Norm-resolvent asymptotics for strongly elliptic systems in the setting of high-contrast homogenisation

Workshop - Threshold phenomena in spectral analysis and their applications to waves and tectonics

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19 July 2022, Mexico city, Mexico

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The structure of the talk

1. Heterogeneous media - from simple composites to metamaterials

2. Strongly elliptic systems in high-contrast - main results

3. Proof ideas and tools

Heterogeneous media - mild contrast

Small parameter (period of material oscillations) $\varepsilon > 0$.

Oscillatory periodic heterogeneous media

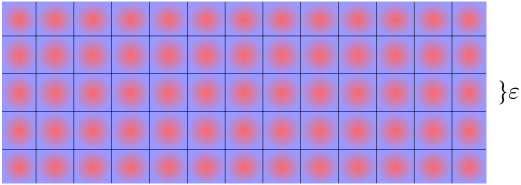
- The material properties are stored in the coefficient matrix function $\mathbb{C}_\varepsilon(x)$
- $\mathbb{C}_\varepsilon(x) = \mathbb{C}(\frac{x}{\varepsilon})$.
- $\mathbb{C}$ is Y-periodic on $\mathbb{R}^d$, $Y = [0,1]^d$.

The matrix $\mathbb{C}$ is symmetric, and $\exists \alpha, \beta > 0$, such that:

$$\alpha|\xi|^2 \leqslant \mathbb{C}(x)\xi \cdot \xi \leqslant \beta|\xi|^2, \quad \forall x, \xi \in \mathbb{R}^d.$$

Elliptic operator:

$$\mathcal{A}_\varepsilon \boldsymbol{u} := -\operatorname{div}\left(\mathbb{C}_\varepsilon(x)\nabla \boldsymbol{u}\right), \quad \mathcal{D}\left(\mathcal{A}_\varepsilon\right) \subset H^1(\mathbb{R}^d).$$



Homogenisation in mild contrast

Heterogeneous problem

$$\int_{\mathbb{R}^d} \mathbb{C}(x/\varepsilon)\nabla u_\varepsilon \cdot \nabla\varphi dx = \int_{\mathbb{R}^d} f_\varepsilon \varphi dx, \quad \forall \varphi \in H^1(\mathbb{R}^d).$$

Homogenised problem

$$\int_{\mathbb{R}^d} \mathbb{C}^{\text{hom}}\nabla u \cdot \nabla\varphi dx = \int_{\mathbb{R}^d} f \varphi dx, \quad \forall \varphi \in H^1(\mathbb{R}^d).$$

$$\mathbb{C}^{\text{hom}}\xi \cdot \eta := \int_Y \mathbb{C}(y)[\xi + \nabla_y w_\xi(y)] \cdot \eta, \quad \int_Y \mathbb{C}(y)[\xi + \nabla_y w_\xi(y)] \cdot \nabla_y v(y) = 0.$$

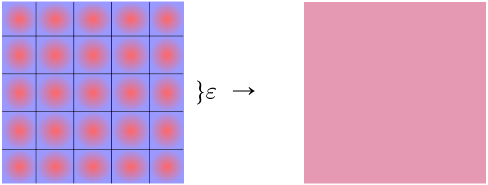
Operator with constant coefficients:

$$\mathcal{A}_0 \boldsymbol{u} := -\operatorname{div}\left(\mathbb{C}^{\text{hom}}\nabla \boldsymbol{u}\right), \quad \mathcal{D}\left(\mathcal{A}_0\right) = H^2(\mathbb{R}^d).$$

Qualitative and quantitative results

Qualitative results given with two-scale convergence

$$(\mathcal{A}_\varepsilon + I)^{-1} f_\varepsilon \xrightarrow{2} (\mathcal{A}_0 + I)^{-1} f, \quad \forall f_\varepsilon \xrightarrow{2} f.$$



Quantitative results are given with the norm-resolvent estimates
(Birman, Suslina 2001., 2005., 2006., ...):

Quantitative result

$$\|(\mathcal{A}_\varepsilon + I)^{-1} - (\mathcal{A} + I)^{-1}\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} \leqslant C\varepsilon,$$

$$\|(\mathcal{A}_\varepsilon + I)^{-1} - (\mathcal{A} + I)^{-1} - \varepsilon\mathcal{R}_{\mathrm{corr}}(\varepsilon)\|_{L^2(\mathbb{R}^d)\to H^1(\mathbb{R}^d)} \leqslant C\varepsilon,$$

$$\|(\mathcal{A}_\varepsilon + I)^{-1} - (\mathcal{A} + I)^{-1} - \varepsilon\widehat{\mathcal{R}}_{\mathrm{corr}}(\varepsilon)\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} \leqslant C\varepsilon^2,$$

