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The Bloch and Fermi varieties for Periodic Operators

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1. Schrödinger and elliptic operators
   — continuous, discrete, in between

   Periodic operators $\quad -\Delta u - k^2 u = \cancel{f}\ 0$

2. Floquet transform $\quad (-\Delta - k^2)u = \cancel{f}\ 0 \qquad -\Delta u = k^2 u$
3. Decomposable operators $\longrightarrow$ periodic operators
4. Dispersion function, Bloch variety, Fermi variety
5. Consequences:
   - Band structure
   - Resolution of rel.
   - Density of states
   - Wannier states
6. (Ir)reducibility of the Fermi (& Bloch) varieties
7. A few words about quasi-periodic operators and/or ergodic ones.

Ideas to follow from K. Cherednichenko's Lecture

- Solutions that can be conceived as "waves" $\rightsquigarrow e^{i(k \cdot x - \omega t)}$

  $\xrightarrow{\text{per}}$ now k only makes sense in discrete variable

- What's the <u>energy</u> of a plane wave?

- "Inhomogeneous media always possess strong dispersion"

- Length scale: wavelength on par with the period

  — near origin: "usual" homogenization $\to$ <u>core</u>

  ~ extreme coefficients: rescale around origin & retain the band-gap structure

  $L/\lambda \to 0 \quad (Lk \to 0)$

  $L/\lambda \to \infty \quad (Lk \to \infty)$

- Eigenvalues for a bounded region

  $\rightsquigarrow$ now diff. bdry conditions dep. on $z = e^{ik}$.

$$\varepsilon(x) e^{i(kx - \omega t)} = r(x) e^{i\theta(x)} e^{i(kn - \omega t)}$$

Real part: $\quad r(x) \cos(\theta(x) + kn - \omega t)$

Overview

Schrödinger or elliptic spectral problems in $\mathbb{R}^3$

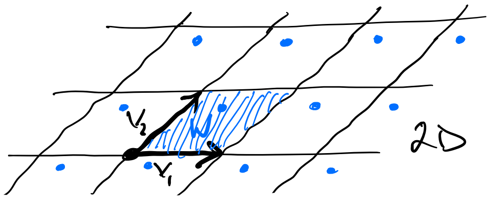
$$-\Delta u + V(x)\,u =: \boxed{Hu = Eu} \qquad E = \text{Energy}$$

$= \text{eigenvalue of Schr. Op.}$

$$\nabla\cdot\tau(x)\nabla u + \omega^2\sigma(x)\,u = 0$$

Periodic: V, τ, σ satisfy $V(x+v_j) = V(x)$

for three independent vectors v_1, v_2, v_3.



2D picture

Floquet-Bloch (quasi-periodic) fields:

$$u(x+v_j) =: \boxed{(S_j u)(x) = z_j\,u(x) = e^{ik_j}u(x)}$$

$k = \langle k_1, k_2, k_3\rangle$ = quasi-momentum = Bloch wavevector

$z_j = e^{ik_j}$ = eigenvalues of the shifts by v_j

H periodic $\iff HS_j = S_jH \sim H$ commutes w/ shifts

$\Longrightarrow$ H & S_j are "simultaneously diagonalizable"

$\Longrightarrow$ H is decomposed on simultaneous eigenstates of H and S_j

Dispersion function $D(z,E) = 0 \iff \exists\, u : Hu = Eu,\ S_ju = z_ju$

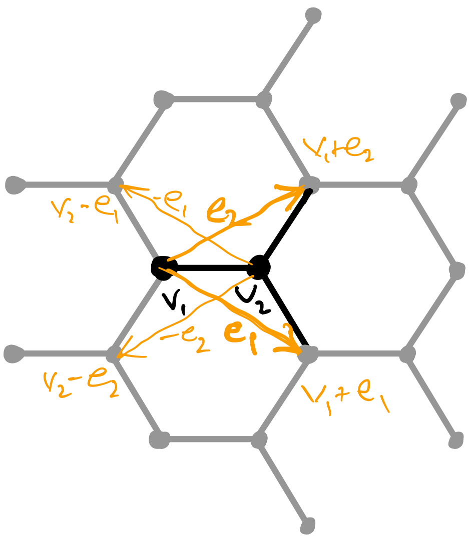
Bloch variety $\mathcal{B} = \{(z,E) \in (\mathbb{C}^*)^d \times \mathbb{C} : D(z,E) = 0\}$

Fermi variety $\Phi_E = \{ z \in (\mathbb{C}^*)^d : D(z,E) = 0\}$

Example

Discrete graphene model

$$(Au)(v_1) = u(v_2) + u(v_2 - e_1) + u(v_2 - e_2)$$
$$(Au)(v_2) = u(v_1) + u(v_1 + e_1) + u(v_1 + e_1)$$



$\rightsquigarrow$ extended periodically

Now let u be such that

$$(1) \quad \begin{cases} u(v+e_1) = z_1 u(v) \\ u(v+e_2) = z_2 u(v) \end{cases} \quad \forall\ v$$

$$(2) \quad Au = Eu$$

$$(1) \implies \begin{bmatrix} (Au)(v_1) \\ (Au)(v_2) \end{bmatrix} = \begin{bmatrix} 0 & 1 + z_1^{-1} + z_2^{-1} \\ 1 + z_1 + z_2 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

