Poroelastic plate model obtained by simultaneous homogenization and dimension reduction

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Recent progress in quantitative analysis of multiscale media, University of Split

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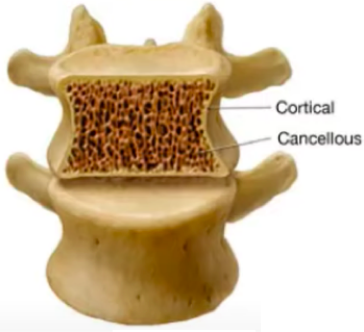
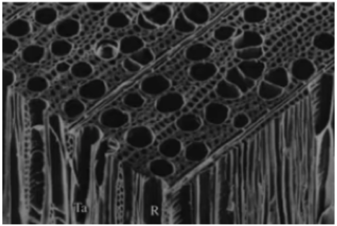
Hrvatska zaklada

Poroelastic materials

Are materials consisting of a skeleton (the solid phase) which is made of a linearly elastic material and the pores saturated by a viscous fluid (the fluid phase). Some examples include:



Bone



Wood

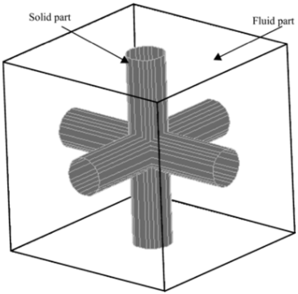
- *Maurice Anthony Biot* (1941, Foundations of theory of poroelasticity). Biot MA (1941). "General theory of three dimensional consolidation". Journal of Applied Physics. 12 (2): 155–164.
- *Andro Mikelic* (2000), was pioneer analysis of multiscale systems, interaction of flow and elastic porous media.
  - **(i)** Mathematically rigurous derivation of Biot's systems (viscid and inviscid flow and elasticity). Fluid-structure interactions in cell tissues and quasi-static Biot's equations in a thin poro-elastic plate.
  - **(iii)** Derivation of transmission laws at interface coupling differents regimes.

- Applications in geosciences: Enviromental cleanup, petroleum production, solid waste disposal...

Interaction of fluid flow with a porous elastic structure

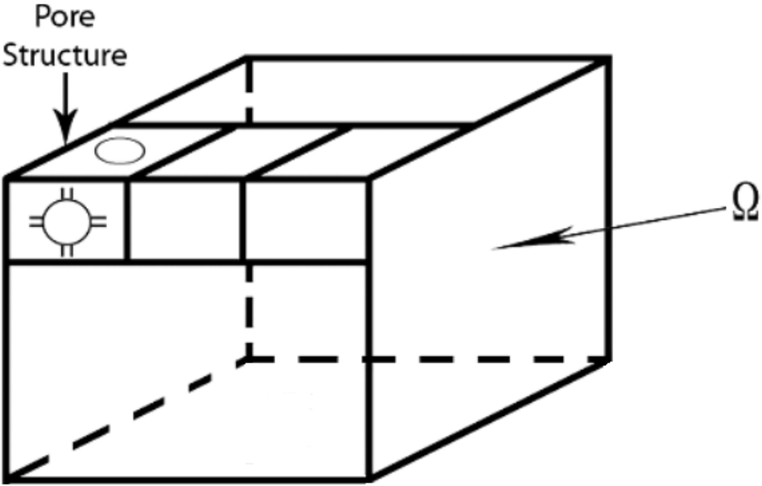
Let $\mathcal{Y} = (0,1)^3$ be the unit cell.

- Let $\mathcal{Y}_s$ (the solid part) be a closed subset of $\overline{\mathcal{Y}}$ and $\mathcal{Y}_f = \mathcal{Y} \setminus \mathcal{Y}_s$ (the fluid part) and we can make periodic repetition of $\mathcal{Y}_s$ over $\mathbb{R}^3$ and set $\mathcal{Y}_s^k = \mathcal{Y}_s + k, k \in \mathbb{Z}^3$.