Analysis of periodic operators

Gelfand transform

For $\boldsymbol{f} \in L^2(\mathbb{R}^d)$ its Gelfand transform $\mathcal{G}\boldsymbol{f} \in L^2_\#(Y; L^2(Q))$, $Q = [-\pi, \pi)^d$

$$\mathcal{G}\boldsymbol{f}(\chi, y) = \left(\frac{1}{2\pi}\right)^{d/2} \sum_{n \in \mathbb{Z}^n} \boldsymbol{f}\big(y+n\big) e^{-i\chi(y+n)}.$$

Spectral resolution:

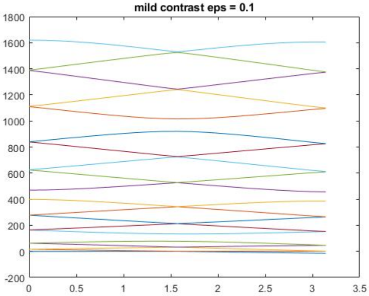
$$L^2(\mathbb{R}^n) = \int_Q^{\oplus} L^2_\#(Y, \chi) d\chi, \quad (\mathcal{A} + zI)^{-1} = \mathcal{G}^{-1} \left(\int_Q^{\oplus} (\mathcal{A}_\chi + zI)^{-1} \, d\chi \right) \mathcal{G}$$

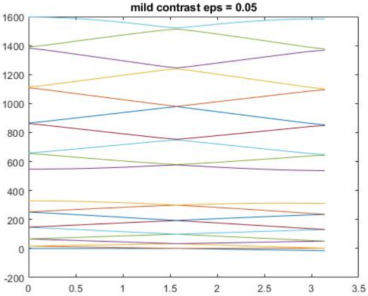
$$\mathcal{A}_\chi := (\nabla + i\chi)^* \mathbb{C}(y)(\nabla + i\chi) : \mathcal{D}(\mathcal{A}_\chi) \subset H^1_\#(Y, \mathbb{C}^3) \to L^2(Y, \mathbb{C}^3)$$

Spectral decomposition and band structure

$$\sigma(\mathcal{A}) = \bigcup_{\chi \in Q} \sigma(\mathcal{A}_\chi) = \bigcup_{n \in \mathbb{N}} \left(\bigcup_{\chi \in Q} \lambda_{n,\chi} \right) = \bigcup_{n \in \mathbb{N}} [\underline{\lambda_n}, \overline{\lambda_n}]$$

Quick numerical experiment in 1D





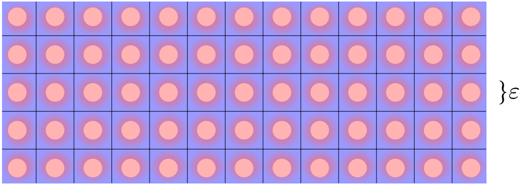
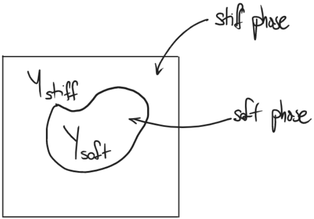


Figure: Depiction of a material with high-contrast inclusions



- Tensor of material coefficients:

$$\mathbb{C}_\varepsilon(y) = \begin{cases} \mathbb{C}_{\text{stiff}}(y), & y \in Y_{\text{stiff}}, \\ \varepsilon^2\, \mathbb{C}_{\text{soft}}(y), & y \in Y_{\text{soft}}. \end{cases}$$

- $\mathbb{C}_{\text{stiff}}$, $\mathbb{C}_{\text{soft}}$ uniformly positive definite.

Homogenisation in high contrast

Heterogeneous problem

$\mathbb{C}_\varepsilon(y) = \varepsilon^2 \chi_{\text{soft}}(y)\mathbb{C}_{\text{soft}}(y) + \chi_{\text{stiff}}(y)\mathbb{C}_{\text{stiff}}(y)$

$$\int_{\mathbb{R}^d} \mathbb{C}_\varepsilon(x/\varepsilon)\nabla u_\varepsilon \cdot \nabla \varphi dx = \int_{\mathbb{R}^d} f\varphi dx, \quad \forall \varphi \in H^1(\mathbb{R}^d).$$

Homogenised problem

$$\begin{cases} \displaystyle\int_{\mathbb{R}^d} \mathbb{C}^{\text{hom}} \nabla u \cdot \nabla \varphi dx = \int_{\mathbb{R}^d}\int_{Y_{\text{stiff}}} f(x,y)\varphi(x) dx, \quad \forall \varphi \in H^1(\mathbb{R}^d). \\ \displaystyle\int_{\mathbb{R}^d}\int_{Y_{\text{soft}}} \mathbb{C}(y)\nabla_y u_0(x,y) \cdot \nabla_y \xi(x,y) dy dx = \int_{\mathbb{R}^d}\int_{Y_{\text{soft}}} f(x,y)\xi(x,y) dy dx, \quad \forall \xi. \end{cases}$$

$$u_\varepsilon(x) \xrightarrow{2} u(x) + u_0(x,y) \in H^1(\mathbb{R}^d) + L^2(\mathbb{R}^d; X),$$

where: $X = \left\{\varphi \in H^1_\#(Y), \quad \varphi = 0 \text{ on } Y_{\text{stiff}}\right\}$

