Asymptotic model for the Rayleigh wave for seismic meta-surfaces

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Contents

Motivation

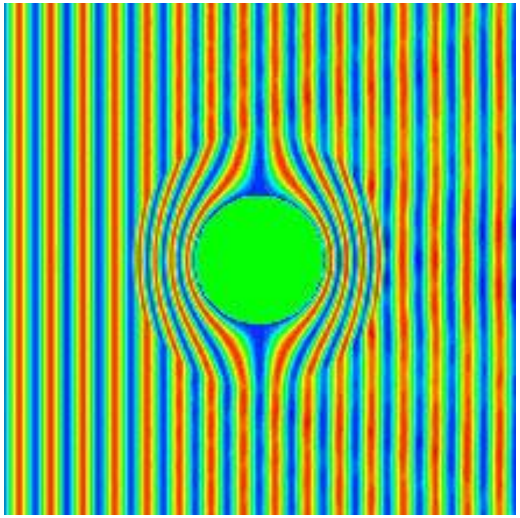
- *Improved seismic protection*

Cloaking discovered in optics,

expanded to acoustic waves

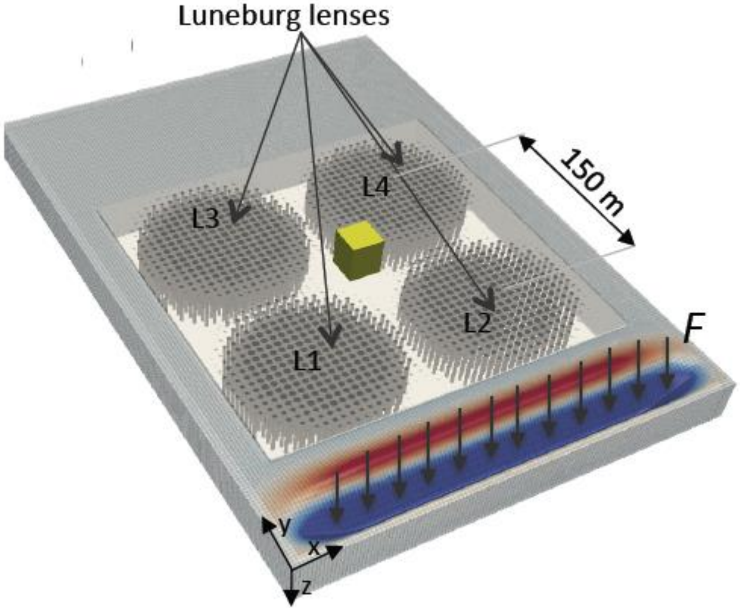
Pendry et al., Science, 2016

Leonhardt, Science, 2016



✓ *Cloaking of Rayleigh waves will give complete seismic protection*

Colombi et al., Sci.Rep., 2016a



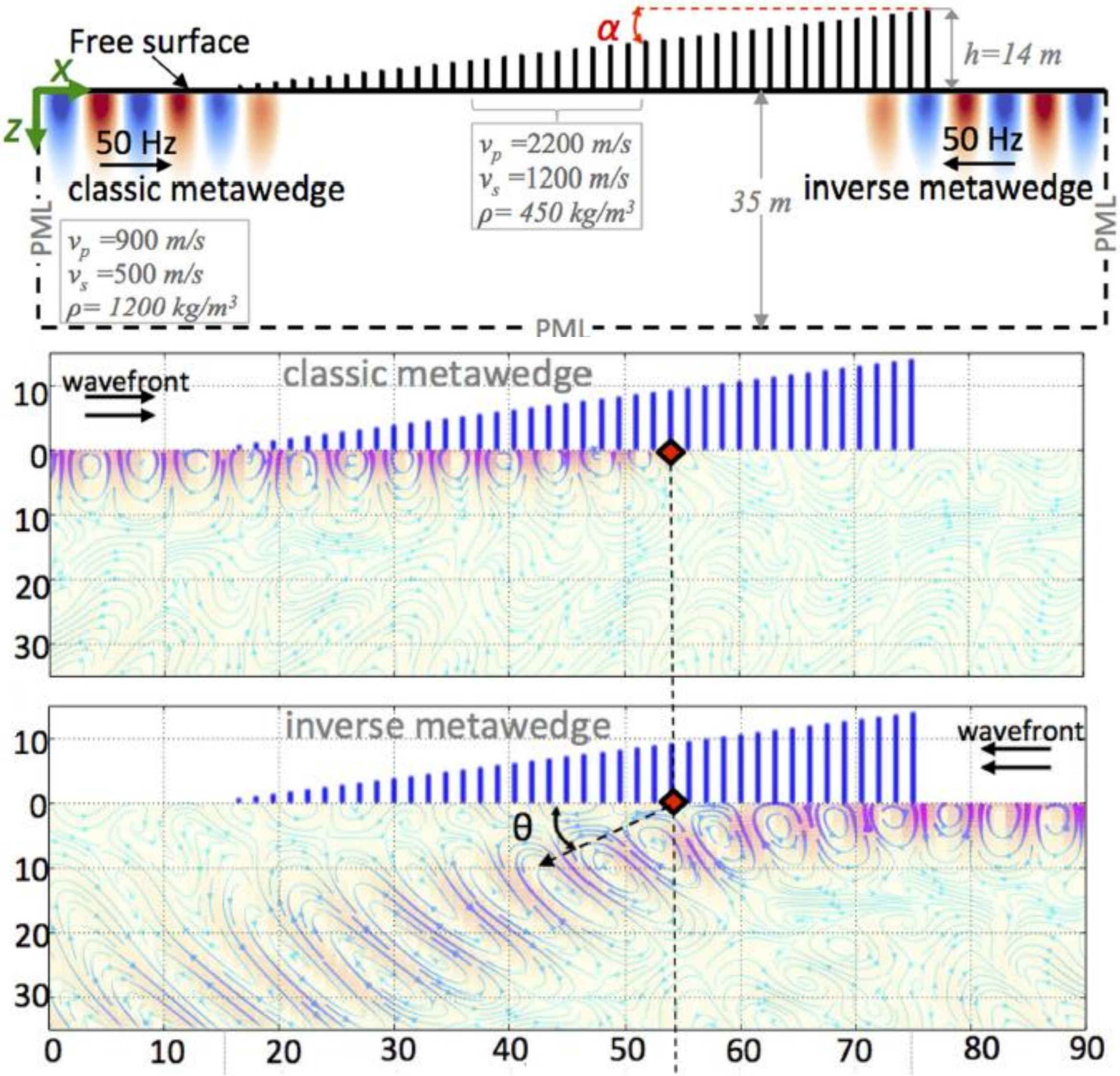
Issue:

- *Not all directions are cloaked*

Introduction

- *Seismic metasurfaces*

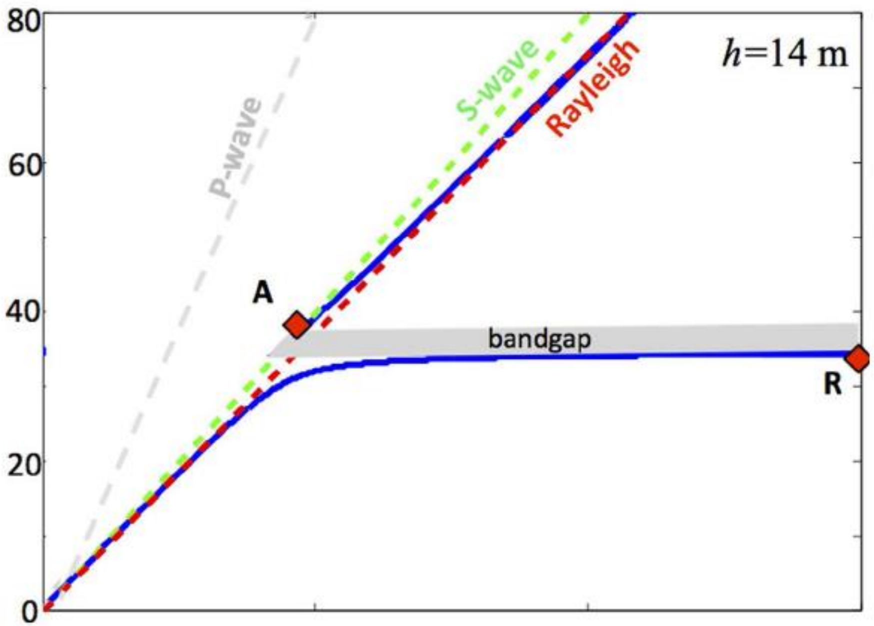
Colombi et al. Sci Rep, 2016b

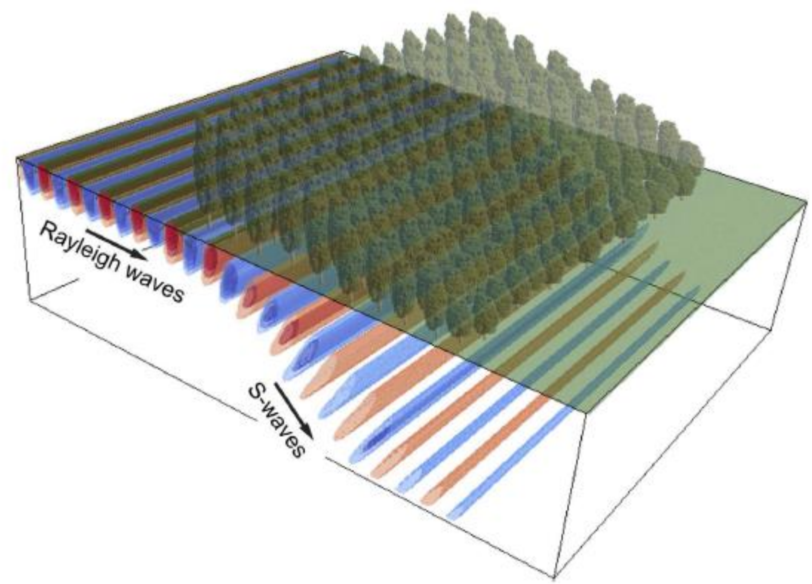


Colombi et al. Sci Rep, 2016c

- *Bandgap revealed*

✓ *Designed forest as a natural seismic metasurface*



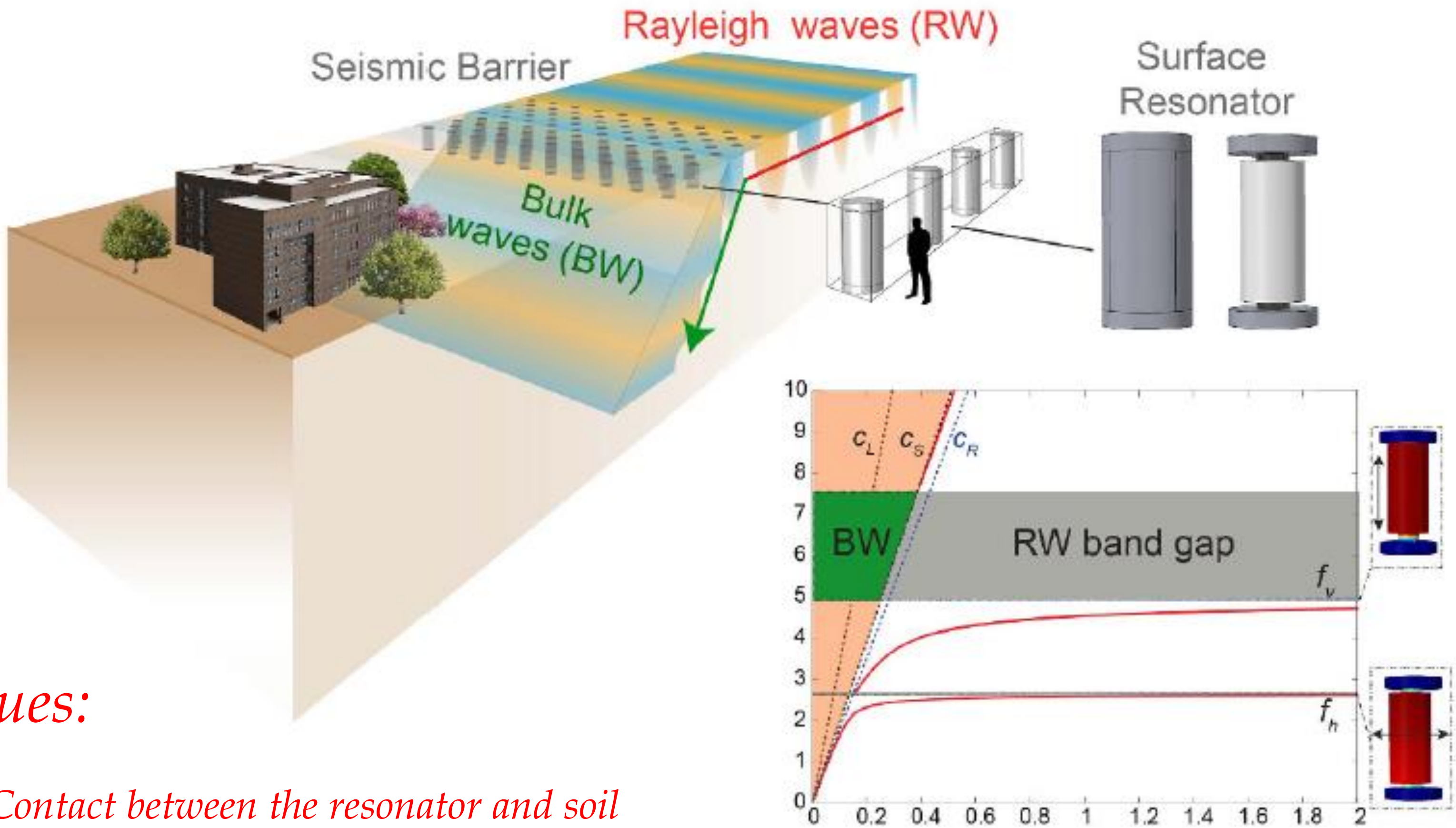


Issues:

- *A rod is not the best model for a tree*
- *Large structures required to realise low-frequency band-gaps*
- *The approach is essentially numerical*

- *Seismic barriers*

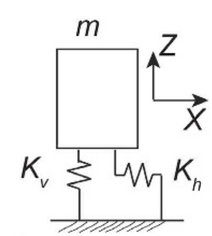
Palermo et al. Sci Rep, 2016



Issues:

- *Contact between the resonator and soil not clearly explained*

- *Horizontal and translational motion neglected in approximate dispersion relation, essentially reducing to mass-spring attached on the surface*



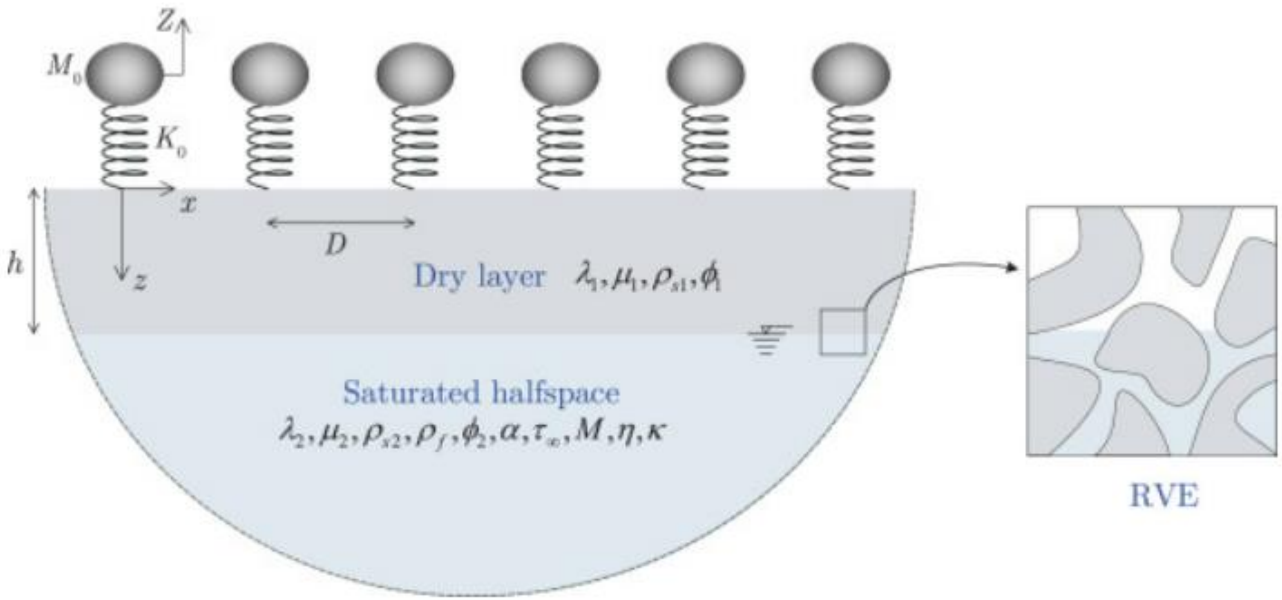
- *Limited possibilities of generalization to 3D (Rayleigh wave is not separated)*

- *More sophisticated formulations,*

i.e. porosity, nonlinearity, etc.

