# Dispersion relations in high-contrast composites

Yi Sheng Lim

22/07/2022

# Goals

1. Explain the boundary triples approach for high-contrast (HC) homogenization.

2. Using the output of this approach – How do waves propagate effectively in our HC composite?

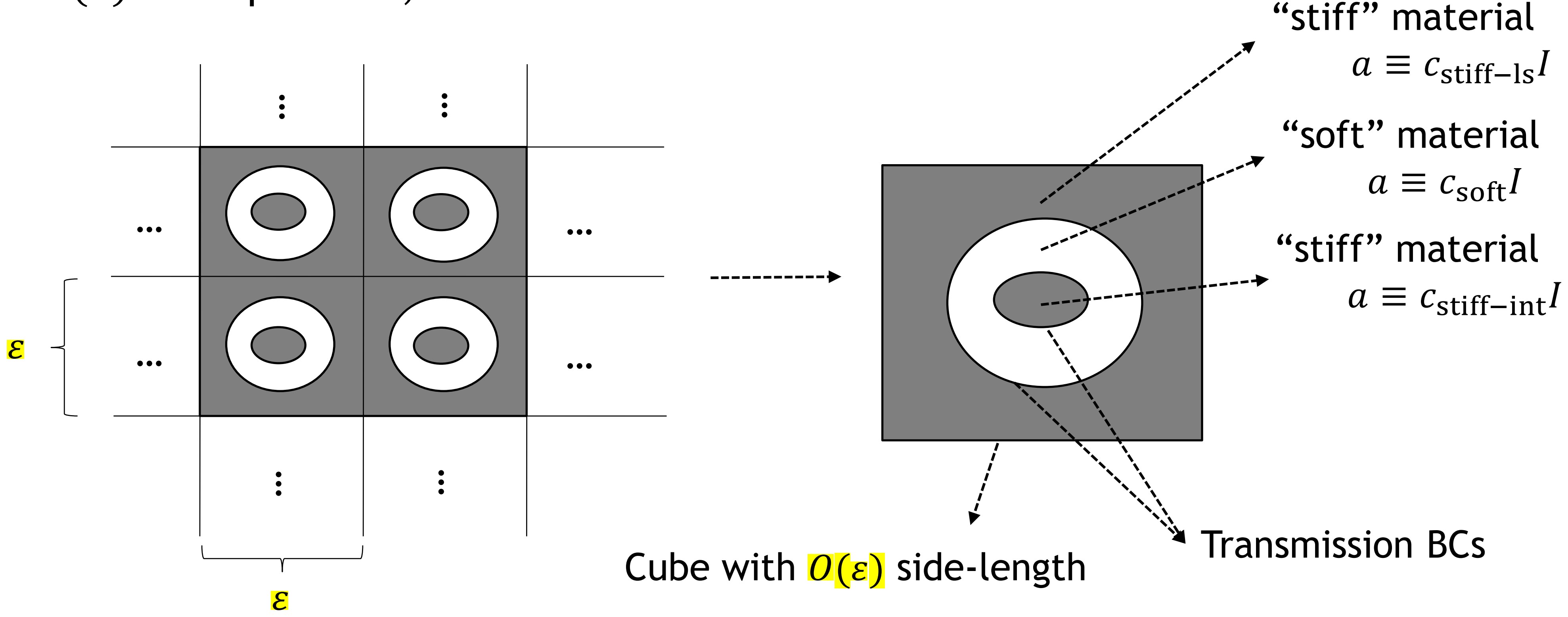
# Problem Setup

$-\mathrm{div}(a(\cdot/\varepsilon)\nabla\,\cdot)$ "$\longrightarrow$" $-\,\mathrm{div}(\mathrm{a}_{\mathrm{hom}}\nabla\,\cdot)$ ?

- Fix dimension $d \geq 2$. Consider the problem

$$-\mathrm{div}\left(\mathrm{a}\left(\tfrac{x}{\varepsilon}\right)\nabla u^{\varepsilon}\right) - \mu u^{\varepsilon} = f, \qquad f \in L^2(\mathbb{R}^d), \qquad \mu \in \mathbb{C}$$

- $a(x)$ is $\mathbb{Z}^d$-periodic, and looks like this:



# Problem Setup

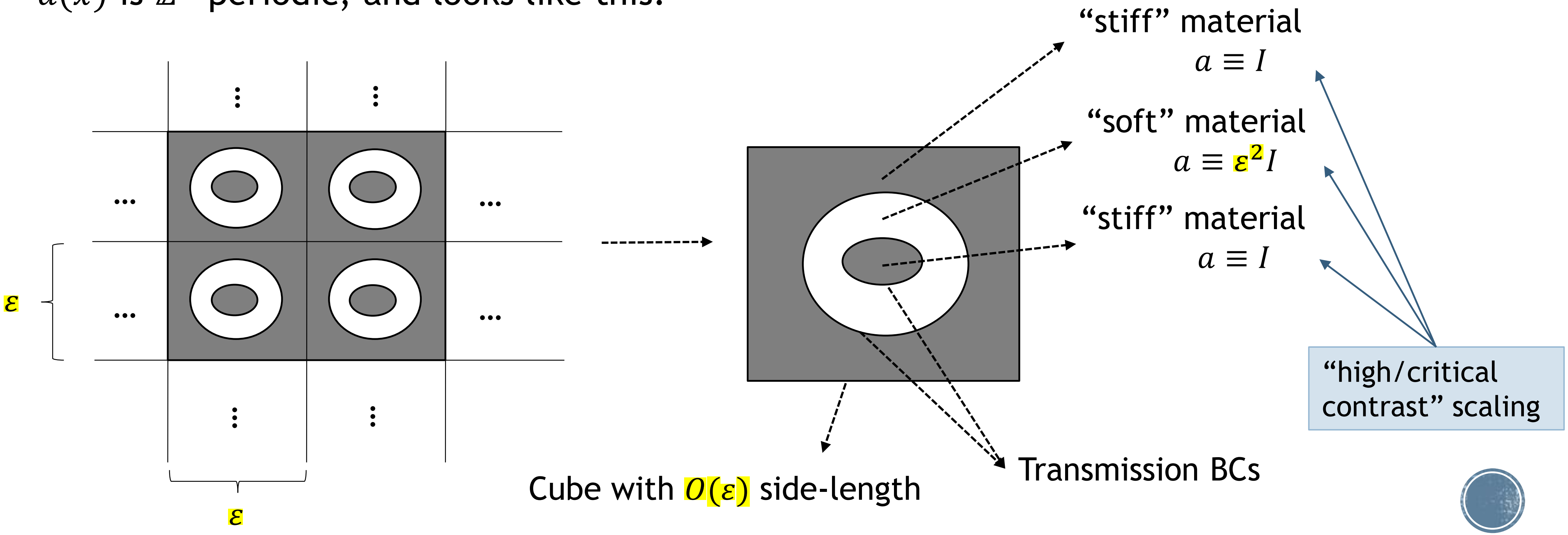
- Fix dimension $d \geq 2$. Consider the problem

$$-\mathrm{div}\left(\mathrm{a}\left(\frac{x}{\varepsilon}\right)\nabla u^{\varepsilon}\right) - \mu u^{\varepsilon} = f, \qquad f \in L^2\left(\mathbb{R}^d\right), \qquad \mu \in \mathbb{C}$$

full space

- Operator POV
- Resolvent eqn $A_\varepsilon u^\varepsilon - \mu u^\varepsilon = \mathrm{f}$
- Find $A_{\mathrm{hom},\varepsilon}$ s.t. $\sigma(A_\varepsilon) \sim \sigma(A_{\mathrm{hom}})$

- $a(x)$ is $\mathbb{Z}^d$-periodic, and looks like this:



# Methods available (non-HC)

(Murat 1978, Tartar 1979) Method of compensated compactness

$$U^\varepsilon \rightharpoonup U^0, V^\varepsilon \rightharpoonup V^0 \text{ in } \left(\mathrm{L}^2(\Omega)^{\mathrm{d}}\right)$$

$$\operatorname{div} U^\varepsilon \to f^0 \in H^{-1} \text{ and } \operatorname{curl} \mathrm{V}^\varepsilon = \boldsymbol{o}$$

Then $U^\varepsilon \cdot V^\varepsilon \rightharpoonup U^0 \cdot V^0$.

