Zero-range models with internal structure in the analysis of strongly inhomogeneous media

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Zero-radius potentials with internal structure

- In many models, an explicit solution can be obtained in a very limited number of special cases (e.g., by separation of variables).
- This deficit has led physicists (E. Fermi, 1934) to the idea to replace potentials with some boundary condition at a point of three-dimensional space, i.e., a *zero-range potential*.
- Rigorously: initiated by Berezin, Faddeev (1961). Shown: the model Hamiltonians are self-adjoint extensions of a Laplacian, restricted to the set of $W^{2,2}$ such that $u(x) = 0$ in $\mathbb{R}^3$. Further development: see the monograph by Albeverio, Kurasov (2000) and references there.
- Drawbacks: spherically symmetric scatterers only. If a more involved structure, the complexity of the model blows up, eliminating the main selling point, i.e., explicit solvability.

Internal structure (B.S. Pavlov, 1980s)

The idea: $A_0 = -\Delta$ restricted to the set of $W^{2,2}$ functions vanishing in a vicinity of a fixed point in $\mathbb{R}^3$, precisely as in Berezin, Faddeev.
A twist: instead of von Neumann extensions, consider the so-called *out of space* extensions, i.e., extensions to self-adjoint operators in a larger Hilbert space (the theory developed by: Neumark, Krasnoselskii, Strauss, 1940s–1970s).
Alongside $H = L^2(\mathbb{R}^3)$, consider an *internal* Hilbert space E (commonly: finite-dimensional), a self-adjoint operator A in it. Let ϕ be its generating vector and consider the restriction A_ϕ of A (non-densely defined) to the space

$$\mathrm{dom}\, A_\phi := \{(A - \mathrm{i})^{-1}\psi : \psi \in E, \langle \phi, \psi \rangle = 0\}.$$

This leads to the symmetric operator $\mathcal{A}_0$ on the Hilbert space $H \oplus E$, defined as $A_0 \oplus A_\phi$ on the domain

$$\mathrm{dom}\, \mathcal{A}_0 := \left\{ \begin{pmatrix} f \\ v \end{pmatrix} : f \in \mathrm{dom}\, A_0, v \in \mathrm{dom}\, A_\phi \right\}.$$

Then $\mathcal{A}_0$: symmetric, non-densely defined, equal deficiency indices. Self-adjoint extensions $\mathcal{A}$: interested in those which non-trivially couple the spaces H and E. I.e., feed the boundary data at x_0 of a function $f \in W^{2,2}(\mathbb{R}^3)$ to A in E (the operator of the "internal structure").

Kuchment's example

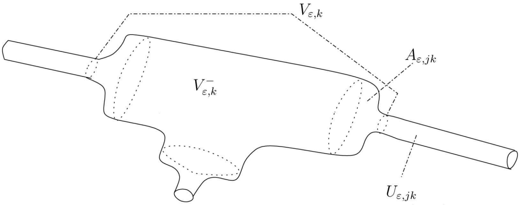

Figure: A "thin" graph-like structure.

The question: asymptotics of spectra of operator sequences associated with domains "shrinking" as $\varepsilon \to 0$ to a metric graph embedded into $\mathbb{R}^d$.
Assume: the rate of shrinking obeys

$$\frac{\mathrm{vol}(V^{\varepsilon}_{\mathrm{vertex}})}{\mathrm{vol}(V^{\varepsilon}_{\mathrm{edge}})} \to \alpha > 0, \quad \varepsilon \to 0. \tag{1}$$

Kuchment's example (cont.)

Then: the spectra of the corresponding Laplacian operators with Neumann BC converge to the spectrum of the following model (in the particular case of the last slide!).

Let $H_{\text{eff}} = H \oplus \mathbb{C}^1$, $H = L^2(\Gamma)$. The operator $\mathcal{A}_{\text{eff}}$ on the space H_{eff}:

$$\operatorname{dom} \mathcal{A}_{\text{eff}} = \left\{ (u, \beta)^\top \in H_{\text{eff}} : \ u \in W^{2,2}(\Gamma), \text{ cont. on } \Gamma, u(V) = \beta/\sqrt{\alpha} \right\}.$$

Here V is the edge junction.

$$\mathcal{A}_{\text{eff}} \binom{u}{\beta} = \begin{pmatrix} -d^2/dx^2 \\ -\frac{1}{\sqrt{\alpha}} \sum_{e \in V} \partial_n u(V), \end{pmatrix}.$$

where we sum up over all edges terminating at V, and ∂_n is the inward normal derivative.

