The interaction between a thin fluid layer and an elastic plate

Andrijana Ćurković

Faculty of Science, University of Split

(joint work with Eduard Marušić-Paloka)

Recent progress in quantitative analysis of multiscale media

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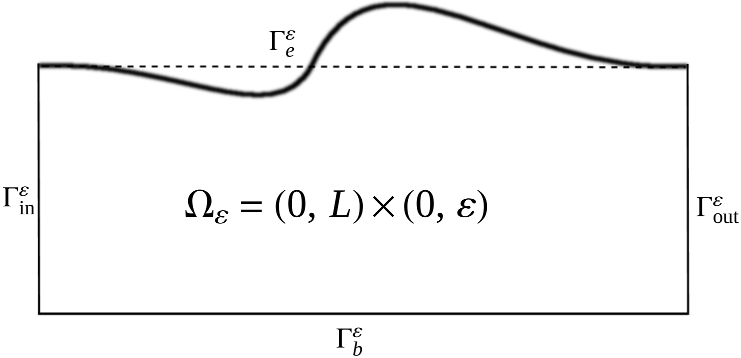
Fluid domain



Fluid flow

$$\begin{aligned} &\rho_f \partial_t \boldsymbol{u}^\varepsilon - \mu \triangle \boldsymbol{u}^\varepsilon + \nabla p^\varepsilon = \boldsymbol{g}^\varepsilon && \text{in } \Omega_\varepsilon \times (0, T_\varepsilon) \\ &\operatorname{div} \boldsymbol{u}^\varepsilon = 0 && \text{in } \Omega_\varepsilon \times (0, T_\varepsilon) \end{aligned}$$

$$\boldsymbol{u}^\varepsilon = 0 \qquad \text{on } \Gamma_b \times (0, T_\varepsilon)$$

$$\begin{aligned} &v^\varepsilon = 0, \;\; p^\varepsilon = 0 && \text{on } \Gamma_{in}^\varepsilon \times (0, T_\varepsilon) \\ &v^\varepsilon = 0, \;\; p^\varepsilon = A^\varepsilon(t) && \text{on } \Gamma_{out}^\varepsilon \times (0, T_\varepsilon) \end{aligned}$$

$$\boldsymbol{u}^\varepsilon(\cdot, 0) = 0 \quad \text{in } \Omega_\varepsilon$$

$$u^\varepsilon = 0, \;\; v^\varepsilon = \partial_t h^\varepsilon \quad \text{on } \Gamma_e^\varepsilon \times (0, T_\varepsilon)$$

Plate equations

$$\vec{n}\cdot \sigma_f^\varepsilon|_{y=\varepsilon}\,\vec{n} = -\rho_s b\partial_t^2 h^\varepsilon - B\partial_x^4 h^\varepsilon + Mb\gamma\partial_x^2 h^\varepsilon + f^\varepsilon \qquad \text{on } \Gamma_e^\varepsilon \times (0, T_\varepsilon)$$

$$h^\varepsilon(0,\cdot) = \partial_x h^\varepsilon(0,\cdot) = 0 \qquad \text{in } (0, T_\varepsilon)$$

$$h^\varepsilon(L,\cdot) = \partial_x h^\varepsilon(L,\cdot) = 0 \qquad \text{in } (0, T_\varepsilon)$$

$$h^\varepsilon(\cdot,0) = \partial_t h^\varepsilon(\cdot,0) = 0 \qquad \text{in } (0, L)$$

Assumption:

$$\lim_{\varepsilon\to 0}(\varepsilon \cdot B(\varepsilon)) = B_0 \in (0, +\infty)$$

$$\lim_{\varepsilon\to 0}(\varepsilon \cdot M(\varepsilon)b(\varepsilon)\gamma(\varepsilon)) = M_0 \in [0, +\infty)$$

Variational formulation

$$-\rho_f \int_0^{T_\varepsilon}\int_{\Omega_\varepsilon} \vec{u}^\varepsilon \cdot \partial_t\varphi \, dxdydt + 2\mu \int_0^{T_\varepsilon}\int_{\Omega_\varepsilon} e(\boldsymbol{u}^\varepsilon) : e(\varphi) \, dxdydt +$$

$$+Mb\gamma \int_0^{T_\varepsilon}\int_{\Gamma_e^\varepsilon} \partial_x\varphi_2 \partial_x h^\varepsilon \, dxdt + B \int_0^{T_\varepsilon}\int_{\Gamma_e^\varepsilon} \partial_x^2\varphi_2 \partial_x^2 h^\varepsilon \, dxdt -$$

$$-\rho_s b \int_0^{T_\varepsilon}\int_{\Gamma_e^\varepsilon} \partial_t\varphi_2 \partial_t h^\varepsilon \, dxdt = \int_0^{T_\varepsilon}\int_{\Gamma_e^\varepsilon} \varphi_2 f^\varepsilon \, dxdt +$$

$$+ \int_0^{T_\varepsilon}\int_{\Omega_\varepsilon} \varphi \cdot \boldsymbol{g}^\varepsilon \, dxdydt - \int_0^{T_\varepsilon}\int_{\Gamma_{out}^\varepsilon} A^\varepsilon \varphi_1 \, dydt$$