- We make the following standard assumptions on $E_s = \bigcup_{k \in \mathbb{Z}^3} \mathcal{Y}_s^k$ and $E_f = \mathbb{R}^3 \setminus E_s$:
  - **(a)** $\mathcal{Y}_s$ is an open connected set of strictly positive measure, with a Lipschitz boundary and $\mathcal{Y}_s$ has strictly positive measure in $\overline{\mathcal{Y}}$ as well.
  - **(b)** The interiors of E_s and E_f are open sets with the boundary of class $C^{0,1}$, which are locally located on one side of their boundary.

Assume that $\Omega = (0, L)^3 \subset \mathbb{R}^3$ is covered with a regular mesh of size ε, each cell being a cube $\mathcal{Y}_i^\varepsilon$, with $1 \le i \le N(\varepsilon) = |\Omega|\varepsilon^{-3}[1 + O(1)]$.



The fluid-solid interface is indicated by $\Gamma^\varepsilon := \partial\Omega_s^\varepsilon \cap \partial\Omega_f^\varepsilon$. The domains Ω_s^ε and Ω_f^ε represent, respectively, the solid an fluid parts of a porous medium Ω.

The equations that describes this previous cases are given respectively

$$\varepsilon^m \rho_f \frac{\partial^2 \boldsymbol{u}^\varepsilon}{\partial t^2} - \mathrm{div}\sigma^{f,\varepsilon} = \rho_f \boldsymbol{F} \quad \text{in } \Omega_f^\varepsilon \times (0,T), \tag{1}$$

$$\nabla \cdot \frac{\partial \boldsymbol{u}^\varepsilon}{\partial t} = 0 \quad \text{in } \Omega_f^\varepsilon \times (0,T), \qquad \left(\kappa_{co} \frac{\partial p^\varepsilon}{\partial t} + \nabla \cdot \frac{\partial \boldsymbol{u}^\varepsilon}{\partial t} = 0 \quad \text{in } \Omega_f^\varepsilon \times (0,T) \right) \tag{2}$$

$$\varepsilon^m \rho_s \frac{\partial^2 \boldsymbol{u}^\varepsilon}{\partial t^2} - \mathrm{div}\, \sigma^{s,\varepsilon} = \rho_s \boldsymbol{F} \quad \text{in } \Omega_s^\varepsilon \times (0,T), \tag{3}$$

$$[\boldsymbol{u}^\varepsilon] = 0 \quad \text{on } \Gamma^\varepsilon \times (0,T) \quad \text{(displacement continuity at the interface)}, \tag{4}$$

$$\sigma^{f,\varepsilon} = -p^\varepsilon I + 2\eta \varepsilon^{-r} e\left(\frac{\partial \boldsymbol{u}^\varepsilon}{\partial t}\right) \quad \text{(fluid stress)}, \tag{5}$$

$$\sigma^{s,\varepsilon} = Ae(\boldsymbol{u}^\varepsilon) \quad \text{(stress in solid)}, \tag{6}$$

$$\sigma^{s,\varepsilon} \cdot \boldsymbol{n} = \sigma^{f,\varepsilon} \cdot \boldsymbol{n} \quad \text{on } \Gamma^\varepsilon \times (0,T). \tag{7}$$

$$\boldsymbol{u}^\varepsilon|_{\{t=0\}} = \partial_t \boldsymbol{u}^\varepsilon|_{\{t=0\}} = 0 \quad \text{on } \Omega. \tag{8}$$

$$\{\boldsymbol{u}^\varepsilon, p^\varepsilon\} \quad \text{is periodic in } (x_1, x_2) \text{ with period } L. \tag{9}$$