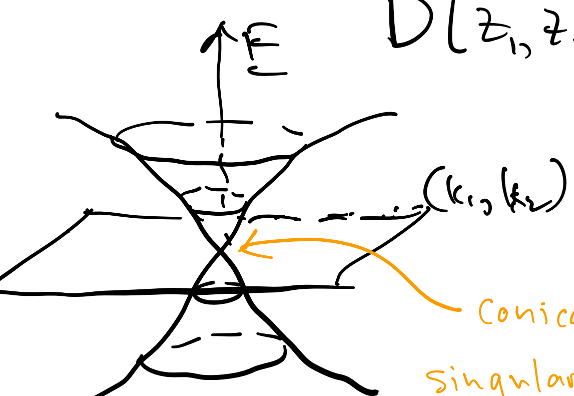
$$(2) \implies \det \begin{bmatrix} -E & 1 + z_1^{-1} + z_2^{-1} \\ 1 + z_1 + z_2 & -E \end{bmatrix} = 0$$

$=$

$$D(z_1, z_2, E) = E^2 - (1 + z_1 + z_2)(1 + z_1^{-1} + z_2^{-1})$$

$$= E^2 - 3 - 2(\cos k_1 + \cos k_2 + \cos(k_1 - k_2))$$

$$\overset{k \text{ real}}{=} E^2 - \left|1 + e^{ik_1} + e^{ik_2}\right|^2$$



Conical (Dirac) singularity at $E = 0$.

1 Schrödinger Operators : continuous, discrete, and in between

In $\mathbb{R}^d$: $\quad H = -(\nabla - iA)^2 + V$

$V =$ electric potential, so electric field $E = -\nabla V$

$A =$ magnetic potential, so magnetic flux fld $B = \nabla \times A$

Quantum mechanical State Space $= L^2(\mathbb{R}^d)$

H is the generator of the time dynamics :

$$i\hbar \frac{d}{dt}\psi = H\psi, \qquad \psi \in L^2(\mathbb{R}^d)$$

Idea of Fourier / spectral analysis : The harmonic solutions act as a sort of basis for all solutions :

$$\psi(t) = \psi(0)e^{-i\omega t} ; \qquad E = \omega\hbar$$

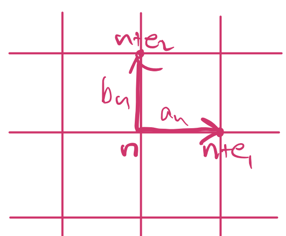
$$\Longrightarrow \qquad H\psi = E\psi \qquad \underline{\text{Spectral problem}} \text{ for } H$$

Discrete Schrödinger operator in $\mathbb{Z}^d$ "tight-binding approximation" :

$$[(\Delta + V)u](n) = -2d\,u(n) + \sum_{|m-n|=1} u(m) + V(n)u(n)$$

Harper model for discrete magnetic Schrödinger operator :

A magnetic potential assigns $\phi(e)$ to an oriented edge $e = (v_1, v_2)$
$\phi(v_1, v_2) = -\phi(v_2, v_1)$. In $\mathbb{Z}^2$, let $a_n = \phi(n, n+e_1)$
and $b_n = \phi(n, n+e_2)$



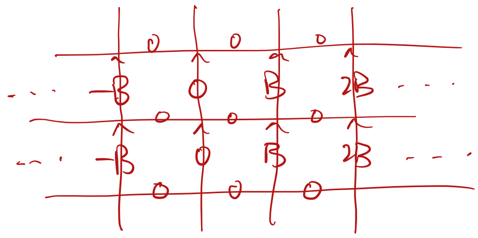
$n = (n_1, n_2)$
$e_1 = (1, 0)$
$e_2 = (0, 1)$

$$(\Delta_\phi u)(n) = -4u(n) + e^{ia_n}u(n+e_1) + e^{-ia_n}u(n-e_1) + e^{ib_n}u(n+e_2) + e^{-ib_n}u(n-e_2)$$

Special case of constant magnetic field B perpendicular to the lattice:

Take $a_n = 0$, $b_n = n_1 B$

Flux thru square $= B$

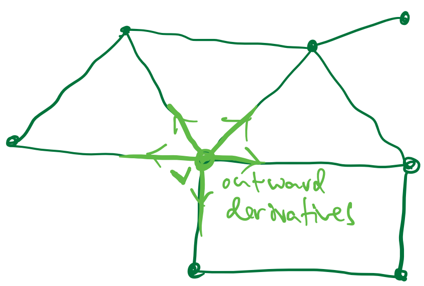


$$(\Delta_B u)(n) = -4u(n) + u(n+e_1) + u(n-e_1) + e^{in_1 B}u(n+e_2) + e^{-in_1 B}u(n-e_2)$$

Notice that this operator has period 1 in n_2 but not necessarily in n_1. If $B = 2\pi\frac{p}{q} \in 2\pi\mathbb{Q}$, then Δ_B has period q in n_1.

If $B \in \mathbb{R}\setminus 2\pi\mathbb{Q}$, then Δ_B is quasi-periodic in n_1.