Pu et al. Int J Eng Sci, 2020

Lou et al. Int J Mech Sci, 2022



- *The approach is even more "numerical"*

- *Would be good to have more explicit results*

✓ *Employ asymptotic model for the Rayleigh wave*

Hyperbolic-elliptic model for the Rayleigh wave

(*J. Kaplunov et. al, IMA J. Appl. Math. 2006*)

- Vertical surface load $\sigma_{xy} = 0, \qquad \sigma_{yy} = P(x,t).$

The pseudo-static *elliptic* equation over the interior

$$\phi_{,yy} + \alpha_R^2 \phi_{,xx} = 0.$$

The boundary condition at $y = 0$ is provided by a *hyperbolic* equation

$$\phi_{,xx} - \frac{1}{c_R^2}\phi_{,tt} = \frac{1+\beta_R^2}{2\mu B}P.$$

The second potential is restored as

$$\psi(x_1 - c_R t, \beta_R y) = \frac{2\alpha_R}{1+\beta_R^2}\phi^*(x_1 - c_R t, \beta_R y).$$

Note: $$u_{,xx} - \frac{1}{c_R^2}u_{,tt} = \frac{\beta_R^4 - 1}{4\mu B}P_{,x} \quad (\text{at } y = 0)$$

- Tangential surface load $\sigma_{xy} = Q(x,t), \qquad \sigma_{yy} = 0.$

The pseudo-static *elliptic* equation over the interior

$$\psi_{,yy} + \beta_R^2 \psi_{,xx} = 0.$$

A *hyperbolic* equation at $y = 0$

$$\psi_{,xx} - \frac{1}{c_R^2}\psi_{,tt} = -\frac{1+\beta_R^2}{2\mu B} Q.$$

The longitudinal potential is found as

$$\phi(x_1 - c_R t, \alpha_R y) = -\frac{2\beta_R}{1+\beta_R^2}\psi^*(x_1 - c_R t, \alpha_R y).$$

- Combined load $\sigma_{xy} = Q(x,t), \qquad \sigma_{yy} = P(x,t).$

The pseudo-static *elliptic* equation over the interior

$$\phi_{,yy} + \alpha_R^2 \phi_{,xx} = 0.$$

The boundary condition at $y = 0$ is provided by a *hyperbolic* equation

$$\phi_{,xx} - \frac{1}{c_R^2}\phi_{,tt} = \frac{1}{2\mu B}\left[2\beta_R Q^* + (1+\beta_R^2)P\right].$$

- ***Alternative formulation via pseudo-differential operator***

(*J. Kaplunov, D.A. Prikazchikov, R.F. Sabirova, Dokl. Phys.,* 2022)

Express the hyperbolic-elliptic formulation

$$\varphi_{yy} + \alpha_R^2 \varphi_{xx} = 0, \qquad \Phi_{xx} - \frac{1}{c_R^2}\Phi_{tt} = \frac{1+\beta_R^2}{2\mu B} P,$$

where $\Phi(x,\ t) = \varphi(x,\ 0,\ t)$, as a hyperbolic equation at a given depth

$$\boxed{\varphi_{xx} - \frac{1}{c_R^2}\varphi_{tt} = \frac{1+\beta_R^2}{2\mu B} e^{-\alpha_R y\sqrt{-\partial_{xx}}}\left[P\right].}$$

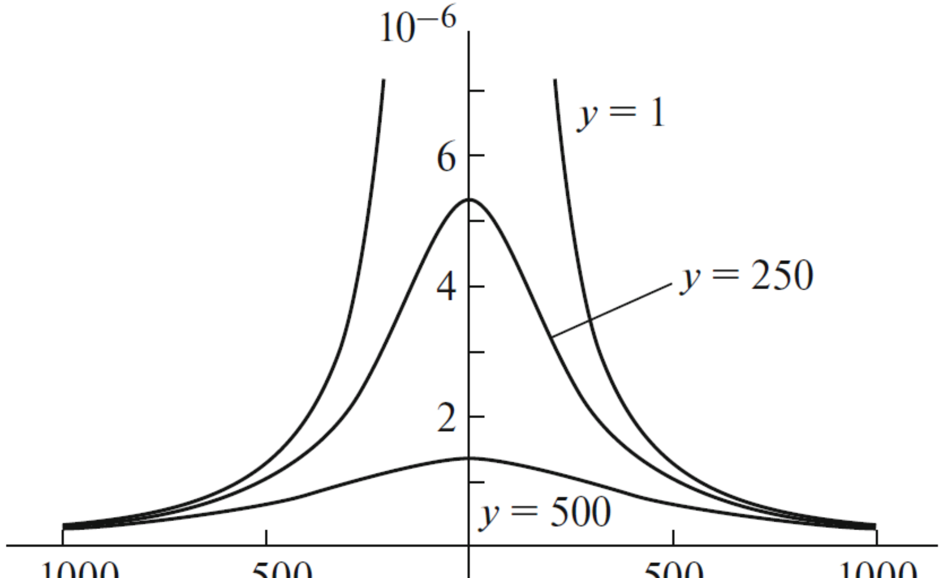
In other words,

$$\varphi_{xx} - \frac{1}{c_R^2}\varphi_{tt} = \frac{1+\beta_R^2}{4\pi\mu B} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} P(x,t) e^{-isx} dx e^{-\alpha_R |s| y} e^{isx} ds$$

- *Example (smoothing surface discontinuities with depth)*

Consider the Lamb's problem, for which $P(x,t)=P_0\delta(t)\delta(x)$.

Distributed delta-function $$e^{-\alpha_R y\sqrt{-\partial_{xx}}}\left[\delta(x)\right]=\delta_y(x)=\frac{\alpha_R y}{\pi\left(x^2+\alpha_R{}^2y^2\right)}$$



Behaviour of the function $\delta_y(x)$ at $y = 1$, $y = 250$, $y = 500$.

Note: $\lim\limits_{y\to 0}\delta_y(x)=\delta(x)$.

Solving the hyperbolic equation $$\varphi_{xx}-\frac{1}{c_R{}^2}\varphi_{tt}=\frac{1+\beta_R{}^2}{2\mu B}P_0\delta_y(x)\delta(t),$$

we result in $$\varphi=\frac{\left(1+\beta_R^2\right)c_R P_0}{4\pi\mu B}\left[\tan^{-1}\frac{x-c_R t}{\alpha_R y}-\tan^{-1}\frac{x+c_R t}{\alpha_R y}\right].$$

• *3D model for the Rayleigh wave (vertical load)*

(*H.H. Dai, J. Kaplunov, D.A. Prikazchikov, Proc. Roy. Soc. A,* 2010)

Surface load

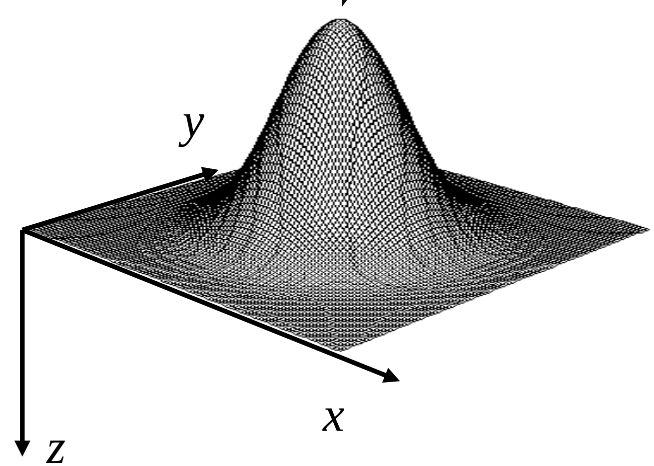
$P = P(x, y, t)$

3D pseudo-static elliptic equations over the interior

$$\phi_{,zz} + \alpha_{\mathrm{R}}^2 \Delta_2 \phi = 0, \qquad \boldsymbol{\psi}_{,zz} + \beta_{\mathrm{R}}^2 \Delta_2 \boldsymbol{\psi} = 0, \qquad \boldsymbol{\psi} = (\psi_1, \psi_2).$$

2D membrane *wave* equation at $z = 0$

$$\Delta_2 \phi - \frac{1}{c_R^2} \phi_{,tt} = \frac{1 + \beta_{\mathrm{R}}^2}{2\mu B} P, \qquad (\Delta_2 = \partial_{xx} + \partial_{yy})$$