- [GILBMIK2000] Monophasis viscoelastic, macroscopic behavior of the fluid and solid matrix. They consider the problem (1)-(9) for $\Omega = (0, L)^3$ with

$$m = r = 0.$$

- [CLFRGILBMIK2001] Diphasic macroscopic behavior of the fluid and solid matrix.
  They consider the problem (1)-(9) for $\Omega = (0, L)^3$ with

$$m = 0 \quad \text{and} \quad r = -2.$$

- [MIKWHE2012] Interface conditions between a poroelastic medium (the pay zone) and an elastic body (the non-pay zone).
  They consider the problem (1)-(9) for $\Omega = (0, L)^3 \cup \Sigma \cup \Omega_{el}$, where $\Omega_{el} = (0, L)^2 \times (-L, 0)$, $\Sigma = (0, L)^2 \times \{0\}$ with

$$m = 1 \quad \text{and} \quad r = -2.$$

- [JAGMIK1996] interaction between porous media and fluid (2d)- derivation of contact conditions;
- [MACMIK2015] derivation of poroelastic plate model starting from 3*d* Biot's equations for isotropic elastic tensor, using dimension reduction techniques.
- [GURWEB2022] analysis of poroelastic plate equation: existence and uniqueness of solution.
- [DuGunHouLee2002] Linear fluid-structure interaction, existence, uniqueness, weak and strong solution.
- [GahJagNeu2022] Regime $\varepsilon \sim h$ plate in fluid; the limit model is not of Biot's type.

Biot's bulk and plate model

$\Omega^\ell = \{(x_1, x_2, x_3) \in \omega_L \times (-\ell/2, \ell/2)\}$, where the mid-surface ω_L is a bounded domain in $\mathbb{R}^2$ with a smooth boundary $\partial\omega_L$ of class C^1.

$$\sigma = 2Ge(\mathbf{u}) + (\frac{2\nu G}{1-2\nu}\,\mathrm{div}\,\mathbf{u} - \alpha p)I \ \text{ in } \ \Omega^\ell,$$
$$-G \triangle \mathbf{u} - \frac{G}{1-2\nu} \bigtriangledown \ \mathrm{div}\,\mathbf{u} + \alpha \bigtriangledown p = 0 \ \text{ in } \ \Omega^\ell,$$
$$\frac{\partial}{\partial t}(\gamma_G p + \alpha\,\mathrm{div}\,\mathbf{u}) - \frac{k}{\eta}\triangle p = 0 \ \text{ in } \ \Omega^\ell.$$

G-shear modulus, ν-Poisson ratio, α- effective stress coefficient, γ_G inverse of Biot's modulus, k-permeability η-viscosity.
The mean velocity (velocity oscillates!) is proportional to the gradient of pressure (in case of absence of volume forces)-Darcy law. The evolution model has memory effects!

The authors additionally scales the constant k/η with ℓ^2.

They impose $\sigma\mathbf{n} = \mathcal{P}^{\pm\ell}$ and a given normal flux $-\frac{k}{\eta}\frac{\partial p}{\partial x_3} = U^\ell$ at $x_3 = \pm\ell/2$.

Biot's bulk and plate model

$$G\Delta_{x_1,x_2}\mathbf{u}^\omega + \frac{G(1+\nu)}{1-\nu}\nabla_{x_1,x_2}\mathrm{div}_{x_1,x_2}\mathbf{u}^\omega + \frac{\alpha(1-2\nu)}{1-\nu}\nabla_{x_1,x_2}N$$
$$+\sum_{j=1}^{2}(\mathcal{P}_j^1 + \mathcal{P}_j^{-1})\mathbf{e}^j = 0,$$

$$(\gamma_G + \frac{\alpha^2(1-2\nu)}{2G(1-\nu)})N = \frac{\alpha(1-2\nu)}{1-\nu}\,\mathrm{div}_{x_1,x_2}(u_1^\omega, u_2^\omega),$$

$$(\gamma_G + \frac{\alpha^2(1-2\nu)}{2G(1-\nu)})\frac{\partial}{\partial t}(p^{eff}+N) - \frac{k}{\eta}\frac{\partial^2}{\partial x_3^2}(p^{eff}+N) = \alpha x_3\frac{1-2\nu}{1-\nu}\frac{\partial}{\partial t}\Delta_{x_1,x_2}w,$$