Limit operator and spectral characterisation

Micro and Macro operators

$$\mathcal{A}_{\rm macro} \longleftrightarrow \int_{\mathbb{R}^d} \mathbb{C}^{\rm hom} \nabla u \cdot \nabla \varphi dx, \quad \mathcal{D}(\mathcal{A}_{\rm macro}) \subset H^1(\mathbb{R}^d)$$

$$\mathcal{A}_{\rm micro} \longleftrightarrow \int_{\mathbb{R}^d} \int_{Y_{\rm soft}} \mathbb{C}(y) \nabla_y u_0(x,y) \cdot \nabla_y \xi(x,y) dy dx, \quad \mathcal{D}(\mathcal{A}_{\rm micro}) \subset L^2(\mathbb{R}^d; X).$$

Self-adjoint, nonnegative operator $\mathcal{A}$ defined through bilinear form:

$$\int_{\mathbb{R}^d} \mathbb{C}^{\rm hom} \nabla u \cdot \nabla \varphi dx + \int_{\mathbb{R}^d} \int_{Y_{\rm soft}} \mathbb{C}(y) \nabla_y u_0(x,y) \cdot \nabla_y \xi(x,y) dy dx,$$

$$\mathcal{D}(\mathcal{A}) \subset H^1(\mathbb{R}^d) + L^2(\mathbb{R}^d; X)$$

Theorem

$$\sigma(\mathcal{A}) = \sigma(\mathcal{A}_{\rm micro}) \cup \{\lambda > 0, \quad \beta(\lambda) \in \sigma(\mathcal{A}_{\rm macro})\},$$

$$\beta(\lambda) = \lambda + \sum_{m=1}^{\infty} \frac{\lambda^2 c_m^2}{\omega_m - \lambda}, \quad \omega_m \in \sigma(\mathcal{A}_{\rm micro}).$$

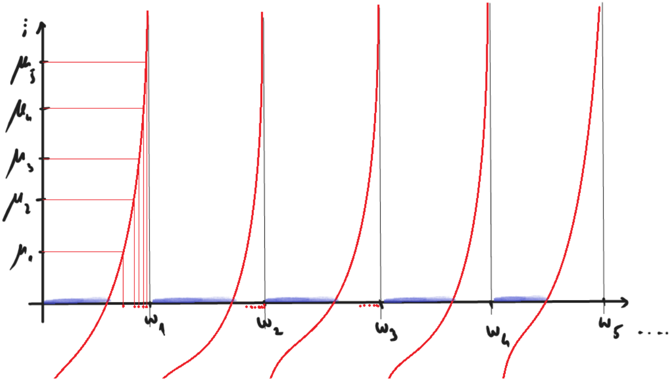
Limit operator

Equivalent formulations of $\mathcal{A}u = \lambda u$

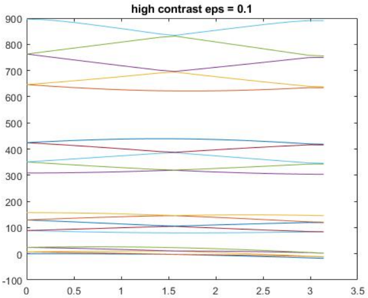
$$\begin{cases} \mathcal{A}_{\text{micro}} u_{\text{micro}} = \lambda(u_{\text{macro}} + u_{\text{micro}}) \\ \mathcal{A}_{\text{macro}} u_{\text{macro}} = \lambda(u_{\text{macro}} + \langle u_{\text{micro}} \rangle) \\ u = u_{\text{micro}} + u_{\text{macro}} \end{cases} \iff \begin{cases} \mathcal{A}_{\text{macro}} u_{\text{macro}} = \beta(\lambda) u_{\text{macro}} \\ u_{\text{macro}} \neq 0 \end{cases}$$

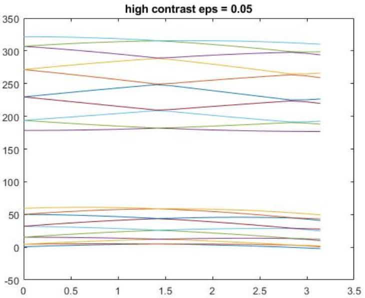
or

$$\begin{cases} \langle u_{\text{micro}} \rangle = u_{\text{macro}} = 0 \\ \mathcal{A}_{\text{micro}} u_{\text{micro}} = \lambda u_{\text{micro}} \end{cases}$$