Relations between the potentials

$$\boldsymbol{\psi}_{,z} = \frac{2}{1 + \beta_{\mathrm{R}}^2} \mathrm{grad}_2 \phi.$$

Displacements $\quad \mathbf{u} = \mathrm{grad}\phi + \mathrm{rot}\boldsymbol{\Psi}, \qquad \boldsymbol{\Psi} = (-\psi_2, \psi_1, 0).$

- ***Tangential load***

(*Ege N., Erbas B., Prikazchikov D.A., ZAMM, 2015*)

Boundary conditions $\sigma_{xz} = Q_1, \quad \sigma_{yz} = Q_2, \quad \sigma_{zz} = 0 \quad \text{at} \quad z = 0.$

Decomposing $Q_1 = Q_{g,x} + Q_{r,y}, \qquad Q_2 = Q_{g,y} - Q_{r,x}.$

Elliptic equations $\varphi_{,zz} + \alpha_R^2 \Delta_2 \varphi = 0, \qquad \boldsymbol{\psi}_{,zz} + \beta_R^2 \Delta_2 \boldsymbol{\psi} = 0.$

2D wave equation $\Delta\boldsymbol{\psi} - \dfrac{1}{c_R^2}\boldsymbol{\psi}_{,tt} = -\dfrac{1+\beta_R^2}{2\mu B}\,\mathrm{grad}_2 Q_g \quad \text{at} \quad z = 0.$

Relation on the surface $\boldsymbol{\psi}_{,z} = \dfrac{2}{1+\beta_\mathrm{R}^2}\,\mathrm{grad}_2 \varphi.$

Note: $\Delta_2 w - \dfrac{1}{c_R^2} w_{,tt} = \dfrac{1-\beta_R^4}{4\mu B}\Delta_2 Q_g \qquad (\text{at } z = 0)$

- ***Example: 3D horizontal loading on the surface of an elastic half-space***

(J.Kaplunov, D.A. Prikazchikov, Adv. Appl. Mech. 2017)

$$\sigma_{xz} = A\delta(x)\delta(y)\delta(t), \quad \sigma_{yz} = \sigma_{zz} = 0 \quad \text{at} \quad z = 0.$$

Decomposing the load

$$Q_1 = A\delta(x)\delta(y)\delta(t), \quad \Rightarrow \quad \Delta_2 Q_g = A\delta'(x)\delta(y)\delta(t).$$

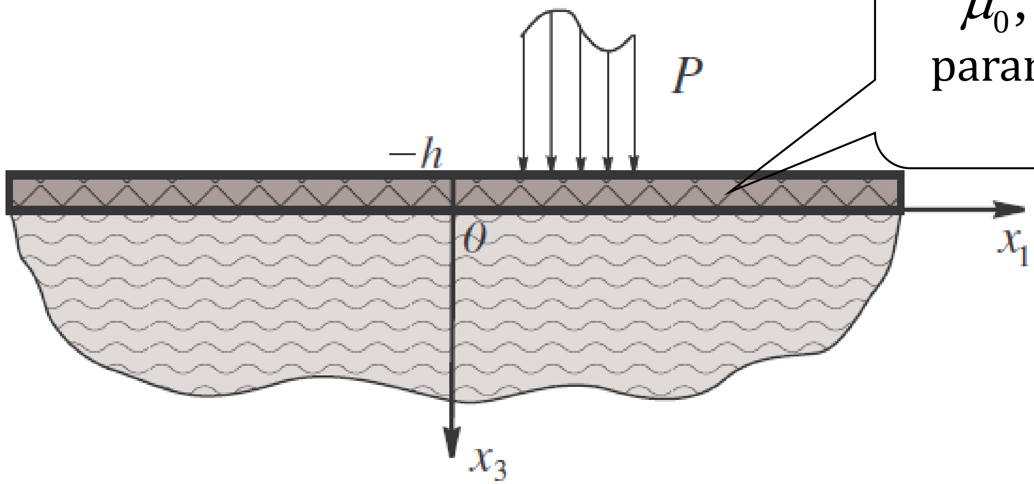
Equation for vertical displacement on the surface

$$\Delta_2 w - \frac{1}{c_R^2} w_{,tt} = \frac{1-\beta_R^4}{4\mu B} A\delta'(x)\delta(y)\delta(t).$$

$$\Longrightarrow \qquad w(x,y,0,t) = \frac{\left(1-\beta_R^4\right) A c_R x}{8\pi\mu B} \frac{H\left(c_R t - \sqrt{x^2+y^2}\right)}{\left(c_R t - \sqrt{x^2+y^2}\right)^{3/2}}.$$

Explicit model for surface wave in a coated half-space

(*H.H. Dai, J. Kaplunov, D.A. Prikazchikov* 2010)



μ_0, β_0, κ_0 – elastic parameters of the coating

Long-wave asymptotic integration $\rightarrow$ efficient boundary conditions

$$\sigma_{i3} = \rho_0 h \left\{ u_{i,tt} - c_{20}^2 \left[u_{i,jj} + 4\left(1-\kappa_0^{-2}\right) u_{i,ii} + \left(3-4\kappa_0^{-2}\right) u_{j,ij} \right] \right\},$$

$$\sigma_{33} = \rho_0 h u_{3,tt} - P, \qquad 1 \le i \ne j \le 2.$$

Resulting formulation

Elliptic equation over the interior

$$\varphi_{,zz} + \alpha_{\mathrm{R}}^2 \Delta_2 \varphi = 0$$

Singularly perturbed equation

$$\Delta_2 \varphi - \frac{1}{c_R^2} \varphi_{,tt} - bh \underline{\sqrt{-\Delta}\left(\Delta \varphi\right)} = \frac{1+\beta_{\mathrm{R}}^2}{2\mu B} P,$$

Pseudo-differential operator

with

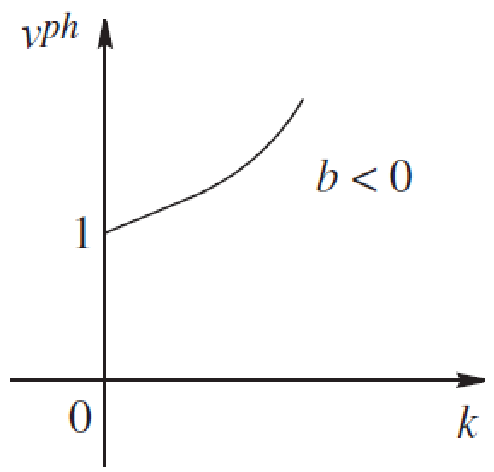
$$b = \frac{\mu_0}{\mu} \frac{(1-\beta_R^2)}{2B} \left[(1-\beta_{R0}^2)(\alpha_R + \beta_R) - 4\beta_R(1-\kappa_0^2)\right]$$

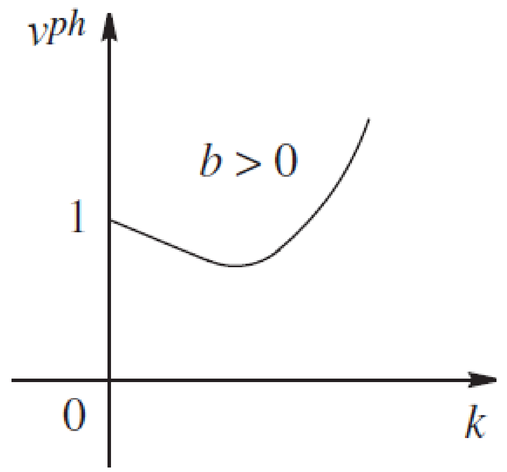
Discussion of the model

- Presence of a coating $\Longrightarrow$ integral form
- At $h=0$ – explicit asymptotic model for the Rayleigh wave in a half-space
- Sign of coefficient b (c_R – local extremum of phase speed)

(*Shuvalov, A. L., & Every, A. G., 2008*)

Dispersion relation $v^{ph} = 1 - \frac{b}{2}|k| + \dots$





- ***Summary and further developments***

- ✓ *Summary* (*J.Kaplunov, D.A. Prikazchikov, Adv. Appl. Mech. 2017*)

- ✓ *Effects of anisotropy, pre-stress and nonlocality*

  (*A.Nobili & D.A.Prikazchikov, Eur. J. Mech. A, 2018; Y.Fu et al., Proc.Roy.Soc.A 2020; D.A.Prikazchikov et al., Mech. Res. Comm., 2018; D.A.Prikazchikov, Vibration, 2023*)