$$\frac{G\ell^3}{6(1-\nu)}\Delta^2_{x_1,x_2}w + \alpha\frac{1-2\nu}{1-\nu}\Delta_{x_1,x_2}\int_{-1/2}^{1/2}x_3p^{eff}\,dx_3 =$$
$$\frac{1}{2}\sum_{i=1}^{2}\frac{\partial}{\partial x_i}(\mathcal{P}_i^1 + \mathcal{P}_i^{-1}) + \mathcal{P}_3^1 + \mathcal{P}_3^{-1},$$

where $w(x_1, x_2, t)$ is the effective transverse displacement of the surface, $\mathbf{u}^\omega = (u_1^\omega, u_2^\omega)$, $u_j^\omega(x_1, x_2, t) - x_3 \dfrac{\partial w}{\partial x_j}$, $j = 1, 2$, are the effective in-plane solid displacements, p^{eff} is the effective fluid pressure and $N = -\int_{-\ell/2}^{\ell/2} p^{eff}\, dx_3$ is the effective stress resultant due to the variation in pore pressure across the plate thickness.

Goal

Our goal is to justify limit model under simultaneous homogenization and dimension reduction (also the evolution). This enables us to understand the limit Darcy's law (which is not seen in continuum model). Moreover it will enable us to derive the contact of poroelastic and elastic plate. We will perform the computations for general elasticity tensor (isotropicity simplifies the limit model which then decouples, which is not true, in general).

Starting rescaled problem

Find $\boldsymbol{u}^h \in H^1(0,T;V^1)$ with $\frac{d^2\boldsymbol{u}^h}{dt^2} \in L^2(0,T;L^2(\Omega)^3)$ such that

$$\frac{d^2}{dt^2}\int_\Omega \eta\kappa^h \boldsymbol{u}^h(t)\varphi\,dx + \frac{d}{dt}\frac{1}{h^4}\int_{\Omega_f^h} 2\varepsilon^2 e_h(\boldsymbol{u}^\varepsilon(t)) : e_h(\varphi)\,dx$$
$$+\frac{1}{h^2}\int_{\Omega_s^h}\mathbb{A}\left(\frac{\hat{x}}{\varepsilon},\frac{x_3}{\frac{\varepsilon}{h}}\right)e_h(\boldsymbol{u}^h(t)) : e_h(\varphi)\,dx - \frac{1}{h^2}\int_{\Omega_f^h} p^h \,\mathrm{div}_h\,\varphi\,dx$$
$$= \int_\Omega \psi^h \boldsymbol{F}^h\varphi\,dx, \quad \forall\varphi\in V^h, \quad \text{a.e. in } (0,T), \tag{10}$$

where,

$$\kappa^h = \kappa_f^0\chi_{\Omega_f^h} + \kappa_s^0\chi_{\Omega_s^h}, \quad \psi^h = \psi_f\chi_{\Omega_f^h} + \psi_s\chi_{\Omega_s^h}.$$

With initial conditions

$$\boldsymbol{u}^h\big|_{\{t=0\}} = \frac{\partial\boldsymbol{u}^h}{\partial t}\bigg|_{\{t=0\}} = 0 \quad \text{on } \Omega.$$

$$\{\boldsymbol{u}^h, p^h\} \quad \text{is periodic in } (x_1,x_2) \text{ with period } \quad L$$

$$V^1 := H^1(\Omega;\mathbb{R}^3). \text{ periodic in } x_1, x_2 \text{ direction }.$$

We consider the cases $\eta \to 0$ and $\eta = 1$.

Assumptions

The tensor $\mathbb{A}$ is assumed to be uniformly positive definite on symmetric matrices, namely: $\exists \nu > 0$ such that

$$\nu|\xi|^2 \leq \mathbb{A}(y)\xi : \xi \leq \nu^{-1}|\xi|^2 \quad \forall \xi \in \mathbb{R}^{3\times 3}_{\mathrm{sym}}, \quad \forall y \in \mathcal{Y}_s, \tag{11}$$

and further assume the following symmetries hold:

$$\mathbb{A}_{ijkl} = \mathbb{A}_{jikl} = \mathbb{A}_{klij}, \quad i, j, k, l \in \{1, 2, 3\}.$$