Schrödinger operators on <u>metric graphs</u> : "quantum graphs"



On each <u>edge</u>: $-\left(\frac{d}{dx} - iA\right)^2 + V$

<u>vertex</u> conditions for domain $\subset \bigoplus_e H^2(e)$

$$\sum_{v\in e} f_e'(v) = \alpha_v f(v)$$

$\boxed{2}$ <u>The Floquet Transform</u> [Gel'fand transform / z-transform]

= Fourier Transform w/r to the shift (translation) group $\mathbb{Z}^d$.

- <u>$\mathbb{Z}^d$ as a translational symmetry group</u> on a measure space $(X, \mathcal{B}, \nu)$

action: $\mathbb{Z}^d \times X \longrightarrow X$

$(n, x) \longmapsto x + n$ ⟵ notation for shift-group action

Free action: $\forall\, x \in X$, $\mathbb{Z}^d \to X :: n \mapsto x+n$ is injective

ie "orbits look like $\mathbb{Z}^d$"

Preserves μ (translational symmetry group):

$$\forall\, S \in \mathcal{B} \quad \forall\, n \in \mathbb{Z}^d, \quad \nu(S+n) = \nu(S)$$

Fundamental domain W = set of one representative point per orbit

Unique representation of $x \in X$:

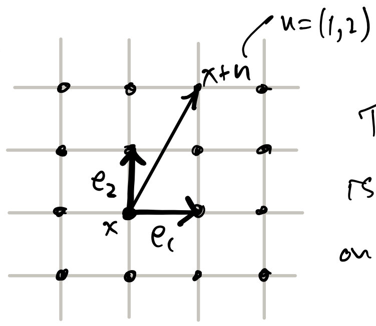
$$x = y + n \quad (y \in W,\ n \in \mathbb{Z}^d)$$

Point of view: $y \in W$ is the local variable

$n \in \mathbb{Z}^d$ is the global variable

Example 1 $X = \mathbb{Z}^2$ with the counting measure; the action of $\mathbb{Z}^d$ is just the addition within $\mathbb{Z}^d$.

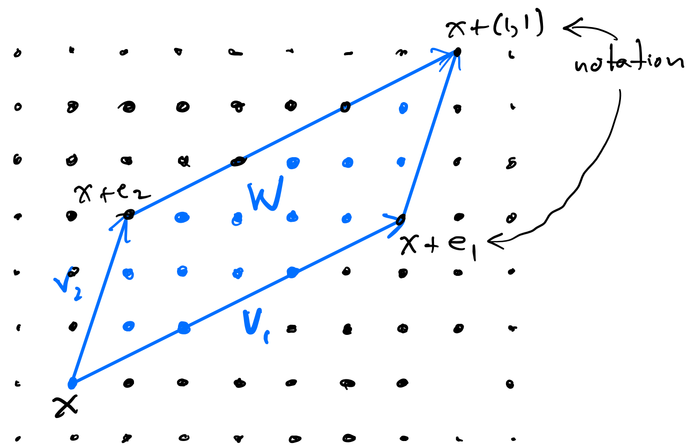
(1)



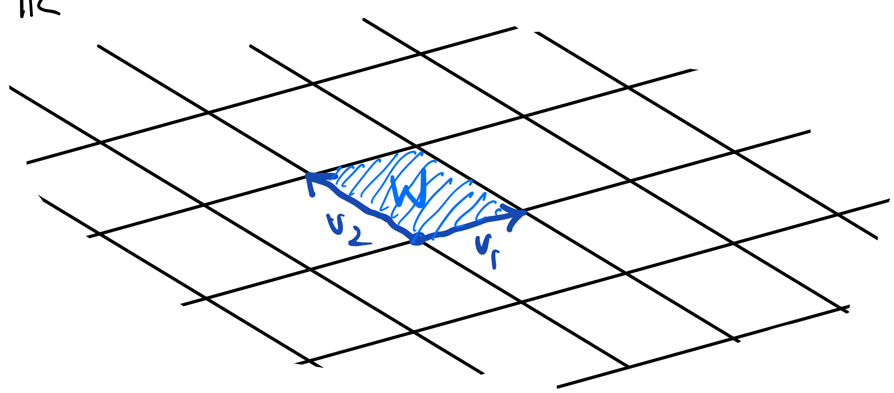
The measure is supported on the vertices.

$v_1 = e_1$, $v_2 = e_2$

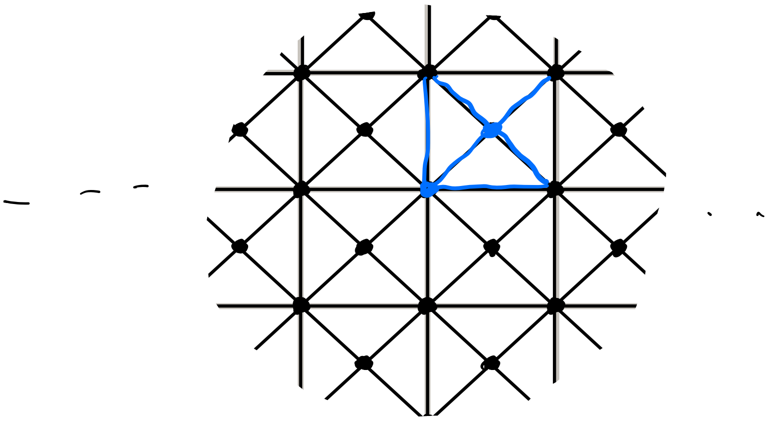
(2)



Example 2 $X = \mathbb{R}^2$



Example 3 A periodic metric (or discrete) graph



W has two vertices and six edges.