- ✓ *Composite theories for plates*

  (*B.Erbas et al., Proc. Roy. Soc. A 2018; Kaplunov et al., Mech. Res. Comm., 2018*)

- ✓ *Surface waves with non-Neumann-type boundary conditions*

  (*J.Kaplunov et al., Phil. Trans. Roy. Soc. A, 2019*)

- ✓ *Refined model*

  (*J.Kaplunov et al., IMA J. Appl. Math. 2020*)

- ✓ *Seismic meta-surfaces*

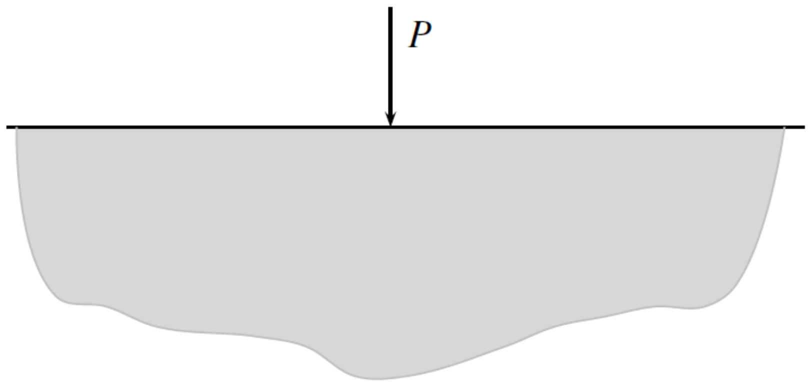
  (*N. Ege et al., JOMMS 2018; P.T. Wootton et al., Proc. Roy. Soc. A 2019; A. Alzaidi et al., ZAMP 2022*)

- ✓ *Models for bending edge waves*

  (*J.Kaplunov et al., Proc.Roy.Soc.A 2016; S.Althobaiti et al., JOMMS 2021*)

- *Developments of the hyperbolic-elliptic model to meta-surfaces*

N. Ege et al., JOMMS, 2018



Start with point time-harmonic vertical force

$$P = -P_0 \delta(x_1) e^{-i\omega t}$$

Hyperbolic equation on the surface

$$\frac{\partial^2 \varphi}{\partial x_1^2} - \frac{1}{c_R^2} \frac{\partial^2 \varphi}{\partial t^2} = -\frac{1+\beta_R^2}{2\mu B} P_0 \delta(x_1) e^{-i\omega t}$$

$$\Longrightarrow \quad \varphi(x_1, 0, t) = i \frac{1+\beta_R^2}{4\mu B} \frac{P_0 c_R}{\omega} e^{i\omega(|x_1|/c_R - t)}$$

(propagating Rayleigh wave patterns)

Then, over the interior

$$\varphi(x_1, x_3, t) = \frac{(1+\beta_R^2) P_0 e^{-i\omega t}}{4\pi \mu B} \int\limits_{-\infty}^{\infty} \frac{e^{-\alpha_R |k| x_3}}{k^2 - \omega^2/c_R^2} e^{-ikx_1} dk \, .$$

Note that $$\int_{-\infty}^{\infty} \frac{e^{-\alpha_R |k| x_3}}{k^2 - \omega^2/c_R^2} e^{-ikx_1} dk = \underbrace{\frac{i\pi c_R}{\omega} e^{(i|x_1| - \alpha_R x_3)\omega/c_R}}_{\text{Contribution of the Rayleigh poles}} + \underbrace{I_2}_{\text{Spurious term}}$$

Here $$I_2 = \frac{c_R}{\omega} \int_0^{\infty} \left(\frac{e^{-\alpha_R k x_3} - e^{-\alpha_R \omega x_3/c_R}}{k - \omega/c_R} - \frac{e^{-\alpha_R k x_3} - e^{-\alpha_R \omega x_3/c_R}}{k + \omega/c_R} \right) \cos(kx_1) dk$$

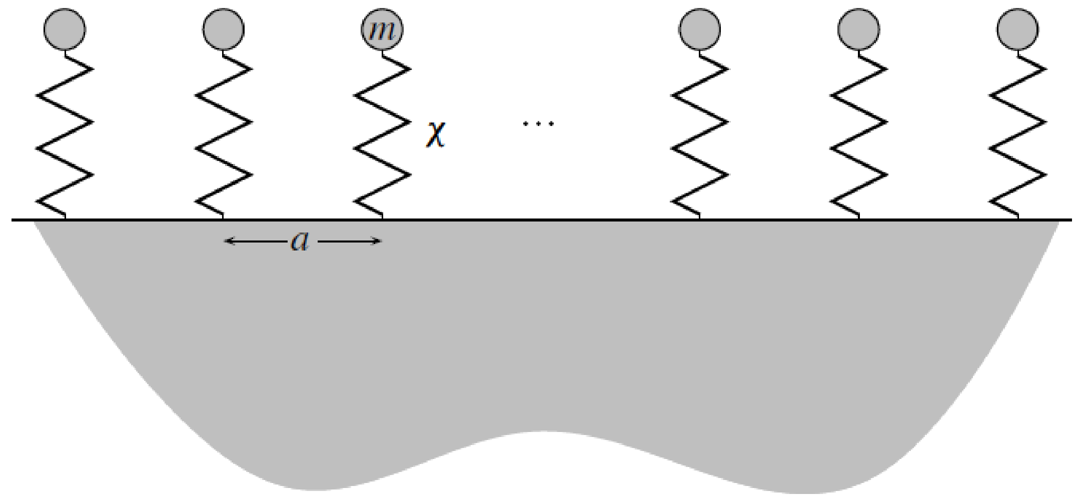
- *The latter is straightforward for numerical integration, however, is an artefact of the asymptotic model, arising due to neglecting bulk waves. A similar term does not appear in analysis of the full problem within linear elasticity.*

- *Moreover, exact analysis shows divergence of the associated Fourier integral, resulting in blow-up of the vertical displacement at the origin*

⇩

✓ *Distributed load should work better*

- *Array of oscillators*



Hyperbolic equation on the surface

$$\frac{\partial^2 \varphi}{\partial x_1^2} - \frac{1}{c_R^2}\frac{\partial^2 \varphi}{\partial t^2} = -\frac{1+\beta_R^2}{2\mu B}\sum_{n=-\infty}^{\infty} p(x_1,t)\delta(x_1+na)$$

($p(x_1, t)$ is the contact force)

Homogenizing the load

$$\sum_{n=-\infty}^{\infty} p(x_1,t)\delta(x_1+na) \approx \frac{1}{a} p(x_1,t).$$

In case of time-harmonic waves

$$p = p_0 \exp i(kx_1 - \omega t)$$

Mass-spring oscillator:

$$m v_{tt} + \chi v = p$$

Rayleigh eigensolution

$$\varphi = -\frac{A}{k\alpha_R} e^{i(kx_1 - \omega t) - k\alpha_R x_3},$$

$$\psi = \frac{2iA}{k(1+\beta_R^2)} e^{i(kx_1 - \omega t) - k\beta_R x_3},$$

$$\Rightarrow \quad u_3 = A\left(e^{-k\alpha_R x_3} - \frac{2e^{-k\beta_R x_3}}{1+\beta_R^2}\right) e^{i(kx_1 - \omega t)}.$$

For the oscillator

$$v = \frac{p_0}{m(\omega_0^2 - \omega^2)} e^{i(kx_1 - \omega t)}, \qquad \omega_0 = \sqrt{\chi / m}.$$