Quick numerical experiment in 1D

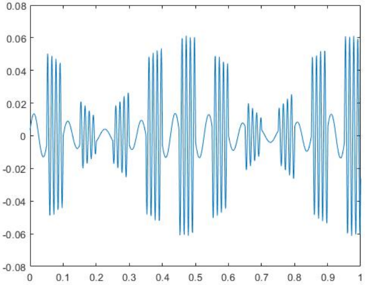


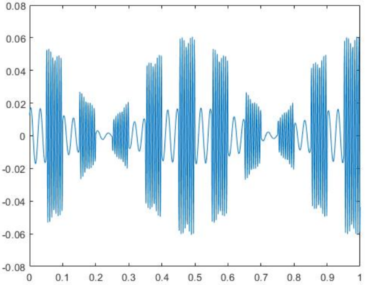


- Gaps remain in the limit spectrum!
- $(\mathcal{A}_\varepsilon + zI)^{-1} \xrightarrow{2} (\mathcal{A} + zI)^{-1}, \quad P_{H^1(\mathbb{R}^d)}(\mathcal{A} + zI)^{-1}|_{H^1(\mathbb{R}^d)} = (\mathcal{A} + \beta(z)I)^{-1}.$
- metamaterials, time non-locality

Quick numerical experiment in 1D

Some eigenfunctions for $\varepsilon = 0.05$.





Problem setting - strongly elliptic systems

Nonnegative selfadjoint operator family

$$\mathcal{A}_\varepsilon \cdot := -\mathrm{div}\left(\mathbb{C}^\varepsilon(\tfrac{x}{\varepsilon})\nabla\cdot\right), \quad \mathcal{D}(\mathcal{A}_\varepsilon) \subset H^1(\mathbb{R}^3,\mathbb{R}^3), \quad \varepsilon > 0,$$

$$a_\varepsilon(\boldsymbol{u},\boldsymbol{v}) := \int_{\mathbb{R}^3} \mathbb{C}^\varepsilon\left(\tfrac{x}{\varepsilon}\right)\nabla\boldsymbol{u} : \nabla\boldsymbol{v}, \quad \boldsymbol{u},\boldsymbol{v} \in H^1(\mathbb{R}^3;\mathbb{R}^3).$$

The tensors of material coefficients $\mathbb{C}^\varepsilon$ with high-contrast structure:

$$\mathbb{C}^\varepsilon(y) = \begin{cases} \mathbb{C}_{\mathrm{stiff}}(y), & y \in Y_{\mathrm{stiff}}, \\ \varepsilon^2\,\mathbb{C}_{\mathrm{soft}}(y), & y \in Y_{\mathrm{soft}}, \end{cases}$$

- The operator $\mathcal{A}_\varepsilon$ is strongly elliptic, $\exists\nu > 0$ such that

$$\nu|\xi|^2|\eta|^2 \leqslant \mathbb{C}_{\mathrm{stiff(soft)}}(y)\left(\xi\eta^T\right) : \left(\xi\eta^T\right) \leqslant \frac{1}{\nu}|\xi|^2|\eta|^2, \quad \forall\xi,\eta \in \mathbb{R}^3, \quad \forall y \in Y.$$

- The coefficients of $\mathbb{C}_{\mathrm{stiff(soft)}}$ are Lipschitz continuous

$$\left[\mathbb{C}_{\mathrm{stiff(soft)}}\right]_{i,j} \in \mathcal{C}^{0,1}(\overline{Y_{\mathrm{stiff(soft)}}}).$$

- The boundary $\Gamma = \partial Y_{\mathrm{soft}}$ is $\mathcal{C}^{1,1}$.

Questions:

- Resolvent asymptotics and norm-resolvent estimates?
- Where do metamaterial properties show up?
- Spectral approximation, other consequences

Main results:

Theorem

Let $z \in \rho(\mathcal{A}_\varepsilon)$*, then we have:*

$$\left\| (\mathcal{A}_\varepsilon - zI)^{-1} - \Theta_\varepsilon^* \left(\mathcal{A}_\varepsilon^{\text{hom}} - zI\right)^{-1} \Theta_\varepsilon \right\|_{L^2(\mathbb{R}^3)\to L^2(\mathbb{R}^3)} \leqslant C\varepsilon^2,$$

where $\mathcal{A}_\varepsilon^{\text{hom}}$ *is pseudodifferential operator, and* Θ_ε *is a partial isometry.*

Theorem

Let $z \in \rho(\mathcal{A}_\varepsilon)$*, then we have:*

$$\left\| P_\varepsilon^{\text{stiff}} (\mathcal{A}_\varepsilon - zI)^{-1} |_{L^2(\Omega_{\text{stiff}}^\varepsilon)} - \Theta_\varepsilon^* (\mathcal{A}_{\text{macro}} - \mathcal{B}(z))^{-1} \Theta_\varepsilon \right\|_{L^2(\Omega_{\text{stiff}}^\varepsilon)\to L^2(\Omega_{\text{stiff}}^\varepsilon)} \leqslant C\varepsilon,$$

where $\mathcal{A}_{\text{macro}}$ *is a strongly elliptic differential operator with constant coefficients,* Θ_ε *is a unitary operator from* $L^2(\Omega_{\text{stiff}}^\varepsilon)$ *to* $L^2(\mathbb{R}^3)$*, and* $\mathcal{B}(z)$ *is a matrix-valued Zhikov function.*

Similar theorems in works of Cherednichenko, Ershova, Kiselev for the case of elliptic operators in high contrast (scalar case).