- Induced action on functions

Let $\mathbb{Z}^d$ act freely on X, preserving the measure μ.

For a function $f : X \to Y$ and $n \in \mathbb{Z}^d$,

$$(n \cdot f)(x) := f(x+n)$$

When Y is a Hilbert space $\mathcal{H}$, $\mathbb{Z}^d$ becomes a unitary group of operators on $L^2(X, \mathcal{H})$.

- Characters, weights, and Floquet modes

The **dual group** of $\mathbb{Z}^d$ is the group of **characters** of $\mathbb{Z}^d$, i.e., all homomorphisms from $\mathbb{Z}^d$ to $\mathbb{C}^*$. A character $\phi : \mathbb{Z}^d \to \mathbb{C}^*$ is determined by the d numbers $\phi(e_i) = z_i$, so the dual of $\mathbb{Z}^d$ is identified with $(\mathbb{C}^*)^d$.

For $z = (z_1, \ldots, z_d) \in (\mathbb{C}^*)^d$, the associated character is

$$\phi_z : \mathbb{Z}^d \longrightarrow \mathbb{C}^* :: n \longmapsto z^n = \prod_{j=1}^{d} z_j^{n_j}$$

$z = (z_1, \ldots, z_d)$ is the **weight vector**,

$z_1, \ldots z_d$ are the **weights**, or **Floquet multipliers**.

The **unitary characters** have $|z_j| = 1 \ \forall j$:

$$\mathbb{T}^d = \{ z \in (\mathbb{C}^*)^d : |z_j| = 1 \ \forall j \}.$$

- The **Raison d'être** of characters : They are all the eigenfunctions of the $\mathbb{Z}^d$ action on $\mathcal{F}(\mathbb{Z}^d, \mathbb{C})$:

$$(n \cdot \phi_z)(m) = \phi_z(m+n) = \phi_z(n)\,\phi_z(m) = z^n \phi_z(m)$$

and the weights $\{z_j\}$ are the eigenvalues of the action of $\{e_j\}$.

These eigenfunctions can be extended to $\mathcal{F}(X, \mathbb{C})$:

Given $f : W \to \mathbb{C}$ (fcn on a fundamental domain),

For $x \in X$, write $x = y + n$ $[y \in W,\ n \in \mathbb{Z}^d]$,

$$f(x) = f(y+n) = \phi_z(n) f(y) = z^n f(y)$$

Eigenspace for weight $z \in (\mathbb{C}^*)^d$:

$$E_z = \{ f : X \to \mathbb{C} \mid f(x+n) = z^n f(x) \ \forall x \in X \ \forall n \in \mathbb{Z}^d \}$$

These are the Floquet-Bloch modes, or quasi-periodic or pseudo-periodic functions.

If $z \in \mathbb{T}^d$ $(|z_j| = 1 \ \forall j)$, then it is a Bloch mode: Put

$$z_1 = e^{ik_1}, \ \ldots, \ z_d = e^{ik_d}$$

So $f(x+n) = f(x) e^{ik \cdot n} = f(x) e^{i(k_1 n_1 + \cdots + k_d n_d)}$

$\rightsquigarrow$ **Bloch wave**

with quasi-momentum, or wavevector $k = (k_1 \ldots k_d)$.

Let $\varepsilon(z, x)$ be a distinguished Floquet mode:

$$\varepsilon(z, \cdot) \in E_z$$

so that each $f \in E_z$ can be written as

$$f(x) = \tilde{f}(x) \varepsilon(z, x), \quad \tilde{f} \in E_1$$

where $\tilde{f}$ is the **periodic factor** of f.

Examples 1 and 3. Take $\mathcal{E}(z, y) = 1$ for $y \in W$,

so $\mathcal{E}(z, y+n) = z^n = e^{ik\cdot n}$

Example 2. For $x \in \mathbb{R}^d$, put $x = \sum_{j=1}^{d} x^j v_j$

The exponential function

$$\mathcal{E}(z, x) = z_1^{x^1} \cdots z_d^{x^d} = e^{i(k_1 x^1 + \cdots + k_d x^d)}$$

interpolates the character ϕ_z across all orbits.

Example 4. For a metric graph, take an embedding into $\mathbb{Z}^d$ and restrict the $\mathcal{E}(z,x)$ to the graph.