(Allaire 1992) Two-scale convergence method: We say $v^\varepsilon \overset{2}{\to} v^0$ if

$$\int_\Omega v^\varepsilon(x)\psi\left(x,\tfrac{x}{\varepsilon}\right)dx \to \iint_{\Omega\times[0,1]} v^0(x,y)\psi(x,y)\,dy\,dx$$

For all $\psi(x,y) \in \mathcal{D}(\Omega; C^\infty_{per}([0,1]))$.

Tartar's method of oscillating test functions (1977)

$\Gamma$-convergence, G-convergence, …

Two-scale expansion method

$$u^\varepsilon(x) = u_0\left(x,\tfrac{x}{\varepsilon}\right) + \varepsilon u_1\left(x,\tfrac{x}{\varepsilon}\right) + \varepsilon^2 u_2\left(x,\tfrac{x}{\varepsilon}\right) + \cdots$$

(Birman-Suslina 2004) "spectral germ"

- Gelfand transform

$$A \cong \int_{[0,1]^d}^{\oplus} A(\tau)\,d\tau$$

- Perturbation theory

$$A(t)\varphi_n(t) = \lambda_n(t)\varphi_n(t), \qquad \tau = t\theta$$

- **Norm-resolvent** approximations!!!

# Why norm-resolvent convergence?

- Definition: Let $A_n$ and $A$ be (unbounded) self-adjoint ops on a Hilbert space $\mathcal{H}$.
- We say that $A_n$ converges to $A$ in the norm-resolvent sense, denoted $A_n \overset{\text{nr}}{\to} A$, if

$$\|(A_n-\lambda)^{-1} - (A-\lambda)^{-1}\|_{op} \to 0, \qquad \text{as } n\to\infty, \qquad \text{for some } \lambda\in\mathbb{C}\setminus\mathbb{R}.$$

- Implies strong convergence of solutions $u^n = (A_n-\lambda)^{-1}f \to (A-\lambda)^{-1}f = u^0$.
- (By functional calculus) $\|g(A_n)-g(A)\|_{op} \to 0$, $g\in C_0(\mathbb{R};\mathbb{C})$
- $A_n \overset{\text{nr}}{\to} A$ implies convergence of spectrum (in some sense), i.e.

$$\sigma\left(\text{nr}-\lim_{n\to\infty} A_n\right) = \lim_{n\to\infty}\sigma(A_n)$$

$$\sigma\left(\text{sr}-\lim_{n\to\infty} A_n\right) \subseteq \lim_{n\to\infty}\sigma(A_n)$$

$g(\lambda) = e^{it\lambda}$ not ok

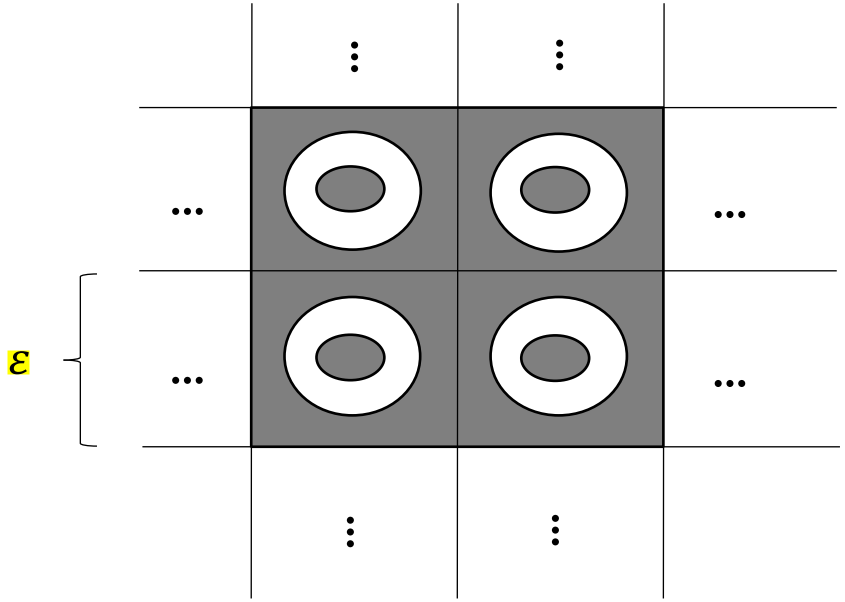
What it *cannot* achieve

- Spectral decomposition
- Might not have limits in general … norm resolvent asymptotics

# Step 0 – unitary transform

- From $A_\varepsilon u^\varepsilon - \mu u^\varepsilon = f$, we apply a sequence of unitary transforms:

$$A_\varepsilon = G_\varepsilon^* \left( \int_{\varepsilon^{-1}Q'}^{\oplus} \Phi_\varepsilon^* A_\varepsilon^{(\varepsilon\theta)} \Phi_\varepsilon \, d\theta \right) G_\varepsilon$$



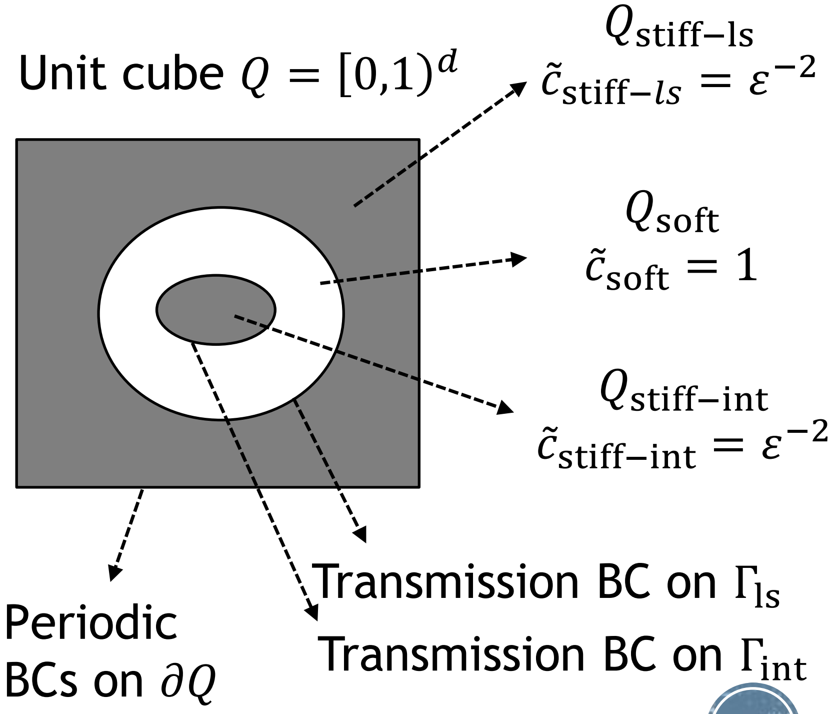

- Gelfand Transform

$$G_\varepsilon : L^2(\mathbb{R}^d) \to L^2(\varepsilon^{-1}Q \times \varepsilon Q)$$

(gives us a family of PDEs on $L^2(\varepsilon Q)$)

