Derivation of reduced models for micro-fluidic channels

Recent progress in quantitative analysis of multiscale media

Hrvatska zaklada za znanost

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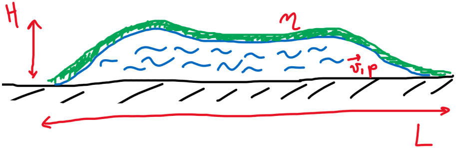
- dynamics of a micro-fluidic channel



- geometric assumption: $H/L \ll 1$

Introduction

Motivation

- wide application area:
  - micro-fluidic chips and soft actuators,
  - blood flow in capillaries,
  - oil flow in long elastic pipes,
  - . . .

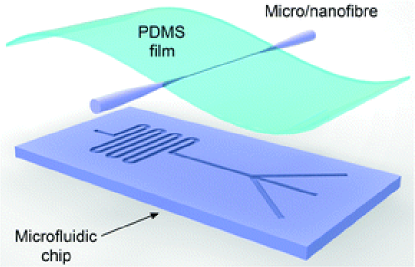
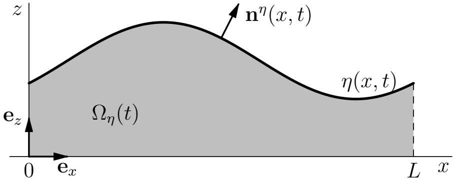


Figure: Zhang et. al. *Lab Chip* **20** (2020), 2572.

Introduction

Problem description



fluid - structure interaction (FSI) system:

$$
\begin{aligned}
\varrho_f\,(\partial_t \boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}) - \operatorname{div}\sigma_f(\boldsymbol{v},p) &= \boldsymbol{f}\,, \quad \Omega_\eta(t)\times(0,\infty)\,,\\
\operatorname{div}\boldsymbol{v} &= 0\,, \quad \Omega_\eta(t)\times(0,\infty)\,,\\
\varrho_s b\partial_{tt}\eta - \mu_s b\,\partial_x^2\partial_t\eta + B\partial_x^4\eta &= -J^\eta\sigma_f(\boldsymbol{v},p)\boldsymbol{n}^\eta\cdot\mathbf{e}_z\,, \quad \omega\times(0,\infty)\,,\\
\boldsymbol{v}(x,\eta(x,t),t) &= (0,\partial_t\eta(x,t))\,, \quad \omega\times(0,\infty)\,,\\
\boldsymbol{v}(x,0,t) &= 0\,, \quad \omega\times(0,\infty)\,.
\end{aligned}
$$

fluid stress tensor:

$$
\sigma_f(\boldsymbol{v},p) = 2\mu_f\,\mathrm{D}(\boldsymbol{v}) - p\mathbb{I}_2\,, \qquad \mathrm{D}(\boldsymbol{v}) = \frac{1}{2}(\nabla\boldsymbol{v} + (\nabla\boldsymbol{v})^T)\,.
$$

Introduction

Nondimensionalization

non-dimensional variables:

$$(\hat{x}, \hat{z}) = \frac{1}{L}(x, z)\,, \quad \hat{\boldsymbol{v}} = \frac{1}{V}\boldsymbol{v}\,, \quad \hat{p} = \frac{p}{P}\,, \quad \hat{t} = \frac{t}{T}\,, \quad \hat{\eta} = \frac{\eta}{L}$$

pressure and time scale: $P = \dfrac{\mu_f V}{L}\,, \quad T = \dfrac{L}{V}$

neglecting hats:

$$\begin{aligned}
\mathrm{Re}\,(\partial_t \boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}) - \Delta\boldsymbol{v} + \nabla p &= \boldsymbol{f}\,, \quad \Omega_\eta(t)\times(0,\infty)\,,\\
\operatorname{div}\boldsymbol{v} &= 0\,, \quad \Omega_\eta(t)\times(0,\infty)\,,\\
\rho\partial_{tt}\eta - \delta\partial_x^2\partial_t\eta + \beta\partial_x^4\eta &= -J^\eta\boldsymbol{\sigma}_f(\boldsymbol{v},p)\boldsymbol{n}^\eta\cdot\mathbf{e}_z\,, \quad \omega\times(0,\infty)\,,\\
\boldsymbol{v}(x,\eta(x,t),t) &= (0,\partial_t\eta(x,t))\,, \quad \omega\times(0,\infty)\,,\\
\boldsymbol{v}(x,0,t) &= 0\,, \quad \omega\times(0,\infty)\,,
\end{aligned}$$