Scaled Gelfand transform

Unitary operator:

$$\mathcal{G}_\varepsilon : L^2(\mathbb{R}^3) \to L^2(Y'; L^2_\#(Y;\mathbb{C}^3)) = \int_{Y'}^{\oplus} L^2_\#(Y;\mathbb{C}^3,\chi)d\chi,$$

$$(\mathcal{G}_\varepsilon \boldsymbol{u})(y,\chi) := \frac{\varepsilon}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}^3} e^{-i\chi(y+n)} \boldsymbol{u}(\varepsilon(y+n)), \quad y \in \mathbb{R}^3, \quad \chi \in Y' = [-\pi,\pi\rangle^3,$$

Resolvent decomposition

$$(\mathcal{A}_\varepsilon + zI)^{-1} = \mathcal{G}_\varepsilon^{-1} \left(\int_{Y'}^{\oplus} \left(\frac{1}{\varepsilon^2} \mathcal{A}_{\chi,\varepsilon} + zI \right)^{-1} d\chi \right) \mathcal{G}_\varepsilon.$$

χ-dependent operator family

$$\mathcal{A}_{\chi,\varepsilon} := (\nabla + iX_\chi)^* \mathbb{C}^\varepsilon(y)(\nabla + iX_\chi) : \mathcal{D}(\mathcal{A}_{\chi,\varepsilon}) \subset H^1_\#(Y,\mathbb{C}^3) \to L^2(Y,\mathbb{C}^3)$$

with the associated bilinear forms

$$a_{\chi,\varepsilon}(\boldsymbol{u},\boldsymbol{v}) := \int_Y \mathbb{C}^\varepsilon(y)(\nabla + iX_\chi)\boldsymbol{u} : \overline{(\nabla + iX_\chi)\boldsymbol{v}}, \quad \boldsymbol{u},\boldsymbol{v} \in H^1_\#(Y;\mathbb{C}^3).$$

Transmission boundary problem

Definition (transmission boundary problem = transformed resolvent problem)

Find $\boldsymbol{u} \in L^2_{\#}(Y;\mathbb{C}^3)$ such that $\boldsymbol{u}|_{Y_{\text{stiff}}} \in H^1(Y_{\text{stiff}},\mathbb{C}^3)$, $\boldsymbol{u}|_{Y_{\text{soft}}} \in H^1(Y_{\text{soft}},\mathbb{C}^3)$ and the following is valid:

$$\begin{aligned}
-\frac{1}{\varepsilon^2}\left(\nabla + iX_\chi\right)^* \mathbb{C}_{\text{stiff}}(y)\left(\nabla + iX_\chi\right)\boldsymbol{u} + z\boldsymbol{u} &= \boldsymbol{f}, \quad \text{on } Y_{\text{stiff}},\\
-\left(\nabla + iX_\chi\right)^* \mathbb{C}_{\text{soft}}(y)\left(\nabla + iX_\chi\right)\boldsymbol{u} + z\boldsymbol{u} &= \boldsymbol{f}, \quad \text{on } Y_{\text{soft}},\\
\boldsymbol{u}_+(y) - \boldsymbol{u}_-(y) &= 0, \quad y \in \Gamma,\\
\frac{1}{\varepsilon^2}\mathbb{C}_{\text{stiff}}(y)\left(\nabla + iX_\chi\right)\boldsymbol{u}_+ \cdot \vec{n}_+ + \mathbb{C}_{\text{soft}}(y)\left(\nabla + iX_\chi\right)\boldsymbol{u}_- \cdot \vec{n}_- &= 0, \quad y \in \Gamma.
\end{aligned}$$

$$\left(\frac{1}{\varepsilon^2}\mathcal{A}_{\chi,\varepsilon} + zI\right)\boldsymbol{u} = \boldsymbol{f}, \text{ on } Y, \quad \boldsymbol{u} \in H^1_{\#}(Y,\mathbb{C}^3)$$

The framework of boundary operators

Harmonic lift operators

$\Pi_\chi^{\mathrm{stiff(soft)}} : L^2(\Gamma) \to L^2(Y_{\mathrm{stiff(soft)}})$ bounded operators (harmonic lifts) defined with:

$$\Pi_\chi^{\mathrm{stiff(soft)}} \boldsymbol{g} := \boldsymbol{u}, \quad \begin{cases} \mathcal{A}_\chi^{\mathrm{soft(stiff)}} \boldsymbol{u} = 0, \\ \boldsymbol{u} = \boldsymbol{g} \text{ on } \Gamma. \end{cases}$$