Time rescaling

$$\tilde{t} = \omega^\varepsilon t, \quad \tilde{t} \in (0, T)$$

$$\begin{aligned} A^\varepsilon(t) &= A(\omega^\varepsilon t) = A(\tilde{t}) \\ f^\varepsilon(\cdot, t) &= f(\cdot, \omega^\varepsilon t) = f(\cdot, \tilde{t}) \\ \boldsymbol{g}^\varepsilon(x, y, t) &= \boldsymbol{g}(x, \frac{y}{\varepsilon}, \omega^\varepsilon t) = \boldsymbol{g}(x, \frac{y}{\varepsilon}, \tilde{t}) \end{aligned}$$

$$\partial_t(\cdot) = \omega^\varepsilon \partial_{\tilde{t}}(\cdot)$$

$$v^\varepsilon = \omega^\varepsilon \partial_t h^\varepsilon \quad \text{on } \Gamma_e^\varepsilon$$

Test function $\varphi = \boldsymbol{u}$

$$2\mu \int_{\Omega_\varepsilon} e(\boldsymbol{u}^\varepsilon) : e(\boldsymbol{u}^\varepsilon)\, \mathrm{d}x\mathrm{d}y + \omega^\varepsilon \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\rho_{\mathrm{f}}}{2} \int_{\Omega_\varepsilon} \boldsymbol{u}^\varepsilon \cdot \boldsymbol{u}^\varepsilon \, \mathrm{d}x\mathrm{d}y + \right.$$

$$\left. + \frac{Mb\gamma}{2} \int_0^L \left|\partial_x h^\varepsilon\right|^2 \mathrm{d}x + \frac{B}{2} \int_0^L \left|\partial_x^2 h^\varepsilon\right|^2 \mathrm{d}x + (\omega^\varepsilon)^2 \frac{\rho_{\mathrm{s}} b}{2} \int_0^L \left|\partial_t h^\varepsilon\right|^2 \mathrm{d}x \right) =$$

$$= - \int_0^\varepsilon A u^\varepsilon(L, \cdot, t)\, \mathrm{d}y + \int_0^L f v^\varepsilon(\cdot, \varepsilon, t)\, \mathrm{d}x + \int_{\Omega_\varepsilon} \boldsymbol{g}^\varepsilon \cdot \boldsymbol{u}^\varepsilon \, \mathrm{d}x\mathrm{d}y$$

$$\|A\|_H^2 = \|A\|_{L^\infty(0,T)}^2 + \int_0^T \left|A'(\tau)\right|^2 \, d\tau$$

$$\|f\|_H^2 = \|f\|_{L^\infty(0,T;L^2(0,L))}^2 + \int_0^T \|\partial_t f(\tau)\|_{L^2(0,L)}^2 \, d\tau$$

A priori estimates

$$\frac{\mu}{2}\,\|\nabla \boldsymbol{u}^{\varepsilon}\|^2_{L^2(\Omega_\varepsilon\times(0,T))^4} + \frac{B\omega^{\varepsilon}}{4}\,\|\partial_{xx}h^{\varepsilon}\|^2_{L^\infty(0,T;L^2(0,L))} \leq$$

$$\leq \left(\frac{\omega^{\varepsilon}L^5}{6B}\,\|A\|^2_H + \frac{\omega^{\varepsilon}L^4}{2B}\,\|f\|^2_H + \right.$$

$$\left. + \frac{\varepsilon^3}{2L\mu}\|A\|^2_{L^2(0,T)} + \frac{\varepsilon^3}{2\mu}\,\|g\|^2_{L^2(\Omega\times(0,T))^2}\right) e^{T}$$

$$\omega^{\varepsilon} = \frac{\varepsilon^2}{\mu}$$

Existence and regularity of a solution

Theorem

If the given functions f, $\mathbf{g}$, A belong to spaces $H^1(0,T;L^2(0,L))$, $H^1(0,T;L^2(\Omega))$ and $H^1(0,T)$ respectively, than for every $\varepsilon > 0$ there is unique weak solution of the problem such that

$$\boldsymbol{u}^\varepsilon \in H^1(0,T;H^1(\Omega_\varepsilon)) \cap C^1([0,T];L^2(\Omega_\varepsilon)),$$

$$h^\varepsilon \in H^2(0,T;L^2(0,L)) \cap H^1(0,T;H_0^2(0,L)),$$

$$p^\varepsilon \in L^2(0,T;L^2(\Omega_\varepsilon)).$$

.