- Unitary rescaling

$$\Phi_\varepsilon : L^2(\varepsilon Q) \to L^2(Q)$$

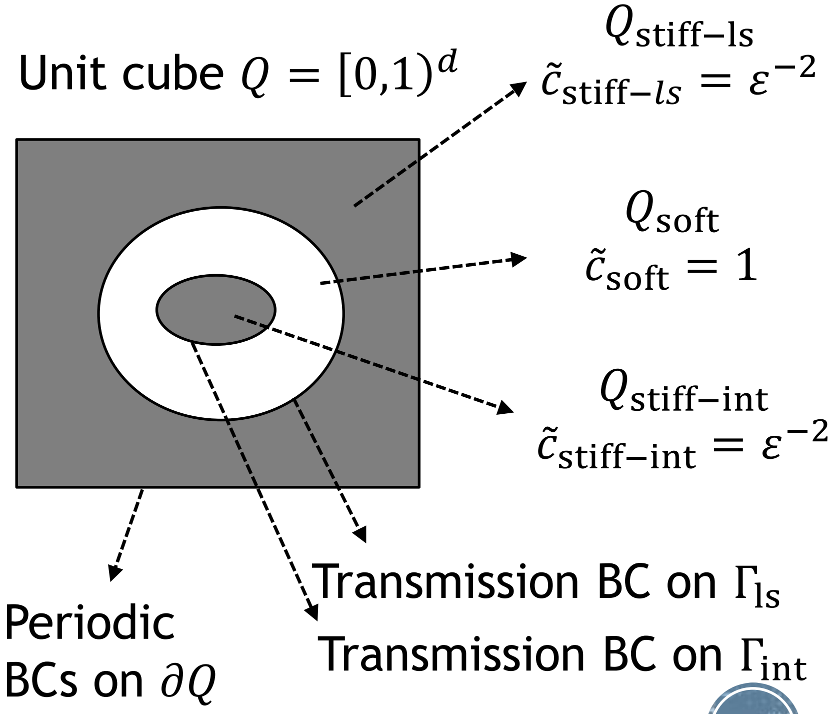


- Write $\tau = \varepsilon\theta \in Q' = [-\pi, \pi)^d$. The resolvent equation

$$\left(A_\varepsilon^{(\tau)} - z\right) u = f \in L^2(Q)$$

has a unique solution $u \equiv u_\varepsilon^{(\tau)} = u_{\text{stiff-int}} + u_{\text{soft}} + u_{\text{stiff-ls}}$ whenever the following BVP can be solved uniquely in the weak sense:

$$\begin{cases} \varepsilon^{-2}\left(\frac{1}{i}\nabla + \tau\right)^2 u_{\text{stiff-int}} - z u_{\text{stiff-int}} = f, & \text{in } Q_{\text{stiff-int}}, \\ \left(\frac{1}{i}\nabla + \tau\right)^2 u_{\text{soft}} - z u_{\text{soft}} = f, & \text{in } Q_{\text{soft}}, \\ \varepsilon^{-2}\left(\frac{1}{i}\nabla + \tau\right)^2 u_{\text{stiff-ls}} - z u_{\text{stiff-ls}} = f, & \text{in } Q_{\text{stiff-ls}}, \\ u_{\text{stiff-int}} = u_{\text{soft}}, \\ \varepsilon^{-2}\left[\frac{\partial u_{\text{stiff-int}}}{\partial n} + i(\tau\cdot n) u_{\text{stiff-int}}\right] + \left[\frac{\partial u_{\text{soft}}}{\partial n} + i(\tau\cdot n) u_{\text{soft}}\right] = 0, & \text{on } \Gamma_{\text{int}}, \\ u_{\text{soft}} = u_{\text{stiff-ls}}, \\ \left[\frac{\partial u_{\text{soft}}}{\partial n} + i(\tau\cdot n) u_{\text{soft}}\right] + \varepsilon^{-2}\left[\frac{\partial u_{\text{stiff-ls}}}{\partial n} + i(\tau\cdot n) u_{\text{stiff-ls}}\right] = 0, & \text{on } \Gamma_{\text{ls}}, \\ u_{\text{stiff-ls}} \text{ periodic} & \text{on } \partial Q \end{cases}$$

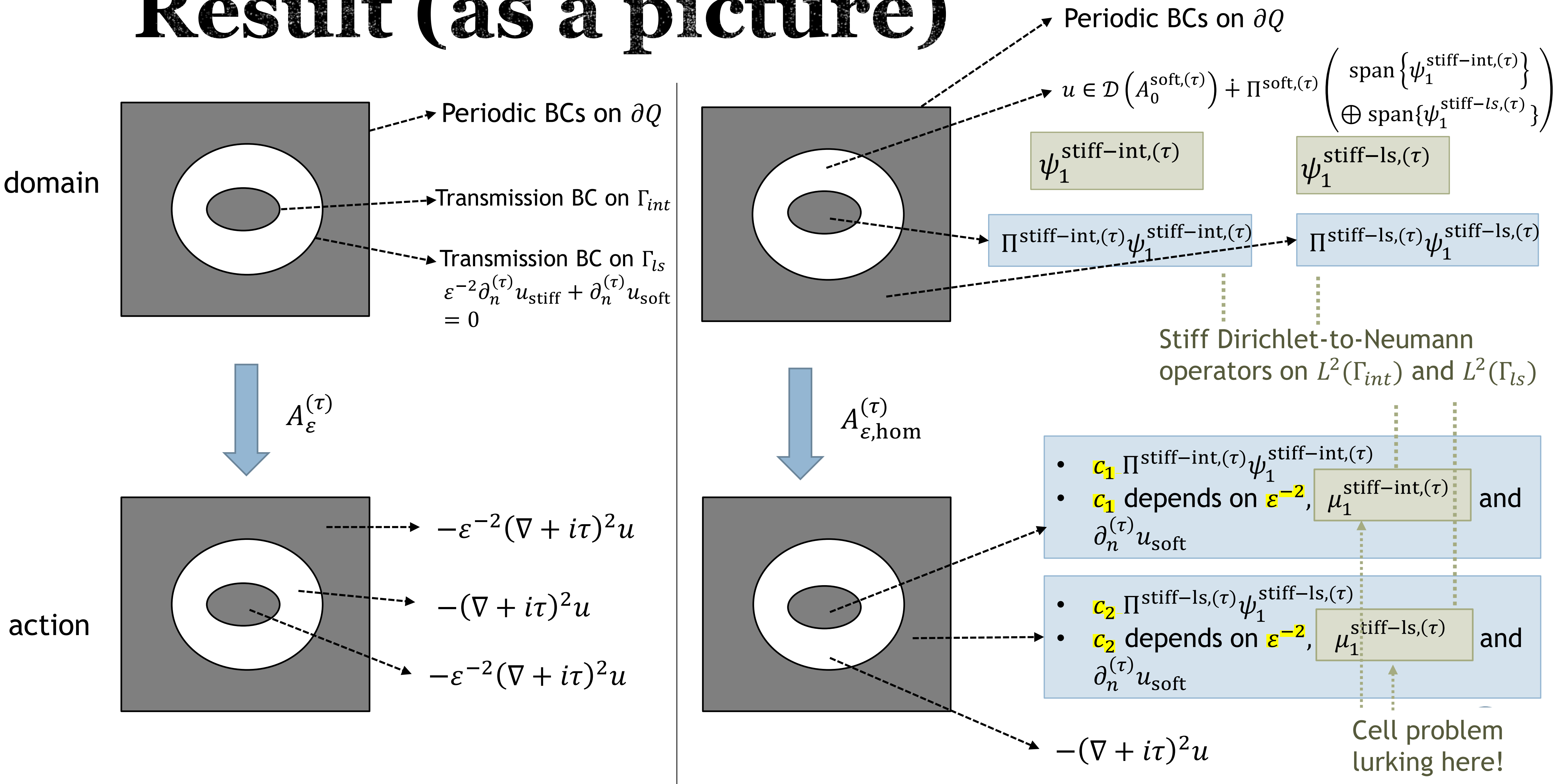


# Our goal

- Find an operator $A_{\varepsilon,hom}^{(\tau)}$ that is
  - self-adjoint on a possibly smaller subspace $L^2(Q_{\text{soft}}) \oplus \widetilde{\mathcal{H}}$ of $L^2(Q)$.
  - Dependence on $\varepsilon$ only allowed in the action of $A_{\varepsilon,hom}^{(\tau)}$ on the stiff component.