where:

$$\mathrm{Re} = \frac{\varrho_f V L}{\mu_f}\,, \quad \rho = \frac{\varrho_s b V}{\mu_f}\,, \quad \delta = \frac{\mu_s b}{\mu_f L}\,, \quad \beta = \frac{B}{L^2\mu_f V}\,.$$

scaling parameter — relative fluid thickness:

$$\varepsilon = \frac{H}{L} \ll 1$$

- system coefficients:

$$\mathrm{Re} \sim O(1)\,, \quad \rho = \frac{\varrho_s b}{\varrho_f L}\mathrm{Re} \sim O(\varepsilon)\,, \quad \delta = \frac{\mu_s b}{\mu_f L} \sim O(\varepsilon^{-r})\,, \quad r \in [1,3]$$

$$\beta = \frac{E b^3}{12(1-\nu^2)L^2 \mu_f V} \sim O(\varepsilon^{-1})$$

- time scaling $\hat{t} = \varepsilon^2 t$
- $\|\boldsymbol{f}\|_{L^\infty(0,\infty;L^\infty(\Omega_\eta(t);\mathbb{R}^2))} \leq C$
- initial displacement: $\eta_0 \sim O(\varepsilon)$

lubrication approximation in fluid channel:

$$\begin{aligned} -\partial_{zz} v_1 + \partial_x p &= 0\,, \\ \partial_z p &= 0\,, \\ \partial_x v_1 + \partial_z v_2 &= 0\,. \end{aligned}$$

Reynolds equation for the pressure: [Bayada and Chambat 1986, Bayada, Chambat and Ciuperca 1998]

$$-\partial_x \left(h^3 \partial_x p\right) = -\partial_t h$$

h – rescaled height of the fluid film

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$$-\partial_x \left(h^3 \partial_x p\right) = -\partial_t h$$

h – rescaled height of the fluid film
pressure balanced by:

- surface tension: $p \sim -h_{xx}$ [Giacomelli and Otto 2003, Günther and Prokert 2008]

$$\partial_t h = -\partial_x \left(h^3 h_{xxx}\right)$$

- structure bending: $p \sim h_{xxxx}$ [Hosoi and Mahadevan 2004, Hewitt et al 2014, Peng and Lister 2020 ...]

$$\partial_t h = \partial_x \left(h^3 h_{xxxxx}\right)$$

Outline of the rest

- briefly on some related results from the literature
- existence of solutions
- uniform estimates
- identification of the reduced model
- further perspectives

Reduced models

Related results for thin FSI problems

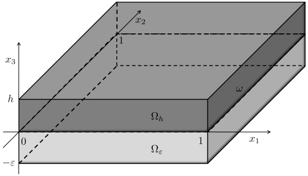
- thin fluid channel (2D) with visco-elastic wall (1D):
  - fixed fluid domain (linear case):

$$\partial_t h = \partial_x^6 h + \text{l.o.t.}$$

h – scaled vertical displacement

[Panasenko and Stavre 2006, Ćurković and Marušić-Paloka 2018]

- thin fluid channel (3D) with membrane wall (2D):
  - fixed cylindrical domain (linear model): [Čanić and Mikelić 2003]
- thin fluid – thin structure interaction (3D-3D):
  - fixed domain (linear model): [B. and Muha 2021]



$$\partial_t h - B\Delta_{2d}^3 h = F\,.$$

  - fixed cylindrical domain (linear model): [Panasenko and Stavre 2020]

$\boldsymbol{v}$ and η classical solutions: for every $t \in (0,T)$

$$\frac{1}{2}\|\boldsymbol{v}(t)\|^2_{L^2(\Omega_\eta(t))} + 2\int_0^t \int_{\Omega_\eta(s)} |\,\mathrm{D}(\nabla \boldsymbol{v})|^2 \mathrm{d}\boldsymbol{x}\mathrm{d}s + \frac{\rho}{2}\|\partial_t \eta(t)\|^2_{L^2(\omega)}$$

$$+ \delta \int_0^t \|\partial_t\partial_x\eta\|^2_{L^2(\omega)}\mathrm{d}s + \frac{\beta}{2}\|\partial_x^2\eta\|^2_{L^2(\omega)} \leq \frac{\beta}{2}\|\partial_x^2\eta_0\|^2_{L^2(\omega)} + \int_0^t\int_{\Omega_\eta(s)} \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}\mathrm{d}s$$