Rescaled functions

Let us introduce the following sequence of functions defined on a fixed domain $\Omega = \Omega_1$

$$\begin{aligned} \boldsymbol{u}(\varepsilon)(x,y) &= \boldsymbol{u}^{\varepsilon}(x,\varepsilon y), \\ p(\varepsilon)(x,y) &= p^{\varepsilon}(x,\varepsilon y). \end{aligned}$$

Every rescaled function satisfies

$$\begin{aligned} \|u(\varepsilon)\|^2_{L^2(\Omega)} &= \varepsilon^{-1}\,\|u^{\varepsilon}\|^2_{L^2(\Omega_\varepsilon)}, \\ \|\partial_x u(\varepsilon)\|^2_{L^2(\Omega)} &= \varepsilon^{-1}\,\|\partial_x u^{\varepsilon}\|^2_{L^2(\Omega_\varepsilon)}, \\ \|\partial_y u(\varepsilon)\|^2_{L^2(\Omega)} &= \varepsilon\,\|\partial_y u^{\varepsilon}\|^2_{L^2(\Omega_\varepsilon)}. \end{aligned}$$

Estimates

$$\frac{1}{\varepsilon}\left\|\partial_y \boldsymbol{u}(\varepsilon)\right\|^2_{L^2(\Omega\times(0,T))^2} + \varepsilon\left\|\partial_x \boldsymbol{u}(\varepsilon)\right\|^2_{L^2(\Omega\times(0,T))^2} +$$

$$+\varepsilon\left\|\partial_{xx} h^{\varepsilon}\right\|^2_{L^\infty(0,T;L^2(0,L))} \leq C\varepsilon^3$$

$$\left\|p(\varepsilon)\right\|_{L^2(\Omega\times(0,T))} \leq C$$

$$\left\|\partial_x p(\varepsilon)\right\|_{H^{-1}((0,L)\times(0,T))} + \frac{1}{\varepsilon}\left\|\partial_y p(\varepsilon)\right\|_{H^{-1}((0,L)\times(0,T))} \leq C$$

Asymptotic expansions

$$
\begin{aligned}
\boldsymbol{u}(\varepsilon) &= \frac{\varepsilon^2}{\mu}\sum_{k\geq 0}\varepsilon^k \boldsymbol{u}^k, \qquad \boldsymbol{u}^k = (u^k, v^k)\\
p(\varepsilon) &= \sum_{k\geq 0}\varepsilon^k p^k\\
h^\varepsilon &= \varepsilon\sum_{k\geq 0}\varepsilon^k h^k
\end{aligned}
$$

Effective equations

$$v^0 = 0$$

$$\partial_x u + \partial_y v = 0 \qquad \text{in } \Omega \times (0, T)$$

$$\partial_{yy} u = \partial_x p - g_1 \qquad \text{in } \Omega \times (0, T)$$

$$p = p(x, t)$$

$$-p = -B_0 \partial_x^4 h + M_0 \partial_x^2 h + f \qquad \text{on } \Gamma_e \times (0, T)$$

$$v = \partial_t h \qquad \text{on } \Gamma_e \times (0, T)$$

Initial-boundary conditions

$$
\begin{aligned}
(u, v) &= 0 \qquad \text{on } \Gamma_b \times (0, T) \\
u &= 0 \qquad \text{on } \Gamma_e \times (0, T)
\end{aligned}
$$

$$
\begin{aligned}
p &= 0 \qquad \text{on } \Gamma_{in} \times (0, T) \\
p &= A \qquad \text{on } \Gamma_{out} \times (0, T)
\end{aligned}
$$

$$
\begin{aligned}
h(0, \cdot) = \partial_x h(0, \cdot) = 0 \qquad \text{in } (0, T) \\
h(L, \cdot) = \partial_x h(L, \cdot) = 0 \qquad \text{in } (0, T)
\end{aligned}
$$

$$
h(\cdot, 0) = 0 \qquad \text{in } (0, L)
$$

Effective equation

$$\partial_t h(x,t) = \frac{1}{12}\partial_{xx} p(x,t) - \int_0^1 \frac{1}{2}\xi(1-\xi)\partial_x g_1(x,\xi,t)\, d\xi$$

$$-p = -B_0 \partial_x^4 h + M_0 \partial_x^2 h + f \quad \text{on } \Gamma_e \times (0,T)$$

$$-12\partial_t h - 6\int_0^1 \xi(1-\xi)\partial_x g_1(x,\xi,t)\, d\xi = -B_0\partial_x^6 h + M_0 \partial_x^4 h + \partial_x^2 f$$

$$\begin{aligned} \left(-B_0\partial_x^4 h + M_0\partial_x^2 h + f\right)\Big|_{x=0} &= 0 \\ \left(-B_0\partial_x^4 h + M_0\partial_x^2 h + f\right)\Big|_{x=L} &= -A \end{aligned}$$