- ## The Floquet Transform

For a function $f : X \to Y$,

$$\hat{f}(z,x) = \sum_{n\in\mathbb{Z}^d} f(x+n)\, z^{-n}, \quad z \in (\mathbb{C}^*)^d, \ x \in X$$

$\left[\text{i.e., } \hat{f}(z,\cdot) = \sum_{n\in\mathbb{Z}^d} (n\cdot f)\, \phi_z(-n)\right]$

Raison d'être : $\hat{f}(z, x+n) = z^n \hat{f}(z,x)$, ie

$$\hat{f}(z,\cdot) \in E_z$$

and $$f(x) = \int_{\mathbb{T}^d} \hat{f}(z,x)\, d\tilde{V}(z)$$ $\left[d\tilde{V} = \frac{dV}{(2\pi)^d} \text{ on the torus } \mathbb{T}^d \right]$

$$= \frac{1}{(2\pi)^d} \int_0^{2\pi} \cdots \int_0^{2\pi} \hat{f}(e^{ik}, x)\, dk_1 \ldots dk_d$$

$$= \frac{1}{(2\pi)^d} \int_0^{2\pi} \cdots \int_0^{2\pi} \tilde{f}(e^{ik}, x) \underbrace{\varepsilon(k,x)}_{\text{Fourier kernel}} dk_1 \ldots dk_d$$

$\rightsquigarrow$ Fourier inversion formula

= decomposition of suitable f into eigenfunctions of the unitary $\mathbb{Z}^d$ action.

<u>Unitarity</u>: $\mathcal{U}: f \longmapsto \hat{f}$ is an L^2 isometric isomorphism

$$\mathcal{U}: \underbrace{L^2(X)}_{\text{Note } L^2(X) = L^2(\mathbb{Z}^d, L^2(W))} \longrightarrow L^2(\mathbb{T}^d, L^2(W)) = L^2(\mathbb{T}^2) \otimes L^2(W)$$

$$=: \int_{\mathbb{T}^d}^{\oplus} L^2(W)\, d\tilde{V}$$

Since $|\varepsilon(z,x)| = 1$ for $z \in \mathbb{T}^d$, $\hat{f} \longrightarrow \tilde{f}$ is unitary, so

$$\tilde{\mathcal{U}}: L^2(X) \longrightarrow L^2(\mathbb{T}^d, L^2(X/\mathbb{Z}^d)) = L^2(\mathbb{T}^d) \otimes L^2(X/\mathbb{Z}^d)$$

is also unitary (the "usual" Fourier transform).

$\boxed{3}$ Periodic Operators

An operator A in $L^2(X)$ is **d-periodic** if it commutes with the translational symmetry group $\mathbb{Z}^d$:

$$A(n \cdot f) = n \cdot Af$$

The Floquet–Bloch theorem says that A is decomposed, or (block) diagonalized by the Floquet transform, i.e., $\exists$ operators $\hat{A}(z)$ in $L^2(W)$ s.th.

$$(Af)^\wedge(z) = \hat{A}(z)\hat{f}(z)$$

i.e., $$(Af)(x) = \int_{\mathbb{T}^d}^{\oplus} \hat{A}(z)\,\hat{f}(z,x)\,d\tilde{V}(z)$$

Decomposable Operators (Reed/Simon vol. IV)

Direct integral of (identical) Hilbert spaces $\mathcal{H}'$ over $(M, \mathfrak{A}, \mu)$

$$\mathcal{H} = \int_M^{\oplus} \mathcal{H}'\,d\mu := L^2(M, \mathcal{H}') \cong L^2(M, \mathbb{C}) \otimes_{\mathbb{C}} \mathcal{H}'$$

$f \in \mathcal{H}$ is of the form $f = \int_M^{\oplus} f(m)\,d\mu$

A bounded decomposable operator in $\mathcal{H}$ is defined through a direct integral of operators

$$A = \int_M^{\oplus} A(m)\,d\mu \in L^\infty(M, \mathcal{L}(\mathcal{H}'))$$

by $$Af = \int_M^{\oplus} A(m)\,f(m)\,d\mu$$

Fundamental Theorem on Decomposable Operators

An operator A in $\mathcal{H} = \int_M^{\oplus} \mathcal{H}' \, d\mu$ is decomposable if and only if A commutes with all decomposable operators of the form $\int_M^{\oplus} \lambda(m) I \, d\mu$, where $\lambda \in L^\infty(M, \mathbb{C})$.

* This can be extended to unbounded operators by applying it to the resolvent $(E-A)^{-1}$. The details are technical and can be found in Reed/Simon.

* Proof that periodic operators are decomposable on $\mathbb{T}^d$:

- Suppose A commutes with $\mathbb{Z}^d$ acting in $L^2(X)$.
- Fourier transform is unitary:

$$\tilde{U} : L^2(X) \longrightarrow \int_{\mathbb{T}^d}^{\oplus} L^2(X/\mathbb{Z}^d) \, d\tilde{V}$$

- $\tilde{A} = \tilde{U} A \tilde{U}^{-1}$ commutes with $\tilde{U} \mathbb{Z}^d \tilde{U}^{-1}$

For $n \in \mathbb{Z}^d$

$$\tilde{U} n \tilde{U}^{-1} = \int_{\mathbb{T}^d}^{\oplus} z^n I \, d\tilde{V} = \int_0^{2\pi} \cdots \int_0^{2\pi} e^{ik\cdot n} I \, dk_1 \cdots dk_d$$