    (e.g. domain $\mathcal{D}\left(A_{\text{hom}}^{(\tau)}\right)$ cannot depend on $\varepsilon$.)
  - Is $O(\varepsilon^2)$ close to $A_\varepsilon^{(\tau)}$ in the norm-resolvent sense. $O(\varepsilon^2)$-error does not depend on $\tau$.
- $A_{\varepsilon,hom}^{(\tau)}$ need not be unique since we are discussing asymptotics.

# Result (as a picture)



# Result

$$\breve{\mathcal{H}}^{\text{stiff-ls},(\tau)} = \text{span}\left\{\Pi^{\text{stiff-ls},(\tau)}\psi_1^{\text{stiff-ls},(\tau)}\right\}$$

$$\Pi^{\text{stiff},(\tau)} = \Pi^{\text{stiff-int},(\tau)} \oplus \Pi^{\text{stiff-ls},(\tau)}$$

$$\hat{u} = \hat{u}_{\text{stiff-int}} + \hat{u}_{\text{stiff-}ls}$$

<u>Theorem</u> The operator $A^{(\tau)}_{\varepsilon,\text{hom}}$ defined by

$$\mathcal{D}\left(A^{(\tau)}_{\varepsilon,\text{hom}}\right) := \left\{\begin{pmatrix} u \\ \hat{u}_{\text{stiff-int}} \\ \hat{u}_{\text{stiff-ls}} \end{pmatrix} \in L^2(Q_{\text{soft}}) \oplus \breve{\mathcal{H}}^{\text{stiff-int},(\tau)} \oplus \breve{\mathcal{H}}^{\text{stiff-ls},(\tau)} : \right.$$

$$\left. u \in \mathcal{D}\left(A_0^{\text{soft},(\tau)}\right) \dot{+} \Pi^{\text{soft},(\tau)}\left(\text{span}\left\{\psi_1^{\text{stiff-int},(\tau)}\right\} \oplus \text{span}\left\{\psi_1^{\text{stiff-}ls,(\tau)}\right\}\right), \quad \hat{u} = \breve{\Pi}^{\text{stiff},(\tau)}\Gamma_0^{\text{soft},(\tau)} u\right\}$$

$$A^{(\tau)}_{\varepsilon,\text{hom}}\begin{pmatrix} u \\ \hat{u}_{\text{stiff-int}} \\ \hat{u}_{\text{stiff-ls}} \end{pmatrix} = \begin{pmatrix} -(\nabla + i\tau)^2 u \\ -\left(\breve{\Pi}^{\text{stiff-int},(\tau)*}\right)^{-1}\mathcal{P}^{(\tau)}_{\text{stiff-int}}\left[\partial_n^{(\tau)} u|_\Gamma\right] \\ -\left(\breve{\Pi}^{\text{stiff-ls},(\tau)*}\right)^{-1}\mathcal{P}^{(\tau)}_{\text{stiff-ls}}\left[\partial_n^{(\tau)} u\Big|_\Gamma + \varepsilon^{-2}\mu_1^{\text{stiff-ls},(\tau)} u\Big|_\Gamma\right] \end{pmatrix}$$

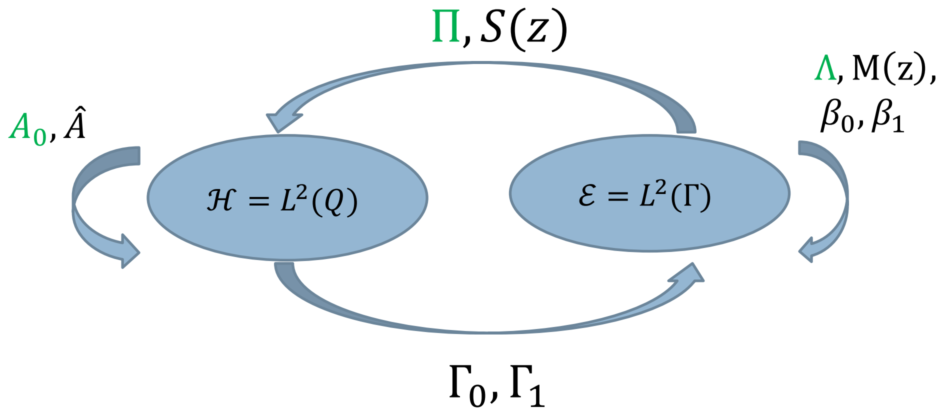
is self-adjoint on $L^2(Q_{\text{soft}}) \oplus \breve{\mathcal{H}}^{\text{stiff-int},(\tau)} \oplus \breve{\mathcal{H}}^{\text{stiff-ls},(\tau)}$, and is $O(\varepsilon^2)$ close to $A^{(\tau)}_\varepsilon$ in the norm-resolvent sense.

This estimate is uniform in $\tau \in Q'$ and $z \in K_\sigma$ (a compact set $\sigma > 0$ distance away from the real line.)

# Boundary triples

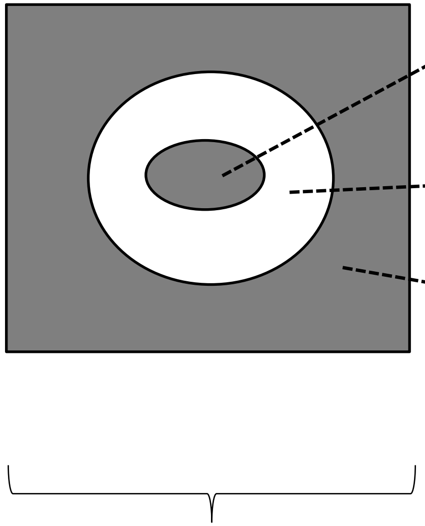
A (Ryzhov) boundary triple $(A_0, \Lambda, \Pi)$ needs:

- Separable Hilbert spaces $\mathcal{H}$ and $\mathcal{E}$ (boundary space).
- (Dirichlet operator) $A_0$ an unbounded SA op on $\mathcal{H}$, with $0 \in \rho(A_0)$.
- (DTN operator) $\Lambda$ an unbounded SA op on $\mathcal{E}$.
- (Lift) $\Pi: \mathcal{E} \to \mathcal{H}$, a bounded injective linear map.
- $\mathcal{D}(A_0) \cap \mathrm{ran}(\Pi) = \{0\}$



# Step 1 – Construct the triples

For each $\tau$, construct the following triples:



$\left(A_{\varepsilon,0}^{\text{stiff-int},(\tau)}, \Lambda_{\varepsilon}^{\text{stiff-int},(\tau)}, \Pi^{\text{stiff-int},(\tau)}\right)$ on $L^2(Q_{\text{stiff-int}})$ and boundary space $L^2(\Gamma_{int})$

$\left(A_{0}^{\text{soft},(\tau)}, \Lambda^{\text{soft},(\tau)}, \Pi^{\text{soft},(\tau)}\right)$ on $L^2(Q_{\text{soft}})$ and boundary space $L^2(\Gamma_{int}) \oplus L^2(\Gamma_{ls})$

$\left(A_{\varepsilon,0}^{\text{stiff-ls},(\tau)}, \Lambda_{\varepsilon}^{\text{stiff-ls},(\tau)}, \Pi^{\text{stiff-ls},(\tau)}\right)$ on $L^2(Q_{\text{stiff-ls}})$ and boundary space $L^2(\Gamma_{ls})$