Energy estimate

Formal calculations

$\boldsymbol{v}$ and η classical solutions: for every $t \in (0,T)$

$$\frac{1}{2}\|\boldsymbol{v}(t)\|^2_{L^2(\Omega_\eta(t))} + 2\int_0^t\int_{\Omega_\eta(s)} |\,\mathrm{D}(\nabla \boldsymbol{v})|^2 \mathrm{d}\boldsymbol{x}\mathrm{d}s + \frac{\rho}{2}\|\partial_t\eta(t)\|^2_{L^2(\omega)}$$
$$+\,\delta\int_0^t \|\partial_t\partial_x\eta\|^2_{L^2(\omega)}\mathrm{d}s + \frac{\beta}{2}\|\partial_x^2\eta\|^2_{L^2(\omega)} \le \frac{\beta}{2}\|\partial_x^2\eta_0\|^2_{L^2(\omega)} + \int_0^t\int_{\Omega_\eta(s)} \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}\mathrm{d}s$$

estimating the force term:

- $$\left|\int_0^t\int_{\Omega_\eta(s)} \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}\mathrm{d}s\right| \le C\,\mathrm{vol}(\Omega_{\eta_0})\int_0^t \|\eta\|^2_{L^\infty(\omega)}\mathrm{d}s + \int_0^t \|\,\mathrm{D}(\boldsymbol{v})\|^2_{L^2(\Omega_\eta(s))}\mathrm{d}s$$
- $C\,\mathrm{vol}(\Omega_{\eta_0})\int_0^t \|\eta\|^2_{L^\infty(\omega)}\mathrm{d}s \le C\,\mathrm{vol}(\Omega_{\eta_0})\left(\int_0^t \|\partial_x^2\eta\|^2_{L^2(\omega)} + \overline{\eta}_0^2 t\right)$

solution spaces: given time horizon $T > 0$

- fluid solution space

$$\mathcal{V}_F(0,T;\Omega_\eta(t)) = L^\infty(0,T;L^2(\Omega_\eta(t))) \cap L^2(0,T;V_F(t))\,,$$

where $V_F(t) = \{\boldsymbol{v} \in H^1(\Omega_\eta(t)) \; : \; \operatorname{div}\boldsymbol{v} = 0,\; \boldsymbol{v}|_{z=0} = 0,\; \boldsymbol{v}$ is $1-$periodic in $x\}$;

- structure solution space

$$\mathcal{V}_S(0,T;\omega) = W^{1,\infty}(0,T;L^2(\omega)) \cap L^\infty(0,T;H^2_{\mathrm{per}}(\omega)) \cap H^1(0,T;H^1(\omega))\,.$$

Definition

$(\boldsymbol{v},\eta) \in \mathcal{V}_F(0,T;\Omega_\eta(t)) \times \mathcal{V}_S(0,T;\omega)$ is a *weak solution* of the FSI problem if for every $(\boldsymbol{\varphi},\psi) \in C^1_c([0,T);\mathcal{V}_F(t) \times H^2_{\mathrm{per}}(\omega))$ satisfying $\boldsymbol{\varphi}(t,x,\eta(t,x')) = \partial_t\psi(t,x)\mathbf{e}_z$ it holds

$$-\int_0^T\int_{\Omega_\eta(t)} (\boldsymbol{v}\cdot\partial_t\boldsymbol{\varphi} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{\varphi}\cdot\boldsymbol{v})\,\mathrm{d}\boldsymbol{x}\mathrm{d}t + 2\int_0^T\int_{\Omega_\eta(t)} \mathrm{D}(\boldsymbol{v}) : \nabla\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}\mathrm{d}t$$
$$-\rho\int_0^T\int_\omega \partial_t\eta\partial_t\psi\,\mathrm{d}x\mathrm{d}t + \delta\int_0^T\int_\omega \partial_x\partial_t\eta\partial_x\psi\,\mathrm{d}x\mathrm{d}t + \beta\int_0^T\int_\omega \partial_x^2\eta\partial_x^2\psi\,\mathrm{d}x\mathrm{d}t$$
$$= \int_0^T\int_{\Omega_\eta(t)} \boldsymbol{f}\cdot\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}\mathrm{d}t\,.$$

existence of weak solutions [Chambolle, Desjardins, Esteban and Grandmont 2005; Muha and Čanić 2013]:

- there exists $T > 0$ and a weak solution on $(0, T)$;
- moreover, either $T = \infty$ or $\lim_{t \uparrow T} \min_{x \in \overline{\omega}} \eta(x, t) = 0$.