- By completeness of $\{e^{ikn}\}_{n\in\mathbb{Z}^d}$ on $\mathbb{T}^d$,

  $\tilde{A}$ commutes with all scalar multiplier operators

$$\int_{\mathbb{T}^d}^{\oplus} \lambda(k_1,\dots k_d)\, I \, dk_1 \cdots dk_d$$

- By the Fundamental Theorem, $\tilde{A}$ is decomposable

$$\tilde{A} = \int_{\mathbb{T}^d}^{\oplus} \tilde{A}(z)\, d\tilde{V}$$

* Proof of Decomposition Theorem

(1) $$\int_M^{\oplus} A(m)\, d\mu \int_M^{\oplus} \lambda(m) I\, d\mu \int_M^{\oplus} \phi(m)\, d\mu = \int_M^{\oplus} A(m)(\lambda(m)\phi(m))\, d\mu$$

$$= \int_M^{\oplus} \lambda(m)(A(m)\phi(m))\, d\mu = \int_M^{\oplus} \lambda(m) I\, d\mu \int_M^{\oplus} A(m)\, d\mu \int_M^{\oplus} \phi(m)\, d\mu$$

(2)

Sps A commutes with all operators $fI \in L^2(M, \mathcal{L}(\mathcal{H}'))$ where $f \in L^2(M, \mathbb{C})$.

Define $\hat{A}(m) \in \mathcal{L}(\mathcal{H}')$ by $\hat{A}(m)\psi := [A(1\psi)](m) \quad \forall\ \psi \in \mathcal{H}'$.

Then for all $f \in L^2(M,\mathbb{C})$, $\psi \in \mathcal{H}'$,

$$A(f\psi) = A\, fI\, (1\psi) = (fI) A\, (1\psi) = f\, A(1\psi)$$

So $[A(f\psi)](m) = f(m)\hat{A}(m)\psi = \hat{A}(m)[f(m)\psi]$

Since A coincides with $\int_M^{\oplus} \hat{A}(m)\,d\mu$ on the dense subset of $\mathcal{H}$ consisting of elements of the form $f\psi$, we obtain

$$A = \int_M^{\oplus} \hat{A}(m)\,d\mu.$$

Decomposition of Spectrum

Let $A = \int_M^{\oplus} A(m)\,d\mu$.

Then $\sigma(A) = \bigcup_{m\in M} \sigma(A(m))$

For a periodic operator A,

$$\sigma(A) = \bigcup_{\substack{k\in\mathbb{R}^d \\ (\mathrm{mod}\ 2\pi)}} \underbrace{\sigma(A(k))} \quad \longleftarrow \quad !!$$

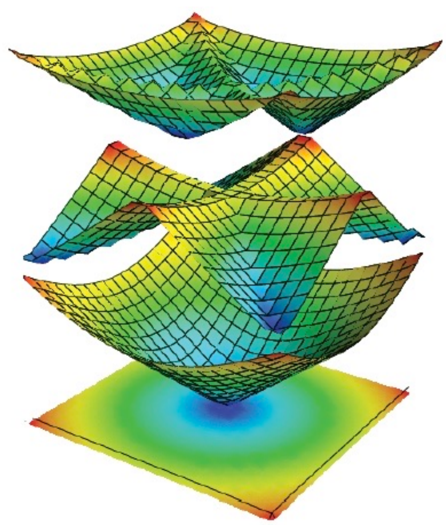
$$\boxed{\begin{array}{l} E\in\sigma(A(k)) \iff D(k,E)=0 \\ E\in\sigma(A) \iff D(k,E)=0 \text{ for some } k\in\mathbb{T}^d \end{array}}$$

5 Consequences

- **Spectral band** structure

$$D(k, E_j(k)) = 0$$

where $E_j(k)$ are implicitly defined energy vs momentum functions



P. Kuchment 2016

In graphene example, there are just two bands

$$E_{1,2}(k) = \text{eigenvalues of } \begin{bmatrix} 0 & 1+e^{-ik_1}+e^{-ik_2} \\ 1+e^{ik_1}+e^{ik_2} & 0 \end{bmatrix}.$$

For PDE and metric graphs, there are infinitely many band functions

$$\{E_j(k)\}_{j\in\mathbb{N}}$$

- <u>Spectral resolution</u>: $\boxed{\text{Bloch's Theorem}}$

Let $\phi_j(k,x)$ be a Bloch eigenstate corresponding to $E_j(k)$.

For any $f \in L^2(X)$, for periodic op. A

$$\hat{f}(k,x) = \sum_{j=1}^{\infty} a_j(k)\,\phi_j(k,x)$$

$$\Longrightarrow \quad f(x) = \int_{\mathbb{T}^d} \sum_{j=1}^{\infty} a_j(k)\,\phi_j(k,x)\,d\tilde{V}(k)$$