We assume that $\boldsymbol{F}^h \in H^2(0, T; L^2(\Omega \times \mathcal{Y})^3)$

$$\boldsymbol{F}^h \in H^2(0, T; L^2(\Omega; \mathbb{R}^3)), \ \boldsymbol{F}^h(0) = 0.$$

$$\left\| \pi_h \partial_t \boldsymbol{F}^h(0) \right\|_{L^2(\Omega;\mathbb{R}^3)} + \left\| \pi_h \partial_{tt} \boldsymbol{F}^h \right\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^3))} \leq C,$$

$$\pi_h \boldsymbol{F}^h := (h\boldsymbol{F}^h_1, h\boldsymbol{F}^h_2, \boldsymbol{F}^h_3) \xrightharpoonup{2} \boldsymbol{F}.$$

We assume periodic boundary conditions in longitudinal direction and Neumann at the transversal boundary (fluid boundary conditions at transversal boundary?)

The following is our main result:

Proposition 1.1

The homogenized equations given by:
Find $(\mathfrak{a}, \mathfrak{b}, p) \in H^1(0, T; H^1(\omega)^2) \times H^1(0, T; H^2(\omega)) \times L^2(0, T; L^2(\Omega))$ *such that*

$$\int_\omega \mathbb{A}^{hom}(e_{\hat{x}}(\mathfrak{a}), \nabla^2_{\hat{x}}\mathfrak{b}) : (e_{\hat{x}}(\boldsymbol{\theta}_*), \nabla^2_{\hat{x}}\theta_3) d\hat{x} - \int_\omega \left(|\mathcal{Y}_f| I - \mathbb{B}^H\right) \overline{p} I : [\iota(e_{\hat{x}}(\boldsymbol{\theta}_*))] \, d\hat{x}$$
$$+ \int_\omega \left(|\mathcal{Y}_f| I - \mathbb{B}^H\right) \overline{x_3 p} I : \left[\iota(\nabla^2_{\hat{x}}\theta_3)\right] d\hat{x} = \int_\omega \overline{\langle \hat{\psi}, \boldsymbol{F} \rangle_{\mathcal{Y}}} \cdot (\boldsymbol{\theta}_*, \theta_3) \, d\hat{x}$$
$$- \int_\omega \overline{\langle \hat{\psi}, x_3 \boldsymbol{F}_* \rangle_{\mathcal{Y}}} \cdot \nabla_{\hat{x}}\theta_3 \, d\hat{x} \quad \forall (\boldsymbol{\theta}_*, \boldsymbol{\theta}_3) \in C^1_c(\omega)^2 \times C^2_c(\omega),$$
$$\frac{\partial}{\partial t} \int_\omega M_0 \overline{p\, \xi} d\hat{x} + \int_\omega \mathbb{K}\, \overline{\partial_3 p\, \partial_3 \xi} d\hat{x} + \int_\omega (|\mathcal{Y}_f| I - \mathbb{B}^H) \overline{\xi} I : \iota \left(e_{\hat{x}} \left(\frac{\partial \mathfrak{a}}{\partial t} \right) \right) d\hat{x}$$
$$- \int_\omega \left(|\mathcal{Y}_f| I - \mathbb{B}^H\right) \overline{x_3 \xi} I : \iota \left(\nabla^2_{\hat{x}} \frac{\partial \mathfrak{b}}{\partial t} \right) d\hat{x} = 0, \quad \forall \xi \in V_1.$$

System has a unique solution.

The main result $\eta \to 0$

- The tensor $\mathbb{A}^{hom}$ corresponds to the perforated domain;
- the constant M_0 and $\mathbb{K}$ are positive, $\mathbb{B}^H$ is symmetric;
- the system doesn't, in general, decouple on membrane and bending equations;
- we can still decouple the last equation by taking ξ independent of x_3 and the ones perpendicular to that ones;
- the limit effective fluid velocity is driven only by $\partial_3 p$, ie.,

$$\boldsymbol{v} = -(\mathbb{K}_1, \mathbb{K}_2, \mathbb{K})\partial_3 p.$$

- for the cases we consider (fluid is stopped at transversal boundary by elastic body) the natural boundary condition at the transverzal boundary is $\partial_3 p = 0$.
- the models of mixed elastic-poroelastic plate can be easily derived.