Solutions

Strong solutions

existence of weak solutions [Chambolle, Desjardins, Esteban and Grandmont 2005; Muha and Čanić 2013]:

- there exists $T > 0$ and a weak solution on $(0, T)$;
- moreover, either $T = \infty$ or $\lim_{t \uparrow T} \min_{x \in \overline{\omega}} \eta(x,t) = 0$.

Theorem ([Grandmond and Hillairet, ARMA 2016])

Let $(\eta_0, \dot{\eta}_0) \in H^3_{\mathrm{per}}(\omega) \times H^1_{\mathrm{per}}(\omega)$ *and* $\boldsymbol{v}_0 \in V_F(0)$. *Then for every* $T > 0$ *there exists a* unique global-in-time strong solution $(\boldsymbol{v}, p, \eta)$ *of the FSI problem in the sense that* $(\boldsymbol{v}, p, \eta)$ *satisfy the FSI system pointwise a.e. Moreover,* $(\boldsymbol{v}, p, \eta)$ *are of the following regularity:*

$$
\begin{aligned}
\eta &\in H^2(0,T; L^2_{\mathrm{per}}(\omega)) \cap L^2(0,T; H^4_{\mathrm{per}}(\omega)) \,, \\
\eta^{-1} &\in L^\infty(0,T; L^\infty(\omega)) \,, \\
\boldsymbol{v} &\in H^1(0,T; H^1(\Omega_\eta(t))) \cap L^2(0,T; H^2(\Omega_\eta(t))) \,, \\
p &\in L^2(0,T; H^1(\Omega_\eta(t))) \,.
\end{aligned}
$$

Theorem ([Schwarzacher and Sroczinski, SIMA 2022])

The strong solution is unique in the class of weak solutions.

Energy estimate revisited

and rescaled

consider FSI(ε) problem + initial data: $\eta_0 = \varepsilon\hat{\eta}_0$, $\dot{\eta}_0 = 0$, $\boldsymbol{v}_0 = 0$

$$
\begin{aligned}
\varepsilon^2 \partial_t \boldsymbol{v}^\varepsilon + (\boldsymbol{v}^\varepsilon \cdot \nabla)\boldsymbol{v}^\varepsilon - \Delta \boldsymbol{v}^\varepsilon + \nabla p^\varepsilon &= \boldsymbol{f}^\varepsilon \,, \quad \Omega_\eta(t) \times (0,\infty)\,, \\
\operatorname{div} \boldsymbol{v}^\varepsilon &= 0\,, \quad \Omega_\eta(t) \times (0,\infty)\,, \\
\rho\varepsilon^5 \partial_{tt}\eta^\varepsilon - \delta\varepsilon^{2-r}\partial_x^2\partial_t\eta^\varepsilon + \beta\varepsilon^{-1}\partial_x^4\eta^\varepsilon &= -J^{\eta^\varepsilon}\boldsymbol{\sigma}_f(\boldsymbol{v}^\varepsilon, p^\varepsilon)\boldsymbol{n}^{\eta^\varepsilon} \cdot \mathbf{e}_z\,, \quad \omega \times (0,\infty)\,, \\
\boldsymbol{v}^\varepsilon(x, \eta^\varepsilon(x,t), t) &= (0, \varepsilon^2 \partial_t \eta^\varepsilon(x,t))\,, \quad \omega \times (0,\infty)\,,
\end{aligned}
$$

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\boldsymbol{v}^\varepsilon(x,\eta^\varepsilon(x,t),t) &= (0,\varepsilon^2\partial_t\eta^\varepsilon(x,t))\,, \quad \omega\times(0,\infty)\,,
\end{aligned}
$$

rescaled energy estimate: for a.e. $t\in(0,T)$

$$
\begin{aligned}
&\frac{1}{2}\|\boldsymbol{v}^\varepsilon(t)\|^2_{L^2(\Omega_\eta(t))} + \frac{\varepsilon^{-2}}{2}\int_0^t\int_{\Omega_\eta(s)}|\nabla\boldsymbol{v}^\varepsilon|^2\mathrm{d}\boldsymbol{x}\mathrm{d}s\\
&+\frac{\rho\varepsilon^5}{2}\|\partial_t\eta^\varepsilon(t)\|^2_{L^2(\omega)} + \delta\varepsilon^{2-r}\int_0^t\|\partial_t\partial_x\eta^\varepsilon\|^2_{L^2(\omega)}\mathrm{d}s + \frac{\beta}{2\varepsilon}\|\partial_x^2\eta^\varepsilon\|^2_{L^\infty(0,t;L^2(\omega))} \le C\varepsilon\,.
\end{aligned}
$$