The take-home message

- The spectral convergence is proved on *compacts* in $\mathbb{C}$.
- Kuchment's example is a particular case of a zero-range model with internal structure (one-dimensional "internal" space!)
- Can be shown to be unitary equivalent to a Laplace operator on Γ with δ' matching at V (Cherednichenko, Kiselev 2017)
- The model is *generic* for inhomogeneous media (see below).
- If we are looking for a convergence *everywhere* in $\mathbb{C}$, we need to relate the spectral parameter z to ε. Then $\Rightarrow$ more involved internal structures.

A PDE problem setup

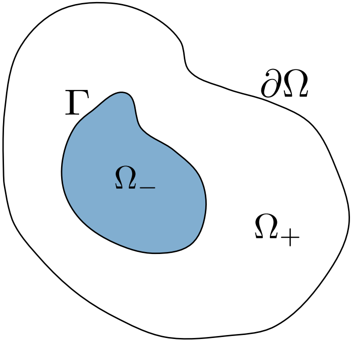

Figure: Domain with a “stiff” inclusion.

Problem setup (maths)

For $a > 0$, $z \in \mathbb{C}$ consider the "transmission" eigenvalue problem

$$\begin{cases} -\Delta u_+ = z u_+ \quad \text{in } \Omega_+, \\ -a\Delta u_- = z u_- \quad \text{in } \Omega_-, \\ u_+ = u_-, \quad \dfrac{\partial u_+}{\partial n_+} + a\dfrac{\partial u_-}{\partial n_-} = 0 \quad \text{on } \Gamma, \\ \dfrac{\partial u_+}{\partial n_+} = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{2}$$

where $n_\pm$ is the exterior normal to the corresponding part of the boundary.
The question: norm-resolvent convergence of the associated family of linear self-adjoint operators as $a \to \infty$?

The leading order term

Let $H_{\text{eff}} = L^2(\Omega_+) \oplus \mathbb{C}$ and

$$\text{dom}\,\mathcal{A}_{\text{eff}} = \left\{ \begin{pmatrix} u_+ \\ \eta \end{pmatrix} \in H_{\text{eff}} : u_+ \in H^2(\Omega_+),\ u_+|_\Gamma = \frac{\eta}{\sqrt{|\Omega_-|}} \mathbb{1}_\Gamma,\ \left.\frac{\partial u_+}{\partial n_+}\right|_{\partial\Omega} = 0 \right\}, \tag{3}$$

where $u|_\Gamma$ is the trace of the function u and $\mathbb{1}_\Gamma$ is the unity function on Γ. On $\text{dom}\,\mathcal{A}_{\text{eff}}$

$$\mathcal{A}_{\text{eff}} \begin{pmatrix} u_+ \\ \eta \end{pmatrix} := \begin{pmatrix} -\Delta u_+ \\ \frac{1}{\sqrt{|\Omega_-|}} \int_\Gamma \frac{\partial u_+}{\partial n_+} \end{pmatrix}.$$

Theorem

Up to a unitary equivalence, we have $(A_a - z)^{-1} \simeq (A_{\text{eff}} - z)^{-1} + O(a^{-1})$ *in operator norm topology, uniformly on compacts* $K_\sigma := \{z \in K : \text{dist}(z, \mathbb{R}) \geq \sigma\}$, *where* $K \subset \mathbb{C}$ *is a compact.*

Remarks

Remark

The operator $\mathcal{A}_{\text{eff}}$ is precisely of Kuchment's form!

Remark

Let P be orthogonal projection from H_{eff} to $L^2(\Omega_+)$. Then the generalised resolvent $P(A_{\text{eff}} - z)^{-1}P$ is the solution operator for

$$\begin{cases} -\Delta u_+ - zu_+ = f, \quad f \in L^2(\Omega_+) \\ u_+|_\Gamma = \frac{1}{|\Gamma|}\int_\Gamma u_+, \quad \frac{|\Gamma|}{|\Omega_-|}\int_\Gamma \frac{\partial u_+}{\partial n_+} = z\int_\Gamma u_+. \end{cases}$$

We have: $P(A_a - z)^{-1}P = P(A_{\text{eff}} - z)^{-1}P + O(a^{-1})$ and A_{eff} is the self-adjoint Neumark-Strauss dilation of the above generalised resolvent. "Impedance BVP" with linear in z BC: Shkalikov'83; Exner-Post'2000s, Kuchment-Zeng'2000s (in quantum graphs); Figotin-Schenker'2000s (problems with memory), et al.