Self-adjoint restrictions of $\mathcal{A}_\chi^{\mathrm{soft(stiff)}}$

$\mathcal{A}_{0,\chi}^{\mathrm{soft(stiff)}} \boldsymbol{u} := (\nabla + iX_\chi)^* \, \mathbb{C}_{\mathrm{soft(stiff)}}(y) \, (\nabla + iX_\chi) \, \boldsymbol{u}$ self-adjoint operators with $\mathcal{D}(\mathcal{A}_{0,\chi}^{\mathrm{soft(stiff)}}) \subset \{\boldsymbol{u} \in H^1_\#(Y_{\mathrm{soft(stiff)}}), \ \boldsymbol{u}|_\Gamma = 0\}$,

Boundary operators

- $\Gamma_{0,\chi}^{\mathrm{stiff(soft)}} \boldsymbol{u} := \boldsymbol{u}|_\Gamma$ left inverses of $\Pi_\chi^{\mathrm{stiff(soft)}}$.
- $\Gamma_{1,\chi}^{\mathrm{stiff(soft)}} \boldsymbol{u} := -\mathbb{C}_{\mathrm{stiff(soft)}}(y) \left(\nabla \boldsymbol{u} + iX_\chi \boldsymbol{u}\right)|_\Gamma \cdot \vec{n}$ traces of normal derivative.

Dirichlet-to-Neumann maps

$\Lambda_\chi^{\mathrm{stiff(soft)}} := \Gamma_{1,\chi}^{\mathrm{stiff(soft)}} \Pi_\chi^{\mathrm{stiff(soft)}}$ self-adjoint operators on $L^2(\Gamma)$ with the domain $H^1(\Gamma)$.

The framework of boundary operators

Decomposition of the space $L^2(Y) = L^2(Y_{\text{stiff}}) \oplus L^2(Y_{\text{soft}})$

Operators related to the transmission problem

- $\mathcal{A}_{0,\chi,\varepsilon} := \frac{1}{\varepsilon^2}\mathcal{A}_{0,\chi}^{\text{stiff}} \oplus \mathcal{A}_{0,\chi}^{\text{soft}}$, self-adjoint operator
- $\Pi_\chi := \Pi_\chi^{\text{stiff}} \oplus \Pi_\chi^{\text{soft}}$, "harmonic lift" from the boundary Γ
- $\Lambda_{\chi,\varepsilon} := \frac{1}{\varepsilon^2}\Lambda_\chi^{\text{stiff}} + \Lambda_\chi^{\text{soft}}$ - jump in the normal derivative on Γ (self-adjoint DTN map)

Krein formula representation of the resolvent (Ryzhov 2009.)

The resolvent associated with the transmission boundary problem:

$$\left(\frac{1}{\varepsilon^2}\mathcal{A}_{\chi,\varepsilon} - zI\right)^{-1} = \left(\mathcal{A}_{0,\chi,\varepsilon} - zI\right)^{-1} - \left(I - z\mathcal{A}_{0,\chi,\varepsilon}^{-1}\right)^{-1}\Pi_\chi M_{\chi,\varepsilon}(z)^{-1}\left(\Pi_\chi^*\left(I - z\mathcal{A}_{0,\chi,\varepsilon}^{-1}\right)^{-1}\right),$$

where

$$M_{\chi,\varepsilon}(z) = \Lambda_{\chi,\varepsilon} + z\left(\Pi_\chi^*\Pi_\chi\right) + z^2\left(\Pi_\chi^*\left(\mathcal{A}_{0,\chi,\varepsilon} - zI\right)^{-1}\Pi_\chi\right).$$

M-function

$M(z)$ is a Dirichlet-to-Neumann map associated with the resolvent problem.

Proposition

(Properties of the Weyl M-function)

- *The following representation holds:*

$$M_{\chi,\varepsilon}(z) = \Lambda_{\chi,\varepsilon} + z\Pi_\chi^* \left(I - z(\mathcal{A}_{0,\chi,\varepsilon})^{-1}\right)^{-1} \Pi_\chi, \quad z \in \rho(\mathcal{A}_{0,\chi,\varepsilon}). \tag{1}$$

- $M_{\chi,\varepsilon}(z)$ *is an analytic operator-valued function with values in the set of closed operators in* $L^2(\Gamma)$ *defined on the* z*-independent domain* $\mathcal{D}(\Lambda_{\chi,\varepsilon})$*.*
- *For* $\boldsymbol{u} \in \ker(\mathcal{A}_{\chi,\varepsilon} - zI) \cap \left\{\mathcal{D}(\mathcal{A}_{0,\chi,\varepsilon}) \dot{+} \Pi_\chi \mathcal{D}(\Lambda_{\chi,\varepsilon})\right\}$*, the following formula holds:*

$$M_{\chi,\varepsilon}(z)\Gamma_{0,\chi,\varepsilon}\boldsymbol{u} = \Gamma_{1,\chi,\varepsilon}\boldsymbol{u}. \tag{2}$$