Conditions on the surface

$\Downarrow$

Dispersion relation

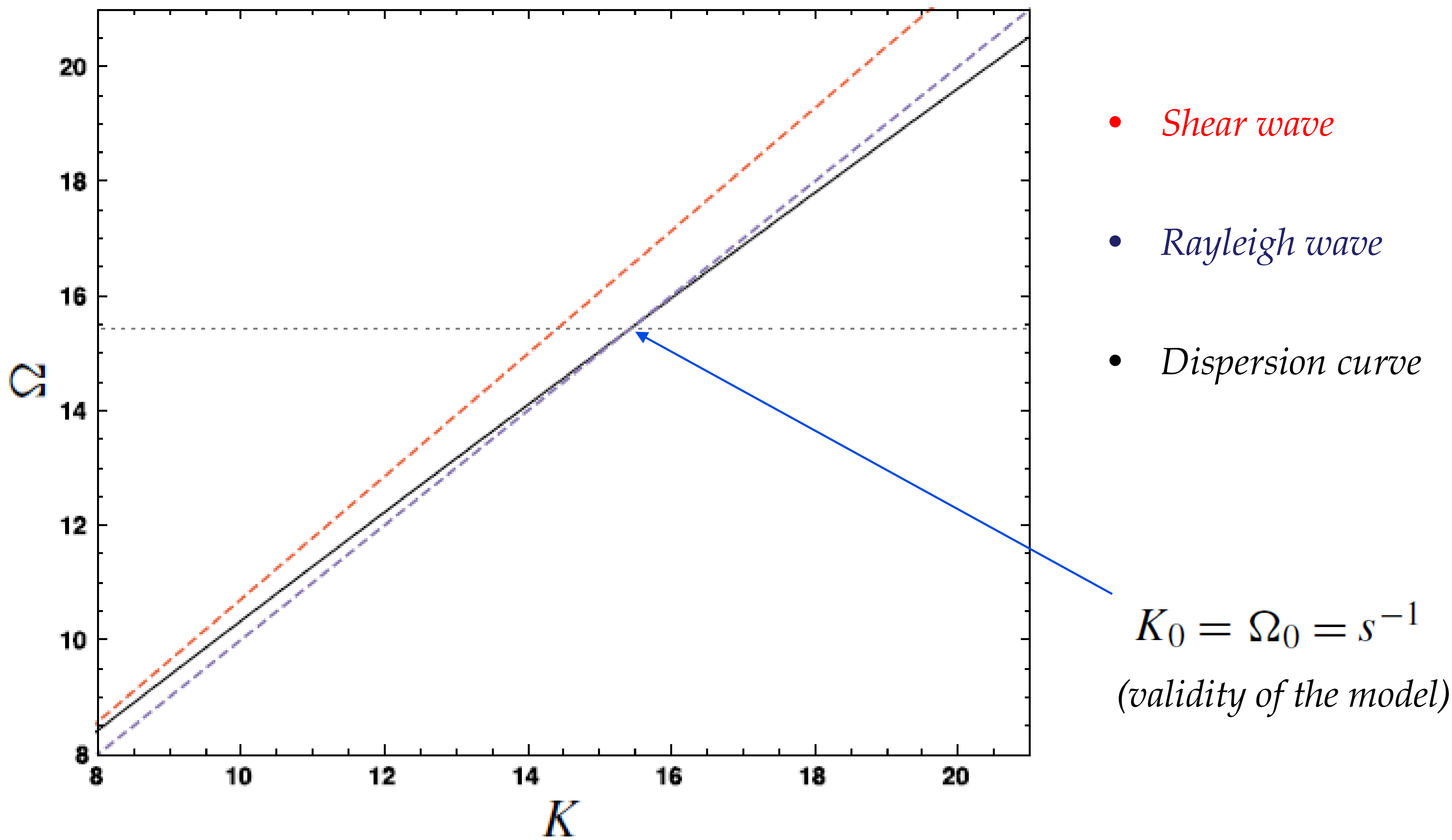
$$\boxed{K^2 - \Omega^2 = r(s^2\Omega^2 - 1)K}$$

LHS - c_R *RHS – ω_0*

Here

$$K = ka, \qquad \Omega = \frac{\omega a}{c_R},$$

$$s = \frac{c_R}{\omega_0 a}, \qquad r = \frac{(1-\beta_R^2)\alpha_R \chi}{2\mu B}.$$

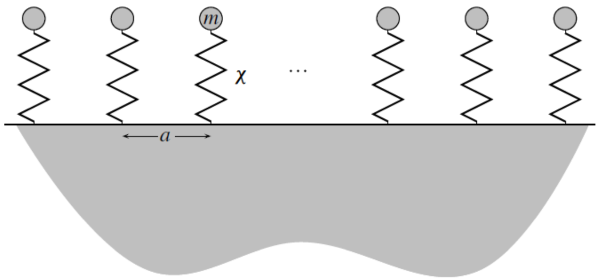


$$\boxed{K^2 - \Omega^2 = r(s^2\Omega^2 - 1)K}$$

- *Extensions to anisotropy (mass-spring system on an orthotropic half-space)*

(D. Prikazchikov et al., submitted)

Asymptotic model for the Rayleigh wave induced by prescribed surface stresses



Formulation within linear elasticity

$$c_{11}u_{1,11} + c_{66}u_{1,22} + \left(c_{12} + c_{66}\right)u_{2,12} = \rho u_{1,tt}$$

and

$$\left(c_{12} + c_{66}\right)u_{1,12} + c_{66}u_{2,11} + c_{22}u_{2,22} = \rho u_{2,tt}$$

subject to boundary conditions at $x_2 = 0$

$$\sigma_{21} = c_{66}\left(u_{1,2} + u_{2,1}\right) = f_1\left(x_1,\ t\right)$$

and

$$\sigma_{22} = c_{12}u_{1,1} + c_{22}u_{2,2} = f_2\left(x_1,\ t\right)$$

Asymptotic formulation: hyperbolic equations for surface displacements

$$\left.\left(u_{1,11}-\frac{1}{c_R^2}u_{1,tt}\right)\right|_{x_2=0}=\frac{Q_1\left(c_R^2\right)}{c_R^2R'\left(c_R^2\right)}\sqrt{-\partial_{,11}}\left[f_1\right]+\frac{Q_2\left(c_R^2\right)}{c_R^2R'\left(c_R^2\right)}f_{2,1}$$

and

$$\left.\left(u_{2,11}-\frac{1}{c_R^2}u_{2,tt}\right)\right|_{x_2=0}=\frac{P_1\left(c_R^2\right)}{c_R^2R'\left(c_R^2\right)}f_{1,1}+\frac{P_2\left(c_R^2\right)}{c_R^2R'\left(c_R^2\right)}\sqrt{-\partial_{,11}}\left[f_2\right]$$

Here

$$Q_1(z)=-c_{22}\eta(z)\zeta(z),\quad Q_2(z)=-P_1(z)=c_{11}-\rho z-c_{12}\eta(z),$$

$$P_2(z)=(\rho z-c_{11})\zeta(z),\quad \zeta(z)=\sqrt{\xi(z)+2\eta(z)},$$

$$R(z)=(c_{11}c_{22}-c_{12}^2-c_{22}\rho z)\eta(z)-\rho z(c_{11}-\rho z)$$

and

$$\xi(z)=\frac{c_{11}c_{22}-c_{12}^2-2c_{12}c_{66}-(c_{22}+c_{66})\rho z}{c_{22}c_{66}},\qquad \eta(z)=\sqrt{\frac{\left(c_{11}-\rho z\right)\left(c_{66}-\rho z\right)}{c_{22}c_{66}}}.$$

Rayleigh eigensolutions

$$u_1 = A\left(e^{-kq_1x_2} - \frac{\gamma+\delta q_1^2}{\gamma+\delta q_2^2} e^{-kq_2x_2} \right) e^{i(kx_1-\omega t)},$$

$$u_2 = A\left(f(q_1)e^{-kq_1x_2} - \frac{\gamma+\delta q_1^2}{\gamma+\delta q_2^2} f(q_2)e^{-kq_2x_2} \right) e^{i(kx_1-\omega t)},$$

For the oscillator

$$v = \frac{p_0}{m(\omega_0^2-\omega^2)} e^{i(kx_1-\omega t)}, \qquad \omega_0 = \sqrt{\chi/m}.$$