The following is our main result:

Proposition 1.2

The homogenized equations given by:
Find $(\mathfrak{a}, \mathfrak{b}, p) \in H^1(0,T;H^1(\omega)^2) \times H^1(0,T;H^2(\omega)) \times L^2(0,T;L^2(\Omega))$ *such that*

$$\int_\omega \frac{\partial^2 \mathfrak{b}}{\partial t^2}\theta_3 + \int_\omega \mathbb{A}^{hom}(e_{\hat{x}}(\mathfrak{a}), \nabla_{\hat{x}}^2 \mathfrak{b}) : (e_{\hat{x}}(\boldsymbol{\theta}_*), \nabla_{\hat{x}}^2\theta_3) d\hat{x} - \int_\omega \left(|\mathcal{Y}_f| I - \mathbb{B}^H\right) \overline{p} I : [\iota(e_{\hat{x}}(\boldsymbol{\theta}_*))]\, d\hat{x}$$

$$+ \int_\omega \left(|\mathcal{Y}_f| I - \mathbb{B}^H\right) \overline{x_3 p} I : \left[\iota(\nabla_{\hat{x}}^2 \theta_3)\right] d\hat{x} = \int_\omega \overline{\langle \hat{\psi}, \boldsymbol{F} \rangle_{\mathcal{Y}}} \cdot (\boldsymbol{\theta}_*, \theta_3)\, d\hat{x}$$

$$- \int_\omega \overline{\langle \hat{\psi}, x_3 \boldsymbol{F}_* \rangle_{\mathcal{Y}}} \cdot \nabla_{\hat{x}} \theta_3 \, d\hat{x} \quad \forall (\boldsymbol{\theta}_*, \boldsymbol{\theta}_3) \in C_c^1(\omega)^2 \times C_c^2(\omega),$$

$$\frac{\partial}{\partial t} \int_\omega M_0 \overline{p}\, \xi d\hat{x} + \int_\omega \mathbb{K}\, \overline{\partial_3 p\, \partial_3 \xi} d\hat{x} + \int_\omega (|\mathcal{Y}_f| I - \mathbb{B}^H) \overline{\xi} I : \iota\left(e_{\hat{x}}\left(\frac{\partial \mathfrak{a}}{\partial t}\right)\right) d\hat{x}$$

$$- \int_\omega \left(|\mathcal{Y}_f| I - \mathbb{B}^H\right) \overline{x_3 \xi} I : \iota\left(\nabla_{\hat{x}}^2 \frac{\partial \mathfrak{b}}{\partial t}\right) d\hat{x} = 0, \quad \forall \xi \in V_1.$$

System has a unique solution.

The main result $\eta = 1$

- there are no memory effects (bending plate is long time evolution);
- Again system is stationary and evolution coupling (decoupling appears in the acse of isotropicity). Elliminating $\mathfrak{a}$ in general causes spatial non-locality. System is hyperbolic-parabolic coupling;
- the models of mixed elastic-poroelastic plate can be easily derived.

Existence of pressure: Pressure is Lagrange multiplier. To ensure its existence in $L^2(L^2)$ one needs forces in $H^1(L^2)$. However to obtain its bound in L^2 one needs forces in $H^2(L^2)$.