Uniform estimates

on structure displacements

energy estimate:

- $\|\eta^\varepsilon\|_{L^\infty(0,T;H^2(\omega))} \leq C\varepsilon,$

Uniform estimates

on structure displacements

energy estimate:

- $\|\eta^\varepsilon\|_{L^\infty(0,T;H^2(\omega))} \leq C\varepsilon$,

uniform no-contact:

- $\dfrac{\eta^\varepsilon(x,t)}{\varepsilon} \geq c$ for a.e. $(x,t) \in \omega \times (0,T)$,
  - test functions: $\psi = \partial_x^2 \eta^\varepsilon$, $\boldsymbol{\varphi} = (-\partial_z \zeta^\varepsilon, \partial_x \zeta^\varepsilon)$, where $\zeta^\varepsilon = \partial_x \eta^\varepsilon \chi\left(\frac{z}{\eta^\varepsilon}\right)$
  - uniform estimate $\left\|\dfrac{1}{\eta^\varepsilon}\right\|_{L^\infty(0,T;L^1(\omega))} \leq \dfrac{C}{\varepsilon}$
  - "distance" estimate
    $$\left\|\frac{\varepsilon}{\eta^\varepsilon}\right\|_{L^\infty(0,T;L^\infty(\omega))} \leq D_{\min}\left(\left\|\frac{\varepsilon}{\eta^\varepsilon}\right\|_{L^\infty(0,T;L^1(\omega))}, \left\|\frac{\eta^\varepsilon}{\varepsilon}\right\|_{L^\infty(0,T;H^2(\omega))}\right)$$

Uniform estimates

on structure displacements

energy estimate:

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uniform no-contact:

- $\dfrac{\eta^\varepsilon(x,t)}{\varepsilon} \geq c \quad$ for a.e. $(x,t) \in \omega \times (0,T)$,
  - test functions: $\psi = \partial_x^2 \eta^\varepsilon$, $\boldsymbol{\varphi} = (-\partial_z \zeta^\varepsilon, \partial_x \zeta^\varepsilon)$, where $\zeta^\varepsilon = \partial_x \eta^\varepsilon \chi\left(\frac{z}{\eta^\varepsilon}\right)$
  - uniform estimate $\left\| \dfrac{1}{\eta^\varepsilon} \right\|_{L^\infty(0,T;L^1(\omega))} \leq \dfrac{C}{\varepsilon}$
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- $\|\partial_t \eta^\varepsilon\|_{L^2(0,T;H^{-1}(\omega))} \leq C\varepsilon$,
- Aubin-Lions lemma implies

$$\frac{\eta^\varepsilon}{\varepsilon} \to h \quad \text{in } C^0([0,T];C^1(\overline{\omega}))$$

Uniform estimates

on fluid velocity and pressure

change of spatial variables:

$$\begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} = \begin{pmatrix} x \\ \dfrac{z}{\eta^\varepsilon(t,x)} \end{pmatrix}, \quad \nabla_\eta^\varepsilon = \begin{pmatrix} \partial_{\hat{x}} - \hat{y}\dfrac{\partial_{\hat{x}}\eta^\varepsilon}{\eta^\varepsilon}\partial_{\hat{y}} \\ \dfrac{1}{\eta^\varepsilon}\partial_{\hat{y}} \end{pmatrix}$$

transformed fluid velocity $\hat{\boldsymbol{v}}^\varepsilon(\hat{x},\hat{y},t) = \boldsymbol{v}^\varepsilon(x,z,t)$ satisfies

$$\int_0^T \int_\Omega |\hat{\boldsymbol{v}}^\varepsilon|^2 \hat{\eta}^\varepsilon \, \mathrm{d}\hat{\boldsymbol{x}}\mathrm{d}s \leq C\varepsilon^2 \int_0^T \int_\Omega |\nabla_\eta^\varepsilon \hat{\boldsymbol{v}}^\varepsilon|^2 \hat{\eta}^\varepsilon \, \mathrm{d}\hat{\boldsymbol{x}}\mathrm{d}s \leq C\varepsilon^4 \, ,$$

which eventually yields $\dfrac{\hat{\boldsymbol{v}}^\varepsilon}{\varepsilon^2} \rightharpoonup \boldsymbol{v}$ in $L^2(0,T;L^2(\Omega))$ (on a subsequence as $\varepsilon \downarrow 0$)