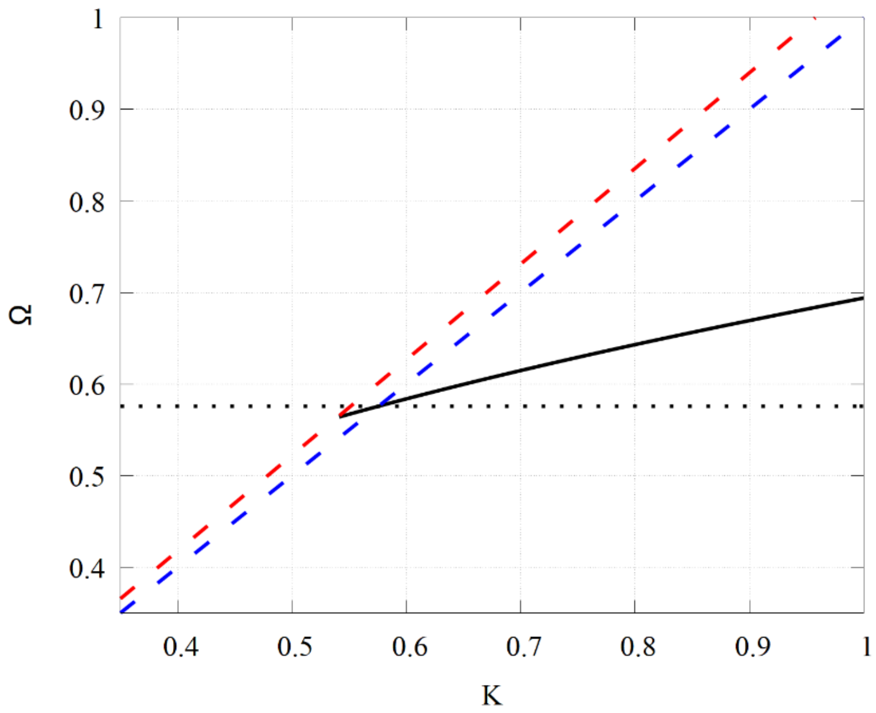
Conditions on the surface

$\Downarrow$

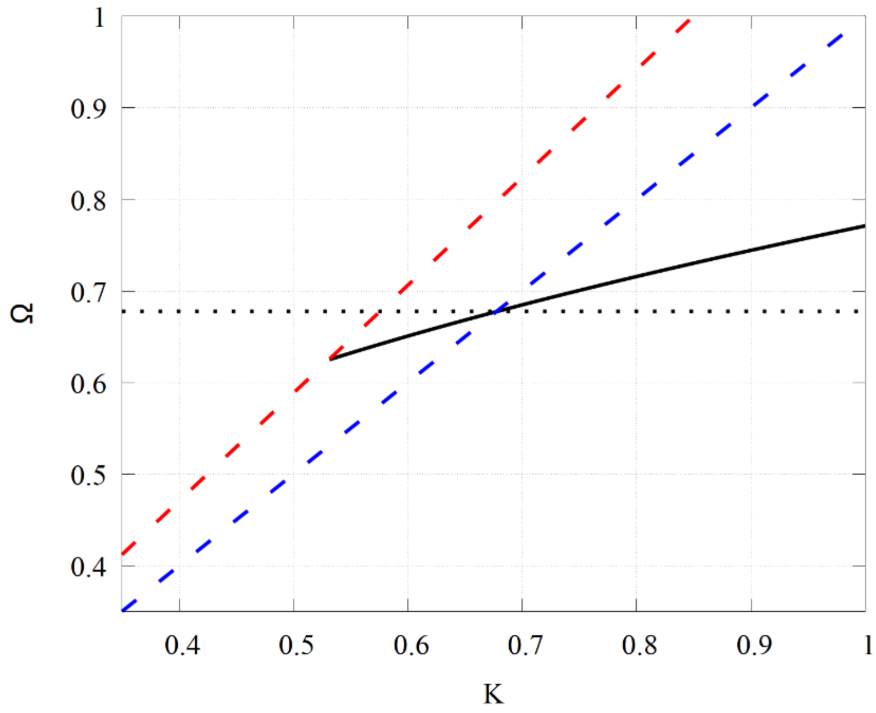
Dispersion relation

$$\boxed{K^2 - \Omega^2 = r(s^2\Omega^2 - 1)K}$$

Dispersion curves

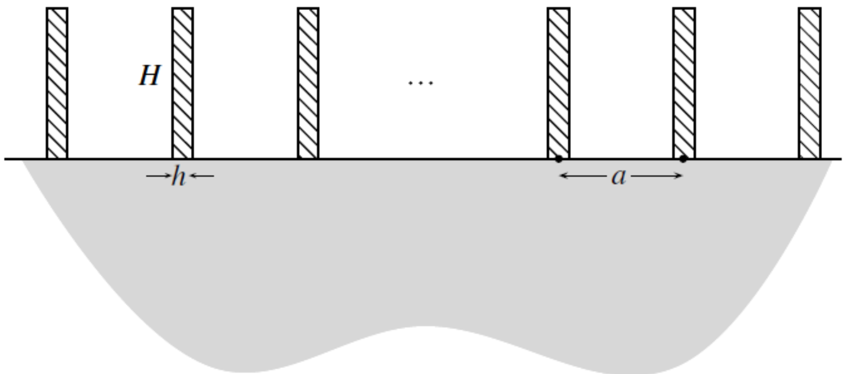


a) pure exponential decay



b) oscillatory decay

- *Array of elastic rods*



Equation of motion

$$E\frac{\partial^2 v}{\partial x_3^2} - m\frac{\partial^2 v}{\partial t^2} = 0,$$

subject to

$$\frac{\partial v}{\partial x_3} = \frac{p}{Eh}, \quad x_3 = 0, \qquad \frac{\partial v}{\partial x_3} = 0, \qquad x_3 = -H$$

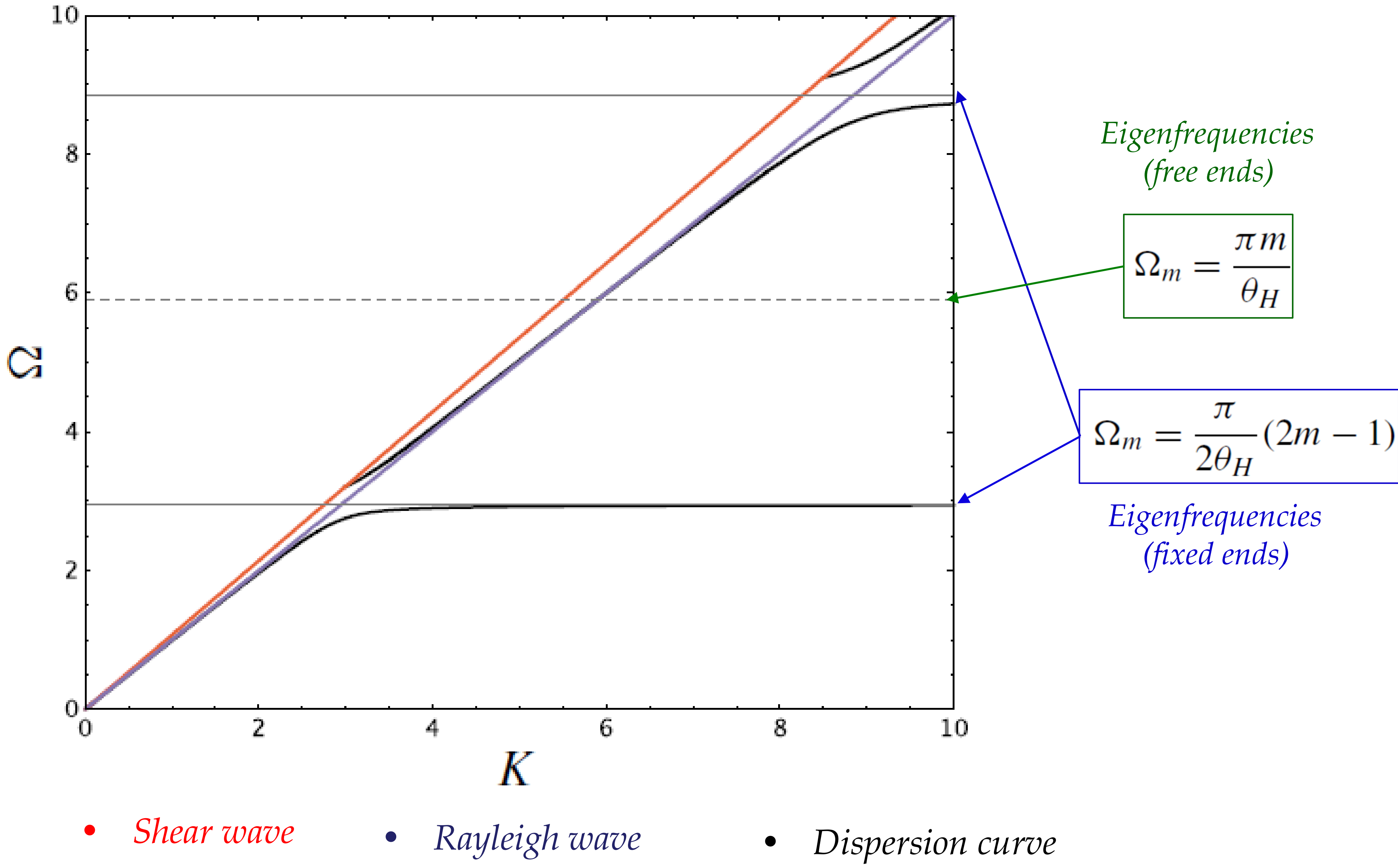
Using the Rayleigh wave eigensolution and the solution for the rod