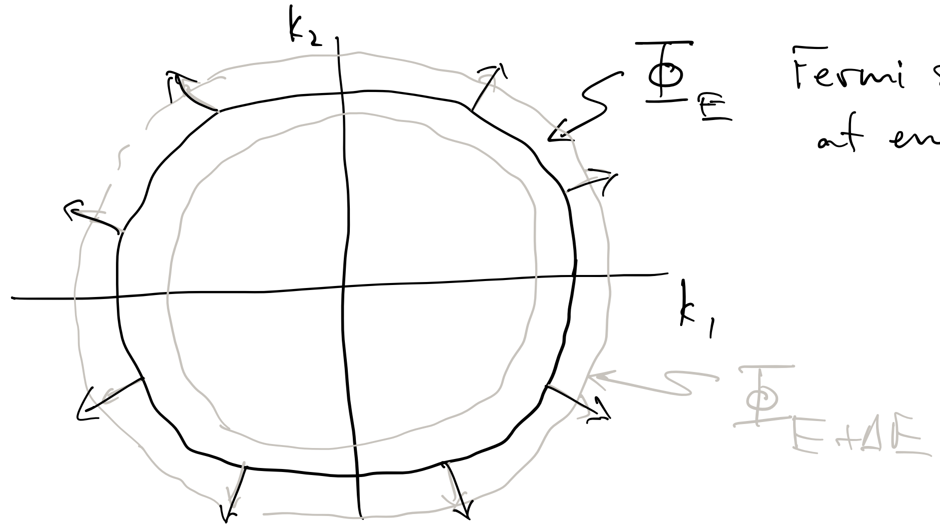
$$= \int_{B_A} a(p)\,\phi(p,x)\,d\tilde{V}(k(p)) \qquad [p=(k,E)\in B_A]$$

$$(Af)(x) = \int_{B_A} E(p)\,a(p)\,\phi(p,x)\,d\tilde{V}(k(p))$$

- <u>Density of states</u>

$$\text{DoS at energy } E = \int_{\Phi_E} \left| \partial E / \partial(k_1, k_2) \right|^{-1} d\tilde{V}(k)$$

Fix E :



- Wannier States

A state formed by a single spectral band [Fix j]

$$w(x) = \int_{\mathbb{T}^d} \phi_j(k,x)\, d\tilde{V}(k)$$

Shifts: $w(x+n) = \int_{\mathbb{T}^d} e^{ik\cdot n} \phi_j(k,x)\, d\tilde{V}(k) \quad (n \in \mathbb{Z}^d)$

$\Rightarrow \{ w(\cdot + n) \}_{n \in \mathbb{Z}^d}$ are dense in $\mathcal{H}^j$

projection to j^{th} band

Question about Wannier states: Are they exponentially localized? [See Nenciu 1983, Kuchment 2009]

$\hookrightarrow$ time-reversal symmetry & isolated band

$\Longrightarrow$ exponential decay

$\boxed{6}$ (Ir)reducibility of the Fermi or Bloch varieties

If a factorisation holds

$$D(z, E) = D_1(z, E) D_2(z, E)$$

where D_1 & D_2 are polynomial (resp. analytic) in z, then the Fermi variety Φ_E is reducible:

$$\Phi_{A,E} = \Phi^1_{A,E} \cup \Phi^2_{A,E} .$$

If D_1 & D_2 are polynomial (analytic / meromorphic) in E also, then the Bloch variety is reducible:

$$\mathcal{B}_A = \mathcal{B}^1_A \cup \mathcal{B}^2_A$$

Consequence of (ir)reducibility [Kuchment/Vainberg 2006]

Solve $(A-E)u = f$ with

- f compact support
- $u \in L^2(X)$
- $E \in \sigma(A)$

If such u exists, then a local forcing will produce a non-radiating response at an energy within the radiation continuum.

NOT typically possible on physical grounds!

Quantum graph case: $\hat{A}(z)$ is a Laurent polynomial. (if it has finite range)

$$(\hat{A}(z) - E)\,\hat{u}(z) = \hat{f}(z)$$

Irreducible: $\hat{u}(z) = \dfrac{R_E(z)}{\cancel{D(z,E)}}\cancel{\hat{f}(z)}$ Make $\hat{f}$ cancel roots of $D(z,E)$ on $\mathbb{T}^d$

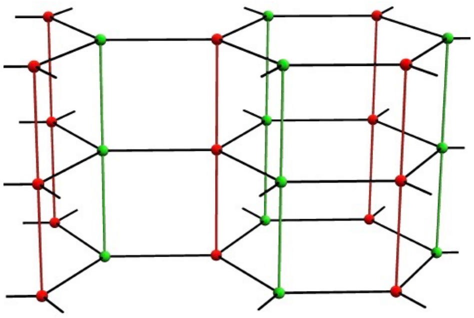
$E \in \sigma(A) \Rightarrow \Phi_{A,E} \cap \mathbb{T}^d \neq \{\}$

$\Rightarrow \hat{u}$ is polynomial $\Rightarrow u$ compact support

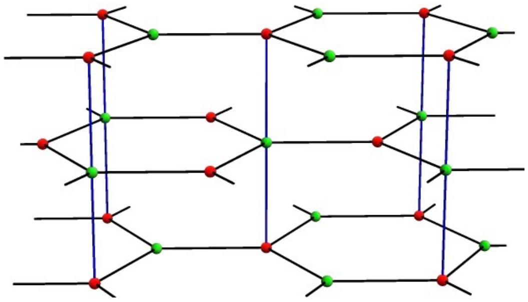
Reducible: $\hat{u}(z) = \dfrac{R_E(z)}{\underbrace{D_1(z,E)}_{\Phi^1_{A,E} \cap \mathbb{T}^d = \{\}}\,\underbrace{\cancel{D_2(z,E)}}_{\Phi^2_{A,E} \cap \mathbb{T}^d \neq \{\}}}\cancel{\hat{f}(z)}$ Make $\hat{f}$ cancel D_2.