$\left(A_{\varepsilon,0}^{(\tau)}, \Lambda_{\varepsilon}^{(\tau)}, \Pi^{(\tau)}\right)$

on $\mathcal{H} = L^2(Q)$ and $\mathcal{E} = L^2(\Gamma_{int}) \oplus L^2(\Gamma_{ls})$

- $A_{\varepsilon,0}^{\text{stiff-int},(\tau)} = -\varepsilon^{-2}(\nabla + i\tau)^2$ with Dirichlet BCs on $\Gamma_{int}$
- $A_{0}^{\text{soft},(\tau)} = -(\nabla + i\tau)^2$ with Dirichlet BCs on $\Gamma_{int}$ and $\Gamma_{ls}$
- $A_{\varepsilon,0}^{\text{stiff-ls},(\tau)} = -\varepsilon^{-2}(\nabla + i\tau)^2$ with Dirichlet BC on $\Gamma_{ls}$ + Periodic BC on $\partial Q$
- $A_{\varepsilon,0}^{(\tau)} = A_{\varepsilon,0}^{\text{stiff-int},(\tau)} \oplus A_{0}^{\text{soft},(\tau)} \oplus A_{\varepsilon,0}^{\text{stiff-ls},(\tau)}$

# Auxillary operators (defn)

To a boundary triple $(A_0, \Lambda, \Pi)$ with spaces $\mathcal{H}$ and $\mathcal{E}$, define the following operators:

- $\hat{A}: \mathcal{H} \supset \mathcal{D}(A_0) \to \mathcal{H}$ with domain $\mathcal{D}(\hat{A}) = \mathcal{D}(A_0) \dotplus \mathrm{ran}\Pi$ and action

$$\hat{A}(\mathrm{A}_0^{-1} f + \Pi\phi) = f, \qquad f \in \mathcal{H}, \phi \in \mathcal{E}$$

- $\Gamma_0: \mathcal{H} \supset \mathcal{D}(\Gamma_0) \to \mathcal{E}$ with domain $\mathcal{D}(\Gamma_0) = \mathcal{D}(A_0) \dotplus \mathrm{ran}\Pi$ and action

$$\Gamma_0(\mathrm{A}_0^{-1} f + \Pi\phi) = \phi, \qquad f \in \mathcal{H}, \phi \in \mathcal{E}$$

- $\Gamma_1: \mathcal{H} \supset \mathcal{D}(\Gamma_1) \to \mathcal{E}$ with domain $\mathcal{D}(\Gamma_1) = \mathcal{D}(A_0) \dotplus \Pi(\mathcal{D}(\Lambda))$ and action

$$\Gamma_0(\mathrm{A}_0^{-1} f + \Pi\phi) = \Pi^* f + \Lambda\phi, \qquad f \in \mathcal{H}, \phi \in \mathcal{E}$$

- (Solution operator) For $z \in \rho(A_0)$, define the bounded op $S(z): \mathcal{E} \to \mathcal{H}$ by

$$S(z)\phi = (I + z(A_0 - z)^{-1})\Pi\phi$$

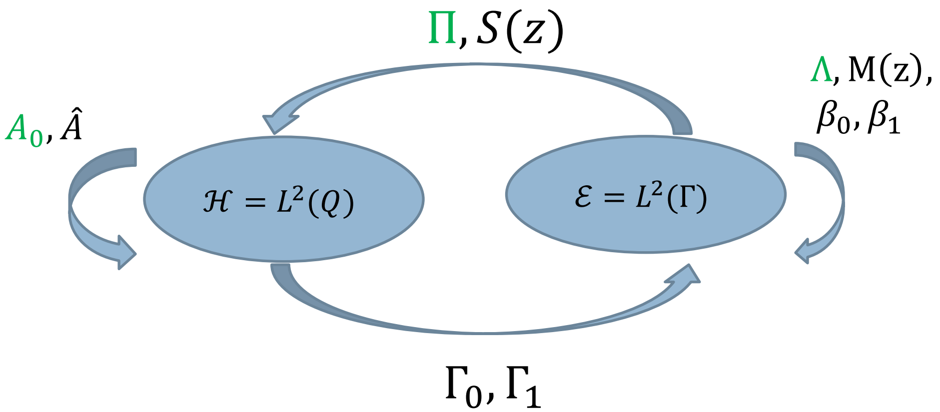
- (M-operator) For $z \in \rho(A_0)$, define the closed op $M(z): \mathcal{E} \supset \mathcal{D}(M(z)) \to \mathcal{E}$ with domain $\mathcal{D}(M(z)) = \mathcal{D}(\Lambda)$ and action

$$M(z)\phi = \Gamma_1 S(z)\phi$$

# Auxillary operators (intuition)

- How to think of the auxillary ops?
- For $\hat{A}$, $\Gamma_0$, and $\Gamma_1$, consider the following "BVP"

$$\begin{cases} (\hat{A} - z)u = f \\ (\beta_0\Gamma_0 + \beta_1\Gamma_1)u = \phi \end{cases}$$




- For $S(z)$, we have $S(0) = \Pi$
- For $M(z)$, we have $M(0) = \Lambda$

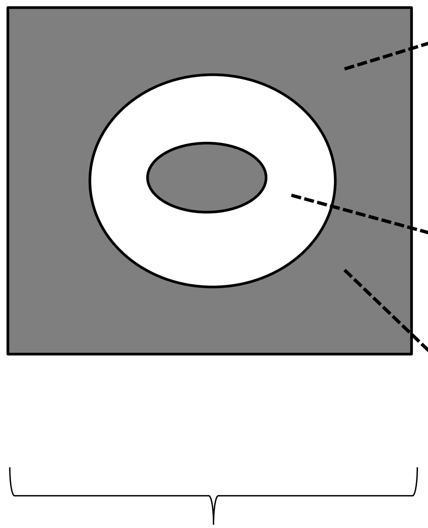
- P.S. $\rho(A_0) \ni z \mapsto S(z)$ and $\rho(A_0) \ni z \mapsto M(z)$ are nice!

# Step 2 – Construct aux ops

Obtain the auxillary ops:

$\left(A_{\varepsilon,0}^{\text{stiff-int},(\tau)}, \Lambda_{\varepsilon}^{\text{stiff-int},(\tau)}, \Pi^{\text{stiff-int},(\tau)}\right)$ and

$\hat{A}_{\varepsilon}^{\text{stiff-int},(\tau)}, \Gamma_{0}^{\text{stiff-int},(\tau)}, \Gamma_{\varepsilon,1}^{\text{stiff-int},(\tau)}, S_{\varepsilon}^{\text{stiff-int},(\tau)}, M_{\varepsilon}^{\text{stiff-int},(\tau)}(z)$



$\left(A_{0}^{\text{soft},(\tau)}, \Lambda^{\text{soft},(\tau)}, \Pi^{\text{soft},(\tau)}\right)$ and

$\hat{A}^{\text{soft},(\tau)}, \Gamma_{0}^{\text{soft},(\tau)}, \Gamma_{1}^{\text{soft},(\tau)}, S^{\text{soft},(\tau)}(z), M^{\text{soft},(\tau)}(z)$

8x4 = 32 operators!!!

$\left(A_{\varepsilon,0}^{\text{stiff-ls},(\tau)}, \Lambda_{\varepsilon}^{\text{stiff-ls},(\tau)}, \Pi^{\text{stiff-ls},(\tau)}\right)$ and

$\hat{A}_{\varepsilon}^{\text{stiff-ls},(\tau)}, \Gamma_{0}^{\text{stiff-ls},(\tau)}, \Gamma_{\varepsilon,1}^{\text{stiff-ls},(\tau)}, S_{\varepsilon}^{\text{stiff-ls},(\tau)}, (z)\ M_{\varepsilon}^{\text{stiff-ls},(\tau)}(z)$

$\left(A_{\varepsilon,0}^{(\tau)}, \Lambda_{\varepsilon}^{(\tau)}, \Pi^{(\tau)}\right)$ and

$\hat{A}_{\varepsilon}^{(\tau)}, \Gamma_{0}^{(\tau)}, \Gamma_{\varepsilon,1}^{(\tau)}, S_{\varepsilon}^{(\tau)}(z), M_{\varepsilon}^{(\tau)}(z)$

# Step 3 – Use results of bdry triple theory

Why boundary triples?