Compactness result

$\varepsilon \ll h$. We assume the above assumption on forces. Let $\boldsymbol{u}^h$ be the variational solution of fluid-structure interaction problem with zero boundary condition. There exist

$$
\begin{aligned}
\mathfrak{a} &\in W^{1,\infty}(0,T;H^1(\omega;\mathbb{R}^2)),\\
\mathfrak{b} &\in W^{1,\infty}(0,T;H^2(\omega)),\\
\boldsymbol{w} &\in W^{1,\infty}(0,T;L^2(\Omega,\dot{H}^1(\mathcal{Y};\mathbb{R}^3))),\\
\boldsymbol{g} &\in W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^3)),\\
\boldsymbol{u}_f^0 &\in W^{1,\infty}(0,T;L^2(\Omega;H^1(\mathcal{Y}_f;\mathbb{R}^3))),\\
p &\in L^\infty(0,T;L^2(\Omega)),
\end{aligned}
$$

which are periodic in (x_1,x_2) with period L, such that for all $t \in (0,T)$ we have (on a subsequence)

$$\mathrm{div}_y \boldsymbol{u}_0^f = 0 \quad \Omega \times \mathcal{Y}_f,$$
$$h^{-1} u_\alpha^h \xrightarrow{L^2} \mathfrak{a}_\alpha - x_3 \partial_\alpha \mathfrak{b}, \quad u_3^h \xrightarrow{L^2} \mathfrak{b},$$
$$h^{-2} \boldsymbol{u}_f^h \xrightarrow{dr-2} \boldsymbol{u}_f^0,$$
$$h^{-1} e_h(\hat{\boldsymbol{u}}^h) \xrightarrow{dr-2} \iota(e_{\hat{x}}(\mathfrak{a}) - x_3 \nabla_{\hat{x}}^2 \mathfrak{b}) + \mathfrak{C}_\infty(\boldsymbol{w}, \boldsymbol{g}),$$
$$\varepsilon h^{-2} e_h(\boldsymbol{u}_f^h) \xrightarrow{dr-2} e_y(\boldsymbol{u}_f^0),$$
$$h^{-1} p^\varepsilon \xrightarrow{t,dr-2,\infty} p$$

$$|\mathcal{Y}_f| \mathrm{div}_{\hat{x}}\, \partial_t \mathfrak{a}(\hat{x}, t) - |\mathcal{Y}_f| x_3 \mathrm{div}_{\hat{x}}\, \nabla_{\hat{x}} \partial_t \mathfrak{b}(\hat{x}, t) + \int_{\mathcal{Y}_f} \partial_t \partial_3 u_{f,3}^0(x, y, t) dy + \int_{\mathcal{Y}_f} \partial_t \left[\mathrm{tr}\, \mathfrak{C}_\infty (\boldsymbol{w}, \boldsymbol{g})(x, y, t)\right] dy = 0,$$

where

$$\mathfrak{C}_\infty(\boldsymbol{w}, \boldsymbol{g})(x, y) \quad := \quad e_y(\boldsymbol{w}) + \mathrm{sym}\,(0|0|\boldsymbol{g}), \quad \boldsymbol{u}^h = \hat{\boldsymbol{u}}^h + \boldsymbol{u}_f^h,$$

$\hat{\boldsymbol{u}}^h$ is the extension.

The proof goes by using the extension operator, testing the equation with $\partial_t \boldsymbol{u}^h$ and using modification of Griso's decomposition for obtaining limits for symmetrized scaled gradients. Notice that the regularity of the limit velocity doesn't allow its third derivative which appears in the above identity!

Limit equation

Next step is to test the equation with suitable test functions for obtaining the limit equations. We choose the test function $\boldsymbol{v}^h\varphi$, where $\varphi \in \mathcal{C}^1([0,T])$ such that $\varphi(T) = 0$ and $\boldsymbol{v}^h$ is defined with

$$\boldsymbol{v}^h(x) = h\boldsymbol{\theta}^h(x) + h\varepsilon\boldsymbol{\zeta}\left(x, \frac{\hat{x}}{\varepsilon}, \frac{x_3}{\frac{\varepsilon}{h}}\right) + h^2\boldsymbol{\xi}\left(x, \frac{\hat{x}}{\varepsilon}, \frac{x_3}{\frac{\varepsilon}{h}}\right) + h^2\int_0^{x_3} \boldsymbol{r}(x)\,dx_3,$$

and integrate the starting equaton over the interval $[0,T]$. Here

$$\boldsymbol{\theta}^h(x) = \begin{pmatrix} \theta_1(\hat{x}) - x_3\partial_1\theta_3(\hat{x}) \\ \theta_2(\hat{x}) - x_3\partial_2\theta_3(\hat{x}) \\ h^{-1}\theta_3(\hat{x}) \end{pmatrix},$$

