm{soft}};\mathbb{R}^3))$ such that

$$\begin{aligned}
&\mathcal{A}^{\mathfrak{b}}_{\mathrm{macro}}\mathfrak{b} + \lambda\mathfrak{b} + \lambda\langle\overline{\mathring{u}_3}\rangle = \langle f_3\rangle,\\
&\mathfrak{a} = \mathfrak{a}^{\mathfrak{b}},\\
&\mathcal{A}_{\mathrm{micro}}\mathring{\boldsymbol{u}} + \lambda(0,0,\mathfrak{b})^T + \lambda\mathring{\boldsymbol{u}} = \boldsymbol{f}.
\end{aligned}$$

Systematic overview of properties

Time scale The level of contrast	Standard time scale: $t \in [0, T]$ The resolvent of unscaled operator $(\mathcal{A}_{\varepsilon_h} + \alpha I)^{-1}$ Unscaled spectrum $\sigma(\mathcal{A}_{\varepsilon_h})$	Long time scale: $\tilde{t} = \frac{t}{h^2} \in [0, \frac{T}{h^2}]$ The resolvent of scaled operator $\left(\frac{1}{h^2}\mathcal{A}_{\varepsilon_h} + \alpha I\right)^{-1}$ Scaled spectrum $\sigma(\frac{1}{h^2}\mathcal{A}_{\varepsilon_h})$
High contrast $\mu_{\varepsilon_h} = {\varepsilon_h}^2$	• Limit resolvent problem exhibits metamaterial properties - is coupled through the spectral parameter. • "bending" deformations are contained in the kernel of the effective operator.	• Limit resolvent problem does not exhibit metamaterial properties. • The spectral parameter appears only with "bending" deformations." • "bending" and "stretching" deformations are coupled through nonlocal operator. The coupling can be removed with additional assumptions on material symmetries.
Very high contrast $\mu_{\varepsilon_h} = {\varepsilon_h}^2 h^2$	No limit.	• Limit resolvent problem exhibits metamaterial properties. • The spectral parameter appears only with "bending" deformations and the "micro" deformations. "

Spectral approximation theorems

Theorem

The sequence of spectra $\sigma(h^{-2}\mathcal{A}_{\varepsilon_h}) = \{h^{-2}\lambda_1^{\varepsilon_h}, h^{-2}\lambda_2^{\varepsilon_h}, \dots\}$ *converges in the sense of Hausdorff to the set* $\sigma(\mathcal{A}^{\mathfrak{b}}_{\mathrm{macro}}) = \{\lambda_1, \lambda_2, \dots\}$*. Moreover,*

$$h^{-2}\lambda_n^{\varepsilon_h} \to \lambda_n, \quad h \to 0, \quad \forall n \in \mathbb{N}.$$

Theorem

Under the additional assumptions on material symmetries, the sequence of spectra $\mathcal{A}_{\varepsilon_h}|_{L^2_{\mathrm{stretch}}}$ *converges in the sense of Hausdorff to the set:*

$$\sigma(\mathcal{A}) = \sigma(\mathcal{A}_{\mathrm{micro}}) \cup \overline{\{\lambda > 0 : \textit{The problem } (1) \textit{ has a nontrivial solution }.\}}.$$

Generalised eigenvalue problem: Find $\lambda > 0$ and $0 \neq \mathfrak{a} \in H^1_{\gamma_{\mathrm{D}}}(\omega, \mathbb{R}^2)$ such that:

$$\mathcal{A}^{\mathrm{stretch}}_{\mathrm{macro}}\mathfrak{a} = \beta^{\mathrm{stretch}}(\lambda)\mathfrak{a}. \tag{1}$$

Theorem

In the case of ***VERY*** *high contrast, the sequence of spectra* $\sigma(h^{-2}\mathcal{A}_{\varepsilon_h})$ *converges in the sense of Hausdorff to the set:*

$$\sigma(\mathcal{A}) = \sigma(\mathcal{A}_{\mathrm{micro}}) \cup \overline{\{\lambda > 0 : \beta^{\mathrm{bend}}(\lambda) \in \sigma(\mathcal{A}^{\mathfrak{b}}_{\mathrm{macro}})\}}.$$

Systematic overview of properties

Time scale The level of contrast	Standard time scale: $t \in [0, T]$ The resolvent of unscaled operator $(\mathcal{A}_{\varepsilon_h} + \alpha I)^{-1}$ Unscaled spectrum $\sigma(\mathcal{A}_{\varepsilon_h})$	Long time scale: $\tilde{t} = \frac{t}{h^2} \in [0, \frac{T}{h^2}]$ The resolvent of scaled operator $\left(\frac{1}{h^2}\mathcal{A}_{\varepsilon_h} + \alpha I\right)^{-1}$ Scaled spectrum $\sigma(\frac{1}{h^2}\mathcal{A}_{\varepsilon_h})$
High contrast $\mu_{\varepsilon_h} = {\varepsilon_h}^2$	• Limit resolvent problem exhibits metamaterial properties. • "bending" deformations are contained in the kernel of the effective operator. • The spectrum is polluted. • "stretching" spectrum has band-gap structure with infinite number of accummulation points.	• Limit resolvent problem does not exhibit metamaterial properties. • The spectral parameter appears only with "bending" deformations." • "bending" and "stretching" deformations are coupled through nonlocal operator. The coupling can be removed with additional assumptions on material symmetries. • The spectrum converges to the spectrum of nonlocal operator (compact resolvent)..
Very high contrast $\mu_{\varepsilon_h} = {\varepsilon_h}^2 h^2$	No limit.	• Limit resolvent problem exhibits metamaterial properties. • The spectral parameter appears only with "bending" deformations and the "micro" deformations. " • The spectrum has band-gap structure with infinite number of accummulation points.