$\Rightarrow \hat{u}$ is rational $\Rightarrow u \in L^2$ but not compact support

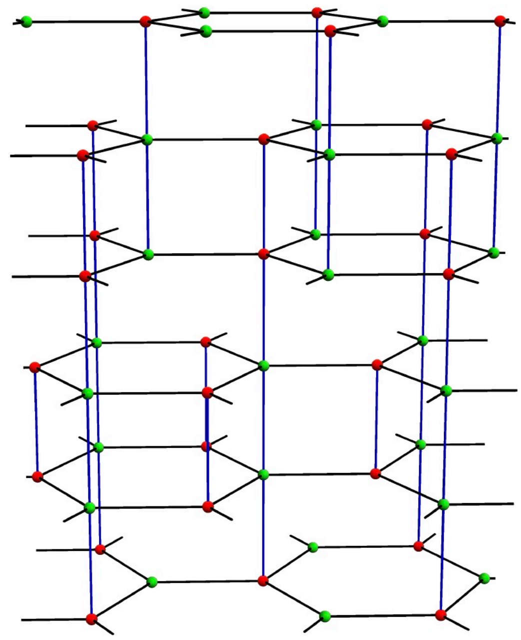
Multi-layered graphene (quantum graph)



AA-stacked

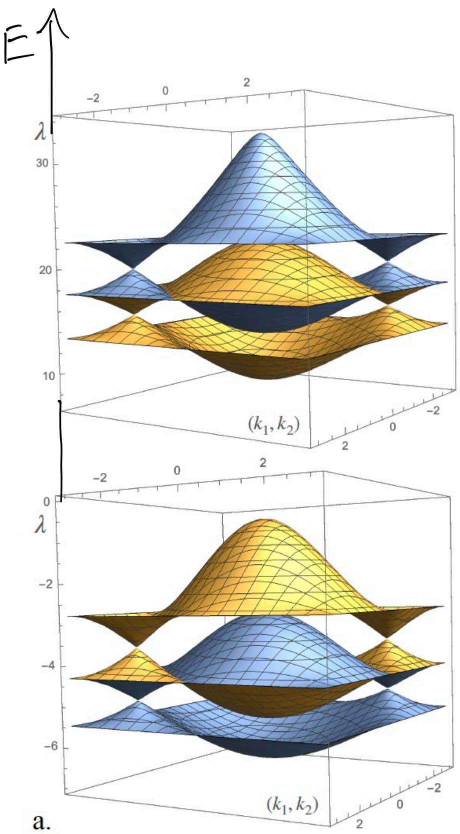


AB-stacked



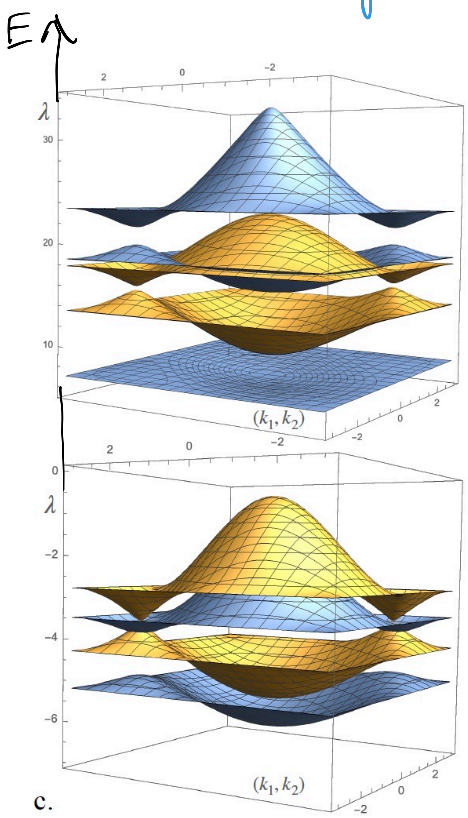
Multiple stacking

Real Bloch surface, AA-stacked (two layers)



a.

Dirac points



c.

No Dirac points because the two connecting edges are different

The Fermi variety is reducible:

$$D(z,E) = D_1(z,E)\, D_2(z,E)$$

Dirac points:

Fefferman-Weinstein 2012

Berkolaiko-Comech 2018

[Fisher, Li, Shipman CommMathPhys 2021]

7. Magnetic operators

Discrete magnetic Laplacian on $\mathbb{Z}^2$:

Harper model : For $n=(n_1,n_2)\in\mathbb{Z}^2$,

$$(Au)(n) = u(n+e_1)+u(n-e_1)+e^{in_1B}u(n+e_2)+e^{-in_1B}u(n-e_2)$$

If $B=2\pi\frac{p}{q}$, A has periods 1 and q.

If $B\notin 2\pi\mathbb{Q}$, A is quasi-periodic ; it is not periodic in n_2.

Floquet transform $u(n_1,n_2) \xrightarrow{\mathcal{U}} \hat{u}(z_1,z_2)$

$$\hat{A}(z_1,z_2)\hat{u} = (z_1+z_1^{-1})\hat{u} + z_2\hat{u}(e^{iB}z_1,z_2) + z_2^{-1}\hat{u}(e^{-iB}z_1,z_2)$$