Decomposition of M-function

$$M_{\chi,\varepsilon}(z) = M_\chi^{\text{soft}}(z) + \frac{1}{\varepsilon^2} M_\chi^{\text{stiff}}(\varepsilon^2 z), \quad z \in \rho(\mathcal{A}_{0,\chi,\varepsilon}).$$

The asymptotics of M-function and Steklov truncation

Stiff component of M-function

$$M_\chi^{\text{stiff}}(\varepsilon^2 z) = \Lambda_\chi^{\text{stiff}} + \varepsilon^2 z\left(\left(\Pi_\chi^{\text{stiff}}\right)^* \Pi_\chi^{\text{stiff}}\right) + \varepsilon^4 z^2\left(\left(\Pi_\chi^{\text{stiff}}\right)^* \left(\mathcal{A}_{0,\chi}^{\text{stiff}} - \varepsilon^2 z I\right)^{-1} \Pi_\chi^{\text{stiff}}\right)$$

$$\frac{1}{\varepsilon^2} M_\chi^{\text{stiff}}(\varepsilon^2 z) = \frac{1}{\varepsilon^2}\Lambda_\chi^{\text{stiff}} + z\left(\Pi_\chi^{\text{stiff}}\right)^* \Pi_\chi^{\text{stiff}} + \sum_{n=1}^{\infty} \varepsilon^{2n} z^n \left(\Pi_\chi^{\text{stiff}}\right)^* \left(\mathcal{A}_{0,\chi}^{\text{stiff}}\right)^{-n} \Pi_\chi^{\text{stiff}}.$$

Lemma (The order in χ of Steklov eigenvalues)

The spectrum of $\Lambda_\chi^{\text{stiff}}$ consists of three lowest eigenvalues of order $\mathcal{O}(|\chi|^2)$ with the rest being of order $\mathcal{O}(1)$.

- $\widehat{P}_\chi$ orthogonal projection on the $3-$dimensional subspace of $L^2(\Gamma)$ (with Steklov eigenvalues of order $|\chi|^2$).
- $\widehat{\Pi}_\chi^{\text{stiff}} := \Pi_\chi^{\text{stiff}} \widehat{P}_\chi, \qquad \widehat{\mathcal{H}}_\chi^{\text{stiff}} = \widehat{\Pi}_\chi^{\text{stiff}} L^2(\Gamma), \qquad \widehat{\Lambda}_\chi^{\text{stiff}} := \widehat{P}_\chi \Lambda_\chi^{\text{stiff}} \widehat{P}_\chi.$

Fiberwise approximation results

Effective operator

$$\mathcal{D}\left(\mathcal{A}_{\chi,\varepsilon}^{\text{hom}}\right) := \left\{(\boldsymbol{u}, \widehat{\boldsymbol{u}}) \in L^2_{\#}(Y_{\text{soft}}) \oplus \widehat{\mathcal{H}}_\chi^{\text{stiff}}, \quad \boldsymbol{u} \in \mathcal{D}(\mathcal{A}_\chi^{\text{soft}}), \quad \widehat{\boldsymbol{u}} = \widehat{\Pi}_\chi^{\text{stiff}} \Gamma_{0,\chi}^{\text{soft}} \boldsymbol{u}\right\},$$

$$\mathcal{A}_{\chi,\varepsilon}^{\text{hom}} \begin{bmatrix} \boldsymbol{u} \\ \widehat{\boldsymbol{u}} \end{bmatrix} := \begin{bmatrix} \mathcal{A}_\chi^{\text{soft}} & 0 \\ -\left(\left(\widehat{\Pi}_\chi^{\text{stiff}}\right)^*\right)^{-1} \Gamma_{1,\chi}^{\text{soft}} & -\frac{1}{\varepsilon^2}\left(\left(\widehat{\Pi}_\chi^{\text{stiff}}\right)^*\right)^{-1} \widehat{\Gamma}_{1,\chi}^{\text{stiff}} \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ \widehat{\boldsymbol{u}} \end{bmatrix}.$$

Theorem (Full resolvent asymptotics)

There exists $C > 0$, such that for the resolvent of the transmission boundary problem we have:

$$\left\| \left(\frac{1}{\varepsilon^2}\mathcal{A}_{\chi,\varepsilon} - zI\right)^{-1} - \Theta_\chi^* \left(\mathcal{A}_{\chi,\varepsilon}^{\text{hom}} - zI\right)^{-1} \Theta_\chi \right\|_{L^2_{\#}(Y) \to L^2_{\#}(Y)} \leqslant C\varepsilon^2,$$

for all $\chi \in Y'$ and $\Theta_\chi : L^2_{\#}(Y) = L^2_{\#}(Y_{\text{soft}}) \oplus L^2_{\#}(Y_{\text{stiff}}) \to L^2_{\#}(Y_{\text{soft}}) \oplus \widehat{\mathcal{H}}_\chi^{\text{stiff}}$ is a projection.