Effective hyperbolic evolution

Real time scale with high contrast

$$\partial_{tt}\big((\mathfrak{a},\mathfrak{b})^\top + \mathring{\boldsymbol{u}}\big)(t) + \mathcal{A}\big((\mathfrak{a},\mathfrak{b})^\top + \mathring{\boldsymbol{u}}\big)(t) = P\boldsymbol{f}(t),$$

$$\big((\mathfrak{a},\mathfrak{b})^\top + \mathring{\boldsymbol{u}}\big)(0) = \boldsymbol{u}_0(x,y), \qquad \partial_t\big((\mathfrak{a},\mathfrak{b})^\top + \mathring{\boldsymbol{u}}\big)(0) = P\boldsymbol{u}_1(x,y).$$

Long time scale with high contrast

$$\partial_{tt}\mathfrak{b}(t) + \mathcal{A}^{\mathfrak{b}}_{\mathrm{macro}}\mathfrak{b}(t) = \mathcal{M}_{\mathrm{bend}}\big(\boldsymbol{f}(t)\big),$$

$$\mathfrak{b}(0) = \mathfrak{b}_0 \in H^2_{\gamma_{\mathrm{D}}}(\omega), \qquad \partial_t\mathfrak{b}(0) = \mathfrak{b}_1 \in L^2(\omega),$$

$$\mathfrak{a}(t) = \mathfrak{a}^{\mathfrak{b}(t)} + \mathfrak{a}^{\boldsymbol{f}_*(t)}, \quad \mathcal{A}_{00}\mathring{\boldsymbol{u}}(t,\hat{x},\cdot) = (\boldsymbol{f}_*(t,\hat{x},\cdot),0)^\top.$$

Long time scale with very high contrast

$$\partial_{tt}\big((0,0,\mathfrak{b})^\top + \mathring{\boldsymbol{u}}\big)(t) + \mathcal{A}\big((0,0,\mathfrak{b})^\top + \mathring{\boldsymbol{u}}\big)(t) = P\boldsymbol{f}(t),$$

$$\big((0,0,\mathfrak{b})^\top + \mathring{\boldsymbol{u}}\big)(0) = \boldsymbol{u}_0(x,y), \qquad \partial_t\big((0,0,\mathfrak{b})^\top + \mathring{\boldsymbol{u}}\big)(0) = P\boldsymbol{u}_1(x,y).$$

Systematic overview of properties

Time scale The level of contrast	Standard time scale: $t \in [0, T]$ The resolvent of unscaled operator $(\mathcal{A}_{\varepsilon_h} + \alpha I)^{-1}$ Unscaled spectrum $\sigma(\mathcal{A}_{\varepsilon_h})$	Long time scale: $\tilde{t} = \frac{t}{h^2} \in [0, \frac{T}{h^2}]$ The resolvent of scaled operator $\left(\frac{1}{h^2}\mathcal{A}_{\varepsilon_h} + \alpha I\right)^{-1}$ Scaled spectrum $\sigma(\frac{1}{h^2}\mathcal{A}_{\varepsilon_h})$
High contrast $\mu_{\varepsilon_h} = {\varepsilon_h}^2$	• Limit resolvent problem exhibits metamaterial properties. • "bending" deformations are contained in the kernel of the effective operator. • The spectrum is polluted. • "stretching" spectrum has band-gap structure with infinite number of accummulation points. • Memory effects are present in the evolution. • "bending" deformations do not show elastic resistance to motion.	• Limit resolvent problem does not exhibit metamaterial properties. • The spectral parameter appears only with "bending" deformations." • "bending" and "stretching" deformations are coupled through nonlocal operator. The coupling can be removed with additional assumptions on material symmetries. • The spectrum converges to the spectrum of nonlocal operator (compact resolvent).. • Standard hyperbolic evolution of the "bending" deformations. Evolution of the "stretching" component is partially quasistatic.
Very high contrast $\mu_{\varepsilon_h} = {\varepsilon_h}^2 h^2$	No limit.	• Limit resolvent problem exhibits metamaterial properties. • The spectral parameter appears only with "bending" deformations and the "micro" deformations. " • The spectrum has band-gap structure with infinite number of accummulation points. • Memory effects are present in the evolution.

Other ratios of ε and h

- $h \ll \varepsilon$ - similar phenomenology to $h \sim \varepsilon$.
- $h \gg \varepsilon$ - inclusions behave like little elastic rods - loss of compactness of normalized eigenfunctions - spectral pollution

The end

Thank you for attention!