Uniform estimates

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$$\int_0^T \int_\Omega |\hat{\boldsymbol{v}}^\varepsilon|^2 \hat{\eta}^\varepsilon \, \mathrm{d}\hat{\boldsymbol{x}}\mathrm{d}s \leq C\varepsilon^2 \int_0^T \int_\Omega |\nabla_\eta^\varepsilon \hat{\boldsymbol{v}}^\varepsilon|^2 \hat{\eta}^\varepsilon \, \mathrm{d}\hat{\boldsymbol{x}}\mathrm{d}s \leq C\varepsilon^4 \,,$$

which eventually yields $\dfrac{\hat{\boldsymbol{v}}^\varepsilon}{\varepsilon^2} \rightharpoonup \boldsymbol{v}$ in $L^2(0,T;L^2(\Omega))$ (on a subsequence as $\varepsilon \downarrow 0$)

pressure estimate:

$$\|p^\varepsilon\|_{H^{-1}(0,T;H^{-1}(\Omega))} \leq C$$

Reduced model

Limit bending model

rescaled FSI(ε) problem involving the pressure term:

$$
\begin{aligned}
-\varepsilon^3 \int_0^T \int_\Omega \boldsymbol{v}^\varepsilon \cdot \partial_t \boldsymbol{\varphi}\, \eta^\varepsilon \mathrm{d}\boldsymbol{x}\mathrm{d}t + \varepsilon^3 \int_0^T \int_\Omega (\boldsymbol{v}^\varepsilon \cdot \nabla_\eta) \boldsymbol{v}^\varepsilon \cdot \boldsymbol{\varphi}\, \eta^\varepsilon \mathrm{d}\boldsymbol{x}\mathrm{d}t \\
+ 2\varepsilon \int_0^T \int_\Omega \mathrm{D}_\eta(\boldsymbol{v}^\varepsilon) : \mathrm{D}_\eta(\boldsymbol{\varphi})\, \eta^\varepsilon \mathrm{d}\boldsymbol{x}\mathrm{d}t - \frac{1}{\varepsilon} \int_0^T \int_\Omega p^\varepsilon (\nabla_\eta \cdot \boldsymbol{\varphi})\, \eta^\varepsilon \mathrm{d}\boldsymbol{x}\mathrm{d}t \\
-\varepsilon^4 \rho \int_0^T \int_\omega \partial_t \eta^\varepsilon \partial_t \psi \,\mathrm{d}x\mathrm{d}t - \delta \varepsilon^{1-r} \int_0^T \int_\omega \partial_t \partial_x \eta^\varepsilon \partial_x \psi \,\mathrm{d}x\mathrm{d}t + \frac{\beta}{\varepsilon^2} \int_0^T \int_0^L \partial_x^2 \eta^\varepsilon \partial_x^2 \psi \,\mathrm{d}x\mathrm{d}t \\
= \frac{1}{\varepsilon} \int_0^T \int_\Omega \boldsymbol{f}^\varepsilon \cdot \boldsymbol{\varphi}\, \eta^\varepsilon \mathrm{d}\boldsymbol{x}\mathrm{d}t
\end{aligned}
$$

Reduced model

Limit bending model

rescaled FSI(ε) problem involving the pressure term:

$$-\varepsilon^3 \int_0^T \int_\Omega \boldsymbol{v}^\varepsilon \cdot \partial_t \boldsymbol{\varphi}\, \eta^\varepsilon \mathrm{d}\boldsymbol{x}\mathrm{d}t + \varepsilon^3 \int_0^T \int_\Omega (\boldsymbol{v}^\varepsilon \cdot \nabla_\eta)\boldsymbol{v}^\varepsilon \cdot \boldsymbol{\varphi}\, \eta^\varepsilon \mathrm{d}\boldsymbol{x}\mathrm{d}t$$
$$+ 2\varepsilon \int_0^T \int_\Omega \mathrm{D}_\eta(\boldsymbol{v}^\varepsilon) : \mathrm{D}_\eta(\boldsymbol{\varphi})\, \eta^\varepsilon \mathrm{d}\boldsymbol{x}\mathrm{d}t - \frac{1}{\varepsilon} \int_0^T \int_\Omega p^\varepsilon (\nabla_\eta \cdot \boldsymbol{\varphi})\, \eta^\varepsilon \mathrm{d}\boldsymbol{x}\mathrm{d}t$$
$$-\varepsilon^4 \rho \int_0^T \int_\omega \partial_t \eta^\varepsilon \partial_t \psi \,\mathrm{d}x\mathrm{d}t - \delta \varepsilon^{1-r} \int_0^T \int_\omega \partial_t \partial_x \eta^\varepsilon \partial_x \psi \,\mathrm{d}x\mathrm{d}t + \frac{\beta}{\varepsilon^2} \int_0^T \int_0^L \partial_x^2 \eta^\varepsilon \partial_x^2 \psi \,\mathrm{d}x\mathrm{d}t$$
$$= \frac{1}{\varepsilon} \int_0^T \int_\Omega \boldsymbol{f}^\varepsilon \cdot \boldsymbol{\varphi}\, \eta^\varepsilon \mathrm{d}\boldsymbol{x}\mathrm{d}t$$

examination of limits as $\varepsilon \downarrow 0$:

- $\displaystyle\int_0^T \int_\Omega p\, \partial_y \phi_2 \,\mathrm{d}\boldsymbol{x}\mathrm{d}t = 0\,,$
- $\displaystyle\chi_{\{r=3\}} \delta \int_0^T \int_\omega h \partial_t \partial_x^2 \psi \,\mathrm{d}x\mathrm{d}t + \beta \int_0^T \int_\omega \partial_x^2 h \partial_x^2 \psi \,\mathrm{d}x\mathrm{d}t = \int_0^T \int_\omega p\psi \,\mathrm{d}x\mathrm{d}t\,,$
- $\partial_y v_2 = 0\,,$
- $\displaystyle\int_0^T \int_\Omega \frac{1}{h} \partial_y v_1 \partial_y \varphi_1 \,\mathrm{d}\boldsymbol{x}\mathrm{d}t - \int_0^T \int_\Omega (ph\, \partial_x \varphi_1 + p \partial_x h\, \varphi_1) \mathrm{d}\boldsymbol{x}\mathrm{d}t = \int_0^T \int_\Omega h f_1 \phi_1 \,\mathrm{d}\boldsymbol{x}\mathrm{d}t$

Reduced model

Reynolds type equation

rescaled divergence-free equation:

$$0 = -\int_0^T \int_\Omega \left(v_1^\varepsilon \partial_x \varphi - \frac{\partial_x \eta^\varepsilon}{\eta^\varepsilon} v_1^\varepsilon \varphi \right) \mathrm{d}\boldsymbol{x} \mathrm{d}t + \int_0^T \int_\omega \frac{\partial_t \eta^\varepsilon}{\eta^\varepsilon} \varphi \, \mathrm{d}x \mathrm{d}t \,,$$

on the limit as $\varepsilon \downarrow 0$ and $\psi = \varphi / \eta^\varepsilon$

$$-\int_0^T \int_\Omega h v_1 \partial_x \psi \, \mathrm{d}\boldsymbol{x} \mathrm{d}t - \int_0^T \int_\omega h \partial_t \psi \, \mathrm{d}x \mathrm{d}t = 0 \,.$$

Reduced model

Reynolds type equation

rescaled divergence-free equation:

$$0 = -\int_0^T \int_\Omega \left(v_1^\varepsilon \partial_x \varphi - \frac{\partial_x \eta^\varepsilon}{\eta^\varepsilon} v_1^\varepsilon \varphi \right) \mathrm{d}\boldsymbol{x}\mathrm{d}t + \int_0^T \int_\omega \frac{\partial_t \eta^\varepsilon}{\eta^\varepsilon} \varphi \, \mathrm{d}x\mathrm{d}t \,,$$

on the limit as $\varepsilon \downarrow 0$ and $\psi = \varphi / \eta^\varepsilon$

$$-\int_0^T \int_\Omega h v_1 \partial_x \psi \, \mathrm{d}\boldsymbol{x}\mathrm{d}t - \int_0^T \int_\omega h \partial_t \psi \, \mathrm{d}x\mathrm{d}t = 0 \,.$$

since formally

$$v_1(x,y,t) = \frac{1}{2} y(y-1) h^2(x,t) \partial_x p(x,t) + h^2(x,t) F_1(x,y,t) \,,$$

where F_1 satisfies $-\partial_y^2 F_1 = f_1$, the Reynolds type equation appears

$$\int_0^T \int_\omega h^3 \left(\frac{1}{12} \partial_x p - \Phi \right) \partial_x \psi \, \mathrm{d}x\mathrm{d}t - \int_0^T \int_\omega h \partial_t \psi \, \mathrm{d}x\mathrm{d}t = 0 \,,$$

where $\Phi = \int_0^1 F_1 \mathrm{d}y$.