Convergence results

$$U = \left\{ \varphi \in L^2(\Omega) : \partial_y \varphi \in L^2(\Omega) \right\}$$

$$\|\varphi\|_U = \|\varphi\|_{L^2(\Omega)} + \|\partial_y \varphi\|_{L^2(\Omega)}$$

$$\begin{aligned}
\frac{\mu}{\varepsilon^2}\boldsymbol{u}(\varepsilon) &\rightharpoonup \boldsymbol{u} = (u, v) \text{ weakly in } L^2(0, T; U) \\
p(\varepsilon) &\rightharpoonup p \text{ weakly in } L^2(\Omega \times (0, T)) \\
\frac{h^\varepsilon}{\varepsilon} &\overset{*}{\rightharpoonup} h \text{ weak} * \text{ in } L^\infty(0, T; H_0^2(0, L))
\end{aligned}$$

Limit (cluster point) properties

$$\begin{aligned} v &= 0 \\ \partial_t h + \partial_x \int_0^1 u(x,y,t)\,dy &= 0 \qquad \text{in the sense of distributions} \\ \frac{\partial_t h^\varepsilon}{\varepsilon} &\rightharpoonup \partial_t h \qquad \text{weakly in } L^2(0,T;H^{-1}(0,L)) \\ \partial_y p &= 0 \end{aligned}$$

$$u = 0 \qquad \text{on } \Gamma_b \cup \Gamma_e \times (0,T)$$

$$h(0,\cdot) = \partial_x h(0,\cdot) = h(L,\cdot) = \partial_x h(L,\cdot) = 0 \qquad \text{in } (0,T)$$

$$h(\cdot,0) = 0 \qquad \text{in } (0,L)$$

Weak formulation of the limiting problem

$$\int_0^T\int_\Omega \partial_y u\,\partial_y\varphi_1\,dxdydt + M_0\int_0^T\int_{\Gamma_e}\partial_x\varphi_2\partial_x h\,dxdt +$$

$$+B_0\int_0^T\int_{\Gamma_e}\partial_x^2\varphi_2\partial_x^2 h\,dxdt - \int_0^T\int_{\Gamma_e}\varphi_2 f\,dxdt - \int_0^T\int_\Omega p\,\mathrm{div}\,\varphi\,dxdydt =$$

$$= \int_0^T\int_\Omega \varphi_1 g_1\,dxdydt - \int_0^T\int_{\Gamma_{out}} A(t)\varphi_1\,dydt$$

Assumptions

$$f \in H^2(0,T;L^2(0,L)) \cap L^2(0,T;H^2(0,L))$$

$$\boldsymbol{g} = (g_1, g_2) \in H^1(0,T;L^2(\Omega)^2)$$

$$g_1 \in L^2(0,T;H^1(\Omega))$$

$$A \in H^2(0,T)$$

$$f(x,0) + \frac{A(0)x}{L} = 0$$

$$\frac{A(0)}{L} - g_1(x,0) = 0, \;\; g_2(x,0) = 0$$

$$G(x,t) = \int_0^1 y(1-y)\partial_x g_1 \, dy$$

Theorem

Under the above assumptions there exists unique solution

$$h \in H^1(0, T; H_0^2(0, L)) \cap L^2(0, T; H^6(0, L))$$

of the initial-boundary-value problem

$$12\partial_t h - B_0 \partial_x^6 h + M_0 \partial_x^4 h = -6G - \partial_x^2 f,$$

$$\begin{aligned} h(0, \cdot) = h'(0, \cdot) = h(L, \cdot) = h'(L, \cdot) &= 0, \\ B_0 \partial_x^4 h(0, \cdot) - M_0 \partial_x^2 h(0, \cdot) &= f(0, \cdot), \\ B_0 \partial_x^4 h(L, \cdot) - M_0 \partial_x^2 h(L, \cdot) &= f(L, \cdot) + A, \end{aligned}$$

$$h(\cdot, 0) = 0.$$

Every convergent subsequence of solutions of the rescaled problem converges weakly, as $\varepsilon \to 0$*, to this unique solution.*

Thank you for your attention!