$$v = -\frac{c_0 p}{Eh\omega}\frac{\cos(\omega(x_3 + H)/c_0)}{\sin(\omega H/c_0)}, \qquad c_0 = \sqrt{E/m},$$

the dispersion relation is obtained in the form

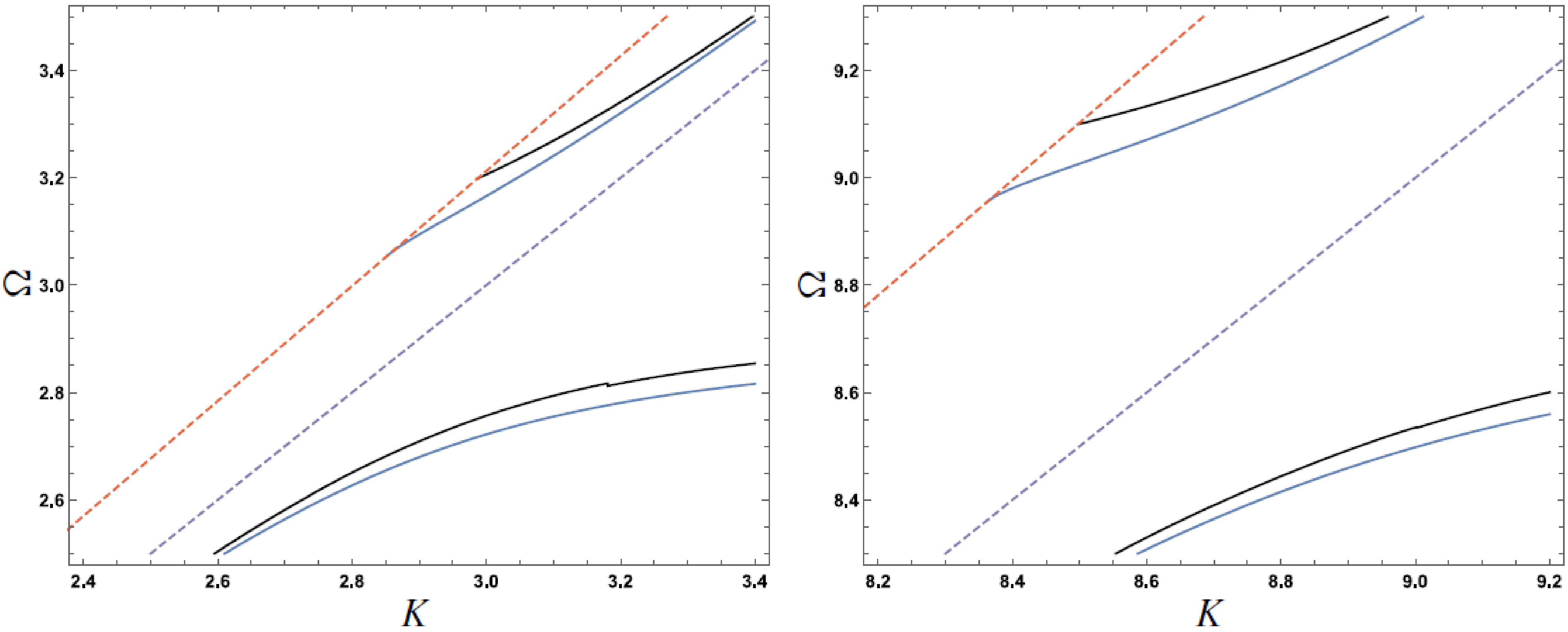
$$\boxed{K^2 - \Omega^2 = q\theta_h K\Omega\tan(\theta_H\Omega),}$$

with $\quad \theta_h = \dfrac{c_R h}{c_0 a}, \qquad \theta_H = \dfrac{c_R H}{c_0 a}, \qquad q = \dfrac{E\alpha_R(1-\beta_R^2)}{2\mu B}.$



Eigenfrequencies (free ends)
$\Omega_m = \frac{\pi m}{\theta_H}$
$\Omega_m = \frac{\pi}{2\theta_H}(2m-1)$
Eigenfrequencies (fixed ends)
Ω
K
0
2
4
6
8
10
• Shear wave
• Rayleigh wave
• Dispersion curve

Comparison with exact solution near the first and second band gaps *(D.J. Colquitt et al., JMPS,* ***2017)***

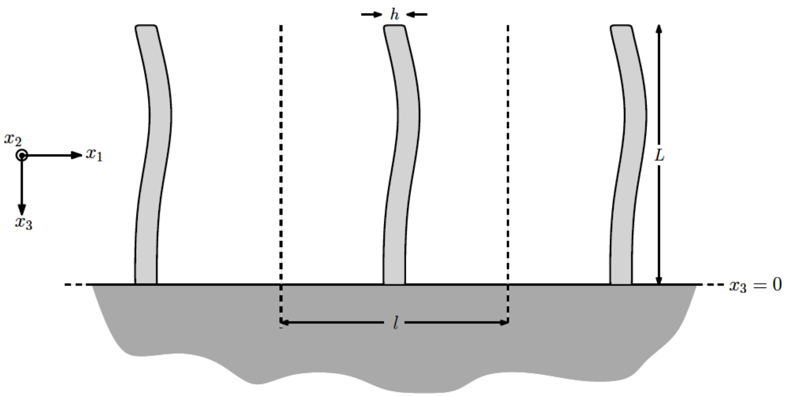


- *Exact solution*
- *Approximate dispersion curves*

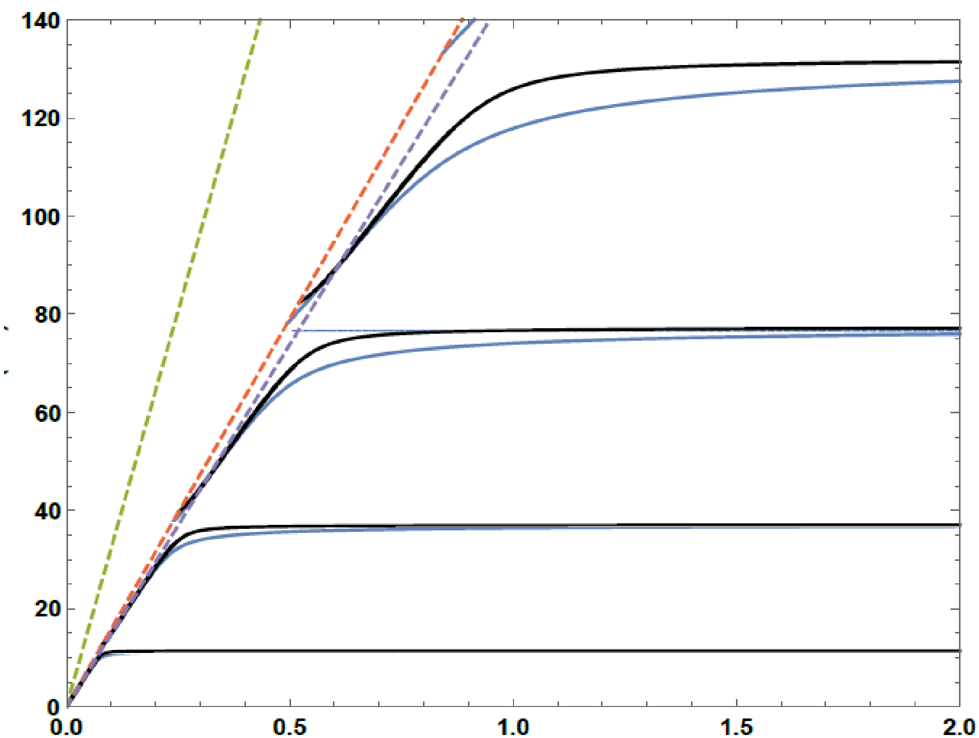
- *Array of Euler-Bernoulli beams*

P. Wootton et al., Proc. Roy. Soc. A, ***2019***

A. Alzaidi et al., ZAMP, ***2022***



Comparison with exact solution



- *Exact*
- *Approximate*

✓ *Several modes of contact analysed (simply supported, horizontal rails, full matching)*

✓ *No band gaps for simply supported*

✓ *Straightforward explicit approximate solutions*

Remarks

- ✓ *The methodology of hyperbolic-elliptic models for surface waves allows various qualitative insights combined with physical understanding*

- ✓ *Prospective explicit approximate solutions for seismic meta-interfaces for various types of embedded resonators are possible, as well as in case of fluid-structure interaction*

- ✓ *More rigorous homogenisation of the contact force, as well as adding randomness factor to distribution of the oscillators, may be beneficial.*