Reduced model

A single evolution equation

Theorem ([B. and Muha, Nonlinearity 2022])

Let $(\boldsymbol{v}^\varepsilon, p^\varepsilon, \eta^\varepsilon)$ *be global strong solutions to FSI*(ε) *problem. Then it holds:*

$$\begin{aligned} \varepsilon^{-2}\boldsymbol{v}^\varepsilon &\rightharpoonup (v_1, 0) \text{ weakly in } L^2(0,T;L^2(\Omega)),\\ p^\varepsilon &\rightharpoonup p \quad \text{weakly in } H^{-1}(0,T;L^2(\Omega)),\\ \varepsilon^{-1}\eta^\varepsilon &\to h \quad \text{strongly in } C^0([0,T];C^1(\overline{\omega})), \end{aligned}$$

where (h,p) *satisfy:*

$$\begin{aligned} \int_0^T\int_\omega h^3\left(\frac{1}{12}\partial_x p - \Phi\right)\partial_x\psi \,\mathrm{d}x\mathrm{d}t &= \int_0^T\int_\omega h\partial_t\psi\,\mathrm{d}x\mathrm{d}t,\\ \chi_{\{r=3\}}\delta\int_0^T\int_\omega h\partial_t\partial_x^2\psi\,\mathrm{d}x\mathrm{d}t + \beta\int_0^T\int_\omega \partial_x^2 h\partial_x^2\psi\,\mathrm{d}x\mathrm{d}t &= \int_0^T\int_\omega p\psi\,\mathrm{d}x\mathrm{d}t. \end{aligned}$$

Moreover, h is unique positive classical solution of

$$\partial_t h = \partial_x\left(h^3\partial_x\left(\frac{\beta}{12}\partial_x^4 h - \chi_{\{r=3\}}\frac{\delta}{12}\partial_t\partial_x^2 h - \Phi\right)\right).$$

Future perspectives

- error estimates for approximate solutions

$$\eta^\varepsilon = \varepsilon h\,,$$

$$p^\varepsilon = \beta\partial_x^4 h - \chi_{\{r=3\}}\delta\partial_t\partial_x^2 h\,,$$

$$v_1^\varepsilon = \frac{\varepsilon^2}{2}y(y-1)h^2\partial_x p^\varepsilon + \varepsilon^2 h^2 F\,, \quad v_2^\varepsilon = 0\,,$$

Future perspectives

- error estimates for approximate solutions

$$\begin{aligned} \eta^\varepsilon &= \varepsilon h\,, \\ \mathsf{p}^\varepsilon &= \beta\partial_x^4 h - \chi_{\{r=3\}}\delta\partial_t\partial_x^2 h\,, \\ \mathsf{v}_1^\varepsilon &= \frac{\varepsilon^2}{2} y(y-1)h^2\partial_x \mathsf{p}^\varepsilon + \varepsilon^2 h^2 F\,, \quad \mathsf{v}_2^\varepsilon = 0\,, \end{aligned}$$

- Navier slip boundary conditions

$$\begin{aligned} (\boldsymbol{v}(x,\eta(x,t),t) - (0,\partial_t\eta(x,t)))\cdot\boldsymbol{n}^\eta &= 0\,, \quad \omega\times(0,\infty)\,, \\ \boldsymbol{v}(x,0,t)\cdot\boldsymbol{e}_z &= 0\,, \quad \omega\times(0,\infty)\,, \\ (2\mu_f\,\mathrm{D}(\boldsymbol{v})\cdot\boldsymbol{n}^\eta + \gamma_\eta\boldsymbol{v})\,(x,\eta(x,t),t)\cdot\boldsymbol{\tau}^\eta &= 0\,, \quad \omega\times(0,\infty)\,, \\ (2\mu_f\,\mathrm{D}(\boldsymbol{v})\cdot\boldsymbol{e}_z + \gamma_0\boldsymbol{v})\,(x,0,t)\cdot\boldsymbol{e}_x &= 0\,, \quad \omega\times(0,\infty)\,. \end{aligned}$$