Focusing on the stiff component

Dispersion function for the stiff component

$$\left(\mathcal{K}_{\chi,\varepsilon}(z) - zI\right)^{-1} := P_{\text{stiff}} \left(\mathcal{A}^{\text{hom}}_{\chi,\varepsilon} - zI\right)^{-1} P_{\text{stiff}}$$

$$\mathcal{K}_{\chi,\varepsilon}(z) = -\left(\left(\widehat{\Pi}^{\text{stiff}}_{\chi}\right)^{*}\right)^{-1} \left(\frac{1}{\varepsilon^2}\widehat{\Lambda}^{\text{stiff}}_{\chi} + \widehat{M}^{\text{soft}}_{\chi}(z)\right) \left(\widehat{\Pi}^{\text{stiff}}_{\chi}\right)^{-1}$$

Lemma

The operator $\widehat{\Lambda}^{\text{stiff}}_{\chi}$ *permits the following asymptotics:*

$$\left\|\widehat{\Lambda}^{\text{stiff}}_{\chi} - (iX_{\chi})^{*}\,\Lambda^{\text{hom}} iX_{\chi}\right\|_{\widehat{\mathcal{H}}^{\text{stiff}}_{\chi}\to\widehat{\mathcal{H}}^{\text{stiff}}_{\chi}} \leqslant C|\chi|^{3},$$

where the matrix Λ^{hom} *does not depend on* χ.

Focusing on the stiff component

What about $\widehat{M}_{\chi}^{\mathrm{soft}}(z)$?

$$\widehat{M}_{\chi}^{\mathrm{soft}}(z) = \widehat{\Lambda}_{\chi}^{\mathrm{soft}} + z\widehat{\Gamma}_{1,\chi}^{\mathrm{soft}}\left(\widehat{\mathcal{A}}_{0,\chi}^{\mathrm{soft}} - zI\right)^{-1}\widehat{\Pi}_{\chi}^{\mathrm{soft}}.$$

Spectral decomposition of the resolvent:

$$\left(\mathcal{A}_{0,\chi}^{\mathrm{soft}} - zI\right)^{-1} = \sum_{k=1}^{\infty} \frac{1}{\lambda_k - z}\mathcal{P}_k^{\chi}, \quad \mathcal{P}_k^{\chi}\boldsymbol{f} := \langle \boldsymbol{f}, \varphi_k\rangle\varphi_k.$$

$$\mathcal{B}_{\chi}(z) := \left(\left(\widehat{\Pi}_{\chi}^{\mathrm{stiff}}\right)^{*}\right)^{-1}\left(z\widehat{\Gamma}_{1,\chi}^{\mathrm{soft}}\left(\mathcal{A}_{0,\chi}^{\mathrm{soft}} - zI\right)^{-1}\widehat{\Pi}_{\chi}^{\mathrm{soft}}\right)\left(\widehat{\Pi}_{\chi}^{\mathrm{stiff}}\right)^{-1} + zI,$$

When written in the basis, we obtain:

$$[\mathbb{B}_{\chi}(z)]_{i,j} := z\sum_{k=1}^{\infty}\frac{\lambda_k}{\lambda_k - z}\langle\widehat{\Pi}_{\chi}^{\mathrm{soft}}\psi_i^{\chi}, \varphi_k^{\chi}\rangle\langle\varphi_k^{\chi}, \widehat{\Pi}_{\chi}^{\mathrm{soft}}\psi_j^{\chi}\rangle + z\delta_{i,j}$$

$\Rightarrow$ Zhikov's matrix beta function

Final result

Theorem (Resolvent asymptotics on the truncated space $\widehat{\mathcal{H}}_\chi^{\text{stiff}}$)

$$\left\| P_{\widehat{\mathcal{H}}_\chi^{\text{stiff}}} \left(\frac{1}{\varepsilon^2} \mathcal{A}_{\chi,\varepsilon} - zI \right)^{-1} |_{\widehat{\mathcal{H}}_\chi^{\text{stiff}}} - \left(\frac{1}{\varepsilon^2} (iX_\chi)^* \mathbb{A}^{\text{hom}} iX_\chi - \mathcal{B}(z) \right)^{-1} \right\|_{\widehat{\mathcal{H}}_\chi^{\text{stiff}} \to \widehat{\mathcal{H}}_\chi^{\text{stiff}}} \leqslant C\varepsilon,$$

for all $\chi \in Y'$.

The end

Thank you for attention!