**Theorem (Ryzhov 2009)** With $\beta_0$ and $\beta_1$ "nice" ops on $\mathcal{E}$, $z \in \rho(A_0)$,

- There is a closed, densely defined operator $\hat{A}_{\beta_0,\beta_1}$ s.t. for $f \in \mathcal{H}$,

$$\begin{cases} (\hat{A} - z)u = f \\ (\beta_0\Gamma_0 + \beta_1\Gamma_1)u = 0 \end{cases} \quad \Longleftrightarrow \quad (\hat{A}_{\beta_0,\beta_1} - z)u = f$$

has a unique solution ⟺ has a unique solution

- (Krein's formula) $R_{\beta_0,\beta_1}(z) := \left(\hat{A}_{\beta_0,\beta_1} - z\right)^{-1} = (A_0 - z)^{-1} - S(z)\left(\overline{\beta_0 + \beta_1 M(z)}\right)^{-1}\beta_1 S(\bar{z})^*$

**Corollary** With our triple $\left(A_{\varepsilon,0}^{(\tau)}, \Lambda_\varepsilon^{(\tau)}, \Pi^{(\tau)}\right)$ we have $A_\varepsilon^{(\tau)} = \hat{A}_{\varepsilon,0,I}^{(\tau)}$ and

$$\left(A_\varepsilon^{(\tau)} - z\right)^{-1} = \left(A_{\varepsilon,0}^{(\tau)} - z\right)^{-1} - S_\varepsilon^{(\tau)}(z)\left(M_\varepsilon^{(\tau)}(z)\right)^{-1} S_\varepsilon^{(\tau)}(\bar{z})^*$$

# Step 3 – Use results of bdry triple theory

Why boundary triples?

- By varying through $\beta_0$ and $\beta_1$ we get a collection of operators $\hat{A}_{\beta_0,\beta_1}$ that is big enough to include all relevant operators that we need.
- Each operator $\hat{A}_{\beta_0,\beta_1}$ has a corresponding "BVP" interpretation
- Krein's formula provides a way to compute norm-resolvent asymptotics in terms of "nicer" objects like $M(z)$ and $S(z)$.

# Step 4 – Using Krein's formula

In a nutshell... $\left(A^{(\tau)}_{\varepsilon,\beta_0,\beta_1} - z\right)^{-1} = \left(A^{(\tau)}_{\varepsilon,0} - -z\right)^{-1} - S^{(\tau)}_{\varepsilon}(z)\left(\overline{\beta_0 + \beta_1 M^{(\tau)}_{\varepsilon}(z)}\right)^{-1} \beta_1 S^{(\tau)}_{\varepsilon}(\bar{z})^*$

... and the crude approximation of $\Lambda = \sum \mu_k \langle \cdot, \psi_k \rangle \psi_k$ by $\mu_1 \langle \cdot, \psi_1 \rangle \psi_1$ is enough, because

**Theorem** The following estimate in the operator norm holds

$$\left(M^{(\tau)}_{\varepsilon}(z)\right)^{-1} = \begin{pmatrix} \left(\mathcal{P}^{(\tau)}\Lambda^{(\tau)}_{\varepsilon}\mathcal{P}^{(\tau)}\right)^{-1} & 0 \\ 0 & 0 \end{pmatrix} + O(\varepsilon^2)$$

2x2 matrix

(abuse of notation

on $\mathcal{P}^{(\tau)}$)

Relative to the decomposition $\mathcal{E} = \mathcal{P}^{(\tau)}\mathcal{E} \oplus \mathcal{P}^{(\tau)}_{\perp}\mathcal{E}$, where $\mathcal{P}^{(\tau)} = \mathcal{P}^{(\tau)}_{\text{stiff-int}} + \mathcal{P}^{(\tau)}_{\text{stiff-}ls}$ .

(e.g. $\mathcal{P}^{(\tau)}_{\text{stiff-}ls}$ = projection onto (1D) eigenspace w.r.t. first (simple) evalue of stiff-DTN)

$\left(\mathcal{P}^{(\tau)}\Lambda^{(\tau)}_{\varepsilon}\mathcal{P}^{(\tau)}\right)^{-1}$ is bounded uniformly in $\varepsilon > 0$, $\tau \in Q'$, and $z \in K_\sigma$.

This estimate is uniform in $\tau \in Q'$, and $z \in K_\sigma$.

# Step 4 – Using Krein's formula … is not enough

**Theorem** The following estimate in the operator norm holds

2x2 matrix

$$\left(M_\varepsilon^{(\tau)}(z)\right)^{-1} = \begin{pmatrix} \left(\mathcal{P}^{(\tau)}\Lambda_\varepsilon^{(\tau)}\mathcal{P}^{(\tau)}\right)^{-1} & 0 \\ 0 & 0 \end{pmatrix} + O(\varepsilon^2)$$

Relative to the decomposition $\mathcal{E} = \mathcal{P}^{(\tau)}\mathcal{E} \oplus \mathcal{P}_\perp^{(\tau)}\mathcal{E}$, where $\mathcal{P}^{(\tau)} = \mathcal{P}_{\text{stiff-int}}^{(\tau)} + \mathcal{P}_{\text{stiff-}ls}^{(\tau)}$ .

(e.g. $\mathcal{P}_{\text{stiff-}ls}^{(\tau)}$ = projection onto (1D) eigenspace w.r.t. first (simple) evalue of stiff-DTN)

$\left(\mathcal{P}^{(\tau)}\Lambda_\varepsilon^{(\tau)}\mathcal{P}^{(\tau)}\right)^{-1}$ is bounded uniformly in $\varepsilon > 0$, $\boldsymbol{\tau} \in \boldsymbol{Q}'$, and $z \in K_\sigma$.

This estimate is uniform in $\boldsymbol{\tau} \in \boldsymbol{Q}'$, and $z \in K_\sigma$.

Needs perturbation theory. For e.g. to prove that

$$\tau \mapsto \Lambda^{(\tau)}\phi,\ \tau \mapsto \mu_1^{(\tau)}\ ,\ \tau \mapsto \psi_1^{\text{stiff-int},(\tau)}, \tau \mapsto \left\| \Pi^{\text{stiff-int},(\tau)}\psi_1^{\text{stiff-int},(\tau)} \right\|, \text{etc.}$$

are continuous.