How does the argument work?

Start with $P(A_a - z)^{-1}P$. It is the solution operator for

$$\begin{cases} -\Delta u_+ - zu_+ = f_+, \quad f_+ \in L^2(\Omega_+) \\ \frac{\partial u_+}{\partial n_+}|_\Gamma = -M^-(z)u_+\big|_\Gamma, \end{cases}$$

where $M^-(z)$ is the Dirichlet-to-Neumann map for Ω_-.
Show: this is $O(a^{-1})$ close to

$$\begin{cases} -\Delta u_+ - zu_+ = f_+, \quad f_+ \in L^2(\Omega_+) \\ \mathcal{P}\frac{\partial u_+}{\partial n_+}|_\Gamma = -\mathcal{P}M^-(z)\mathcal{P}u_+\big|_\Gamma, \; \mathcal{P}^\perp u_+|_\Gamma = 0, \end{cases}$$

where $\mathcal{P}$ is the orthoprojection onto constants on Γ. But because of large a,

$$\mathcal{P}M^-(z)\mathcal{P} = -z\frac{|\Omega_-|}{|\Gamma|}\mathcal{P} + O(a^{-1}),$$

and that's it.

Is it the end of the story?

Assume we want to drop the requirement that z is in a compact. Then consider the next order term:

$$\mathcal{P}M^{-}(z)\mathcal{P} = -z\frac{|\Omega_{-}|}{|\Gamma|}\mathcal{P} - \frac{z^2}{a}T + O\left(\frac{z^3}{a^2}\right),$$

leading to a *quadratic* impedance in BCs. Further terms $\Rightarrow$ *polynomial*.
Then the "internal" space of the effective model acquires more dimensions, and the internal structure becomes more involved!

The result for polynomial impedance

Let now $H_{\text{eff}} = L^2(\Omega_+) \oplus \mathbb{C}^k$; in this space take the self-adjoint Neumark-Strauss dilation A_{eff} of the generalised resolvent pertaining to the *polynomial of the order* k impedance boundary conditions on Γ.

Theorem

Up to a unitary equivalence, we have

$$(A_a - z)^{-1} \simeq \mathfrak{P}(A_{\text{eff}} - z)^{-1}\mathfrak{P} + O(\max\{a^{-1}, |z|^{k+1}a^{-k}\})$$

in operator norm topology.
Here $\mathfrak{P}$ *is the orthogonal projection of* H_{eff} *onto* $L^2(\Omega_+) \oplus \mathbb{C}$ *(i.e., the* H_{eff} *of the previous result).*

The quadrupole case

It the case where $k = 2$ ("quadrupole regime") we have, in particular:

- $H_{\text{eff}} = L^2(\Omega_+) \oplus \mathbb{C}^2$;
-

$$\mathrm{dom}\,\mathcal{A}_{\text{eff}} = \left\{ \begin{pmatrix} u_+ \\ \vec{\eta} \end{pmatrix} \in H_{\text{eff}} : u_+ \in H^2(\Omega_+),\ u_+|_\Gamma = \frac{\eta_1}{\kappa} \mathbb{1}_\Gamma,\ \left.\frac{\partial u_+}{\partial n_+}\right|_{\partial\Omega} = 0 \right\},$$

where $\vec{\eta} = (\eta_1, \eta_2) \in \mathbb{C}^2$, $u|_\Gamma$ is the trace of the function u and $\mathbb{1}_\Gamma$ is the unity function on Γ.

- On $\mathrm{dom}\,\mathcal{A}_{\text{eff}}$

$$\mathcal{A}_{\text{eff}} \begin{pmatrix} u_+ \\ \eta_1 \\ \eta_2 \end{pmatrix} := \begin{pmatrix} -\Delta u_+ \\ \frac{1}{\kappa} \int_\Gamma \frac{\partial u_+}{\partial n_+} + a(B^2 D^{-1} \eta_1 + B\eta_2) \\ a(B\eta_1 + D\eta_2) \end{pmatrix}.$$

Here B, D and κ are real parameters, explicitly computed in terms of coefficients defining the impedance.

An explicit spectral decomposition for $\mathcal{A}_{\text{eff}}$

In [4], we have used functional model techniques + the boundary triples theory, to establish an *explicit* spectral representation for operators of the form $\mathcal{A}_{\text{eff}}$.
In particular, in the "dipole" case (i.e., when the "internal" space E is one-dimensional), it is realised as $L^2(\mathbb{R}; d\mu)$, where μ is a Clarke measure.

References

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Thank you for your attention!