$\boldsymbol{\theta}_* \in \mathcal{C}_c^1(\omega)^2$, $\boldsymbol{\theta}_3 \in \mathcal{C}_c^2(\omega)$, $\boldsymbol{\zeta} \in \mathcal{C}_c^1(\Omega;\mathcal{C}_{\text{per}}^1(\mathcal{Y}))^3$, $\boldsymbol{\xi} \in \mathcal{C}_c^1(\Omega;\mathcal{C}_{\text{per}}^1(\mathcal{Y}))^3$ with $\operatorname{supp}\boldsymbol{\xi} \subset \Omega\times\mathcal{Y}_f$ and $\operatorname{div}_y\boldsymbol{\xi} = 0$ in $\Omega\times\mathcal{Y}_f$, $\boldsymbol{r} \in \mathcal{C}_c^1(\Omega)^3$.

From this we conclude for a.e. $t \in [0, T]$

$$\int_\Omega \int_{\mathcal{Y}_s} \mathbb{A}\left[\iota(e_{\hat{x}}(\mathfrak{a}) - x_3 \nabla^2_{\hat{x}} \mathfrak{b}) + \mathfrak{C}_\infty(\boldsymbol{w}, \boldsymbol{g})\right] : \left[\iota(e_{\hat{x}}(\boldsymbol{\theta}_*) - x_3 \nabla^2_{\hat{x}} \theta_3)\right] \, dy \, dx$$
$$+ |\mathcal{Y}_f| \int_\omega \overline{x_3 p} \, (\mathrm{div}_{\hat{x}} \nabla_{\hat{x}} \theta_3) \, d\hat{x} - |\mathcal{Y}_f| \int_\omega \overline{p} \, \mathrm{div}_{\hat{x}} \, (\boldsymbol{\theta}_*) \, dx$$
$$= \int_\omega \overline{\langle \hat{\psi}, \boldsymbol{F} \rangle_{\mathcal{Y}}} \cdot (\boldsymbol{\theta}_*, \theta_3) \, d\hat{x} - \int_\omega \overline{\langle \hat{\psi}, x_3 \boldsymbol{F}_* \rangle_{\mathcal{Y}}} \cdot \nabla_{\hat{x}} \theta_3 \, d\hat{x};$$
$$\int_I \int_{\mathcal{Y}_s} \mathbb{A}\left[\iota(e_{\hat{x}}(\mathfrak{a}) - x_3 \nabla^2_{\hat{x}} \mathfrak{b}) + \mathfrak{C}_\infty(\boldsymbol{w}, \boldsymbol{g})\right] : \mathfrak{C}_\infty(\zeta, \boldsymbol{r}) \, dy \, dx_3$$
$$- \int_I \int_{\mathcal{Y}_f} p \, (\mathrm{div}_y \, \zeta + r_3) \, dy \, dx_3 = 0,$$
$$2 \int_\Omega \int_{\mathcal{Y}_f} e_y \left(\partial_t \boldsymbol{u}_f^0\right) : e_y(\boldsymbol{\xi}) dy dx - \int_\Omega \int_{\mathcal{Y}_f} p \, \partial_3 \xi_3 \, dx \, dy = 0,$$
$$\forall \boldsymbol{\xi} \in L^2(\omega; H^1(I; H^1(\mathcal{Y}; \mathbb{R}^3)))$$

such that $\boldsymbol{\xi} = 0$ on $\Omega \times \mathcal{Y}_s$ and $\mathrm{div}_y \, \boldsymbol{\xi} = 0$ on $\Omega \times \mathcal{Y}_f$.

How to deal with this? How to obtain the uniqueness of the solution?

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Thank you for your attention!