# Result (again)

$\breve{\mathcal{H}}^{\text{stiff-ls},(\tau)} = \text{span}\left\{\Pi^{\text{stiff-ls},(\tau)}\psi_1^{\text{stiff-ls},(\tau)}\right\}$

$j_{\text{stiff-ls}}^{(\tau)}$ is a unitary map for

$\breve{\mathcal{H}}^{\text{stiff-ls},(\tau)} \cong \mathbb{C}_{\text{stiff-int}}$

Theorem The operator $A_{\varepsilon,\text{hom}}^{(\tau)}$ defined by

$$\mathcal{D}\left(A_{\varepsilon,\text{hom}}^{(\tau)}\right) := \left\{\begin{pmatrix} u \\ \beta_{\text{stiff-int}} \\ \beta_{\text{stiff-ls}} \end{pmatrix} \in L^2(Q_{\text{soft}}) \oplus \mathbb{C}_{\text{stiff-int}} \oplus \mathbb{C}_{\text{stiff-ls}}\right.$$

$$u \in \mathcal{D}\left(A_0^{\text{soft},(\tau)}\right) \dot{+} \Pi^{\text{soft},(\tau)}\left(\text{span}\left\{\psi_1^{\text{stiff-int},(\tau)}\right\} \oplus \text{span}\left\{\psi_1^{\text{stiff-}ls,(\tau)}\right\}\right),$$

$$\left.\beta_{\text{stiff-int}} = j_{\text{stiff-int}}^{(\tau)}\Pi^{\text{stiff-int},(\tau)}\Gamma_0^{\text{soft},(\tau)}u,\ \beta_{\text{stiff-ls}} = j_{\text{stiff-ls}}^{(\tau)}\Pi^{\text{stiff-ls},(\tau)}\Gamma_0^{\text{soft},(\tau)}u\right\}$$

$$A_{\varepsilon,\text{hom}}^{(\tau)}\begin{pmatrix} u \\ \beta_{\text{stiff-int}} \\ \beta_{\text{stiff-ls}} \end{pmatrix} = \begin{pmatrix} -(\nabla + i\tau)^2 u \\ -\left(\left(j_{\text{stiff-int}}^{(\tau)}\breve{\Pi}^{\text{stiff-int},(\tau)}\right)^*\right)^{-1}\mathcal{P}_{\text{stiff-int}}^{(\tau)}\left[\Gamma_1^{\text{soft},(\tau)}u + \varepsilon^{-2}\mu_1^{\text{stiff-int},(\tau)}\left(j_{\text{stiff-int}}^{(\tau)}\breve{\Pi}^{\text{stiff-int},(\tau)}\right)^{-1}\beta_{\text{stiff-int}}\right] \\ -\left(\left(j_{\text{stiff-ls}}^{(\tau)}\breve{\Pi}^{\text{stiff-ls},(\tau)}\right)^*\right)^{-1}\mathcal{P}_{\text{stiff-ls}}^{(\tau)}\left[\Gamma_1^{\text{soft},(\tau)}u + \varepsilon^{-2}\mu_1^{\text{stiff-ls},(\tau)}\left(j_{\text{stiff-ls}}^{(\tau)}\breve{\Pi}^{\text{stiff-ls},(\tau)}\right)^{-1}\beta_{\text{stiff-ls}}\right] \end{pmatrix}$$

is self-adjoint on $L^2(Q_{\text{soft}}) \oplus \mathbb{C}_{\text{stiff-int}} \oplus \mathbb{C}_{\text{stiff-ls}}$, and (upon a unitary transformation) is $O(\varepsilon^2)$ close to $A_\varepsilon^{(\tau)}$ in the norm-resolvent sense.

This estimate is uniform in $\tau \in Q'$ and $z \in K_\sigma$ (a compact set $\sigma > 0$ distance away from the real line.)

# Wave propagation

# Spectral properties of DTN

**Proposition** For all $\tau \in Q' = [-\pi, \pi)^d$, the DTN ops are SA, semibounded from *above*, and have compact resolvent. If we order evalues in descending order (counting multiplicities, then)

- The evalues of $\tilde{\Lambda}^{\text{stiff}-\text{int},(\tau)}$ (unweighted) satisfies

$$\text{For all } \tau,\ 0 = \mu_1^{\text{stiff}-\text{int},(\tau)} > \mu_2^{\text{stiff}-\text{int},(\tau)} \geq \mu_3^{\text{stiff}-\text{int},(\tau)} \geq \cdots \to -\infty$$

- The evalues of $\tilde{\Lambda}^{\text{stiff}-ls,(\tau)}$ satisfies

$$\text{If } \tau = 0, \text{ then } 0 = \mu_1^{\text{stiff}-\text{int},(\tau)} > \mu_2^{\text{stiff}-\text{int},(\tau)} \geq \mu_3^{\text{stiff}-\text{int},(\tau)} \geq \cdots \to -\infty$$

$$\text{If } \tau \neq 0, \text{ then } 0 > \mu_1^{\text{stiff}-\text{int},(\tau)} > \mu_2^{\text{stiff}-\text{int},(\tau)} \geq \mu_3^{\text{stiff}-\text{int},(\tau)} \geq \cdots \to -\infty$$

The first evalue admits an asymp expansion in $\tau$, with quadratic leading order term

$$\mu_1^{\text{stiff}-\text{ls},(\tau)} = \mu_*^{\text{stiff}-\text{ls},(\tau)} \tau \cdot \tau + O(|\tau|^3), \qquad \mu_*^{\text{stiff}-\text{ls},(\tau)} \text{ is a (strictly) neg-definite matrix}$$

# Plug in the values…

Vanishes because

$\mu_1^{(\tau)} = 0$ for all $\tau$!

$$A_{\varepsilon,\mathrm{hom}}^{(\tau)}\begin{pmatrix} u \\ \beta_{\text{stiff−int}} \\ \beta_{\text{stiff−ls}} \end{pmatrix} = \begin{pmatrix} -(\nabla + i\tau)^2 u \\ -\left(\left(j_{\text{stiff−int}}^{(\tau)} \breve{\Pi}^{\text{stiff−int},(\tau)}\right)^*\right)^{-1} \mathcal{P}_{\text{stiff−int}}^{(\tau)}\left[\Gamma_1^{\text{soft},(\tau)} u + \varepsilon^{-2}\mu_1^{\text{stiff−int},(\tau)}\left(j_{\text{stiff−int}}^{(\tau)} \breve{\Pi}^{\text{stiff−int},(\tau)}\right)^{-1}\beta_{\text{stiff−int}}\right] \\ -\left(\left(j_{\text{stiff−ls}}^{(\tau)} \breve{\Pi}^{\text{stiff−ls},(\tau)}\right)^*\right)^{-1} \mathcal{P}_{\text{stiff−ls}}^{(\tau)}\left[\Gamma_1^{\text{soft},(\tau)} u + \varepsilon^{-2}\mu_1^{\text{stiff−ls},(\tau)}\left(j_{\text{stiff−ls}}^{(\tau)} \breve{\Pi}^{\text{stiff−ls},(\tau)}\right)^{-1}\beta_{\text{stiff−ls}}\right] \end{pmatrix}$$

Quadratic in $\tau$

$$=: \begin{pmatrix} -(\nabla + i\tau)^2 u \\ T_{\varepsilon,\text{stiff−int}}^{(\tau)}(u, \beta_{\text{stiff−int}}, \beta_{\text{stiff−}ls}) \\ T_{\varepsilon,\text{stiff−}ls}^{(\tau)}(u, \beta_{\text{stiff−int}}, \beta_{\text{stiff−}ls}) \end{pmatrix}$$

# Finding the "dispersion reln"

How do waves propagate in the stiff-interior region and the stiff-landscape region?

Let's start with stiff-interior. Consider the following operator from $\mathbb{C}_{\text{stiff}-\text{int}}$ to $\mathbb{C}_{\text{stiff}-\text{int}}$:

$$P_{\mathbb{C}_{\text{stiff}-\text{int}}}\left(A_{\varepsilon,\text{hom}}^{(\tau)}-z\right)^{-1}P_{\mathbb{C}_{\text{stiff}-\text{int}}}$$

Suppose that we can write the "resolvent eqn"

$$P_{\mathbb{C}_{\text{stiff}-\text{int}}}\left(A_{\varepsilon,\text{hom}}^{(\tau)}-z\right)^{-1}P_{\mathbb{C}_{\text{stiff}-\text{int}}}\beta-z\beta=\delta,\qquad \delta\in\mathbb{C}_{\text{stiff}-\text{int}}$$

In the form

$$(K(\tau,z)-z)\beta=\delta$$

We will call $K$ the "dispersion function"

Then,

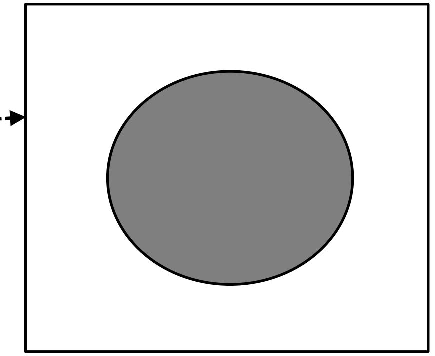
$$P_{\mathbb{C}_{\text{stiff}-\text{int}}}\left(A_{\varepsilon,\text{hom}}^{(\tau)}-z\right)^{-1}P_{\mathbb{C}_{\text{stiff}-\text{int}}}=M_{(K(\tau,z)-z)^{-1}}$$

# Dispersion fct for stiff-int

$$v \coloneqq \Pi^{\mathrm{soft},(\tau)}\left(\psi_1^{\mathrm{stiff-int},(\tau)}, 0\right), \qquad w \coloneqq \Pi^{\mathrm{soft},(\tau)}\left(0,\ \psi_1^{\mathrm{stiff-ls},(\tau)}\right), \qquad \Psi_1^{\mathrm{stiff-ls},(\tau)} \coloneqq \Pi^{\mathrm{stiff-ls},(\tau)} \psi_1^{\mathrm{stiff-ls},(\tau)}$$

Then (omitting dependence on $\varepsilon$ and $\tau$ for brevity),

Dispersion fct for this setup



$$K_{\mathrm{stiff-int}}(\tau, z) = \frac{1}{\left\|\Psi_1^{stiff-int}\right\|} T_{\mathrm{stiff-int}} \begin{pmatrix} z\left(A_0^{\mathrm{soft}} - z\right)^{-1} v + v \\ \left\|\Psi_1^{\mathrm{stiff-int}}\right\| \\ 0 \end{pmatrix} +$$

$$\frac{1}{z\left\|\Psi_1^{\mathrm{stiff-ls}}\right\|\left\|\Psi_1^{\mathrm{stiff-int}}\right\|} T_{\mathrm{stiff-int}} \begin{pmatrix} z\left(A_0^{\mathrm{soft}} - z\right)^{-1} w + w \\ 0 \\ \left\|\Psi_1^{\mathrm{stiff-ls}}\right\| \end{pmatrix} T_{\mathrm{stiff-ls}} \begin{pmatrix} z\left(A_0^{\mathrm{soft}} - z\right)^{-1} v + v \\ \left\|\Psi_1^{\mathrm{stiff-int}}\right\| \\ 0 \end{pmatrix} \Bigg/ \left[1 - \frac{1}{z\left\|\Psi_1^{\mathrm{stiff-ls}}\right\|} T_{\mathrm{stiff-ls}} \begin{pmatrix} z\left(A_0^{\mathrm{soft}} - z\right)^{-1} w + w \\ 0 \\ \left\|\Psi_1^{\mathrm{stiff-ls}}\right\| \end{pmatrix}\right]$$

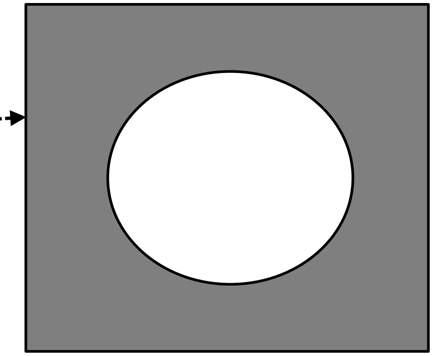
(Negligible) correction term

# Dispersion fct for stiff-ls

$$v := \Pi^{\text{soft},(\tau)}(\psi_1^{\text{stiff-int},(\tau)}, 0), \qquad w := \Pi^{\text{soft},(\tau)}(0,\ \psi_1^{\text{stiff-ls},(\tau)}), \qquad \Psi_1^{\text{stiff-ls},(\tau)} := \Pi^{\text{stiff-ls},(\tau)}\psi_1^{\text{stiff-ls},(\tau)}$$

Then (omitting dependence on $\varepsilon$ and $\tau$ for brevity),

Dispersion fct for this setup



$$K_{\text{stiff-ls}}(\tau, z) = \frac{1}{\|\Psi_1^{\text{stiff-ls}}\|} T_{\text{stiff-ls}} \begin{pmatrix} z\left(A_0^{\text{soft}} - z\right)^{-1} w + w \\ 0 \\ \|\Psi_1^{\text{stiff-ls}}\| \end{pmatrix} +$$

$$\left(\widetilde{K}(\varepsilon\theta, z) - z\right)^{-1} = \left(a^{\text{hom}}\,\theta\cdot\theta - \beta(z)\right)^{-1}$$

$$\frac{1}{z\|\Psi_1^{\text{stiff-ls}}\|\|\Psi_1^{\text{stiff-int}}\|} T_{\text{stiff-int}} \begin{pmatrix} z\left(A_0^{\text{soft}} - z\right)^{-1} w + w \\ 0 \\ \|\Psi_1^{\text{stiff-ls}}\| \end{pmatrix} T_{\text{stiff-ls}} \begin{pmatrix} z\left(A_0^{\text{soft}} - z\right)^{-1} v + v \\ \|\Psi_1^{\text{stiff-int}}\| \\ 0 \end{pmatrix} \Bigg/ \left[1 - \frac{1}{z\|\Psi_1^{\text{stiff-int}}\|} T_{\text{stiff-int}} \begin{pmatrix} z\left(A_0^{\text{soft}} - z\right)^{-1} v + v \\ \|\Psi_1^{\text{stiff-int}}\| \\ 0 \end{pmatrix}\right]$$

Correction term

# Justification of term “dispersion function”

- For const. coefficient PDEs

Spacetime Fourier transform

$$\partial_{tt}u - \Delta u = 0 \dashrightarrow \omega^2 + |\xi|^2 = 0$$

“seek *plane-wave* solutions $e^{i(\xi \cdot x - \omega t)}$”.

- Alternatively, notice that if $p(\xi) = |\xi|^2$ and $D = -i\nabla$ and $\mathcal{F}$ =FT in space, then $-\Delta = p(D)$ and

$$p(D) = \mathcal{F}^{-1} M_p \mathcal{F}$$

The poly $p$ gives us the “spatial part” of the dispersion reln in !!!

So, we call $p$ the “dispersion function”

(e.g. if $\partial_{tt}u$ is replaced by $i\partial_t u$ (Schrodinger eqn), then $p$ coincides the dispersion relation.)

# Justification

- Generalize the definition to our case:
  - Write $A = A_{pp} \oplus A_{ac} \oplus A_{sc}$. Suppose that we can find a unitary $U$ such that $A_{ac} = U^* M_p U$ where $M_p$is the mult. operator by $p(x)$ on $L^2(\Omega, \textbf{Leb})$. Then we call $p(x)$ the "dispersion function".
- Observe that with $f_z(\lambda) = \frac{1}{(\lambda - z)}$ ,

$$(A_{ac} - z)^{-1} = f_z(A_{ac}) = U^* f_z\left(M_p\right) U = U^* M_{f_z \circ p} U = U^* M_{(p(\lambda)-z)^{-1}} U$$

- In our case, we had for each $\tau$ the following operator on $\mathbb{C}$:

$$P_{\mathbb{C}} \left(A_{\varepsilon,\mathrm{hom}}^{(\tau)} - z\right)^{-1} P_{\mathbb{C}} = M_{(K(\tau,z)-z)^{-1}}$$

In general, we allow the this fct to depend on spectral parameter $z$. This corresponds to incorporating the "temporal part" of the dispersion reln.

Therefore, on $\int_{Q'}^{\oplus} \mathbb{C} d\tau \cong L^2(Q';\mathbb{C})$, we will have a mult. operator by $\tau \mapsto (K(\tau, z) - z)^{-1}$

Hence, by definition , $K$ is our dispersion fct.

# Thank you!