Limiting absorption principle and virtual levels of operators in Banach spaces

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ABSTRACT. We review the concept of the limiting absorption principle and its connection to virtual levels of operators in Banach spaces.

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1 Virtual levels, virtual states

Scattering of neutrons on protons [Wigner [33]]

just a year after discovery of neutron [Chadwick [32]].

- $\boldsymbol{p}^{\uparrow} + \boldsymbol{n}^{\uparrow} \quad \Rightarrow$ deuteron (Deuterium's nucleus); $E_{\mathbf{binding}} \sim 2.2\,\mathrm{MeV}$

- $\boldsymbol{p}^{\uparrow} + \boldsymbol{n}^{\downarrow} \quad \Rightarrow \quad E_{\mathbf{binding}} \sim 0$

  [Fermi [35]]: **real** or **virtual**?

  [Amaldi & Fermi [36]]: **virtual**, $E_{\mathbf{binding}} \sim -67\,\mathrm{KeV}$.

2 Radiation principle. Vladimir Ignatowsky

Helmholtz equation: $(-\Delta - z)u(x) = f(x) \in L^2(\mathbb{R}^3), \quad x \in \mathbb{R}^3, \quad z \in \mathbb{C}$

If $z \notin \overline{\mathbb{R}_+}$: unique L^2-solution,

$$u(x) = (-\Delta - zI)^{-1} f = \frac{e^{-|x|\sqrt{-z}}}{4\pi|x|} * f, \qquad \mathrm{Re}\sqrt{-z} > 0.$$

If $z = k^2$, $k \geq 0$: could be no L^2-solution; then a solution is not unique.

Radiation principle: a way to specify a unique solution.

[Smirnov [41]] credits [Ignatowsky [05]] and [Sommerfeld [12]].

- **Limiting absorption principle**, or LAP [Ignatowsky [05]], [Sveshnikov [50]]:

$$u(x) = \lim_{\varepsilon\to 0+} (-\Delta - (k+\mathbf{i}\boldsymbol{\varepsilon})^2 I)^{-1} f(x), \qquad u(x) \sim \lim_{\varepsilon\to 0+} e^{+\mathrm{i}(k+\mathrm{i}\varepsilon)r} \sim e^{\mathrm{i}kr}.$$

[Sveshnikov [50]], [Povzner [53]], [Eidus [62]], [Vainberg [66]]...

- **Sommerfeld radiation condition** [Sommerfeld [12]]:

$$\lim_{r\to\infty} r\Big(\frac{\partial u}{\partial r} - \mathrm{i}ku\Big) = 0; \qquad u \sim e^{\mathrm{i}kr}$$

- **Limiting amplitude principle** [Tikhonov & Samarskii [48]]

$$\partial_t^2\psi - \Delta\psi = f(x)e^{-\mathrm{i}kt}, \quad (\psi,\,\partial_t\psi)\big|_{t=0} = (0,0); \qquad \psi \sim e^{\mathrm{i}k(r-t)};$$

$$u(x) = \lim_{t\to+\infty} \psi(x,t)e^{\mathrm{i}kt} \sim e^{\mathrm{i}kr}.$$

Vladimir Sergeevich Ignatowsky

1 April 1875 – 30 January 1942

Tiflis–Dresden–Kiev

Reitz-Rotermann factory, Revel (1895–1897)

St. Petersburg University (1898–1906)

Universität Gießen (PhD., 1909)

TH Berlin (1911–1914)

Schneider–Creusot, Paris (1914)

GOMZ (LOMO) (1914–1935)

Phototechnology Institute (1918–1923)

ITMO (1918–1928)

SPBU, Physics/Matmech (1920–1941)

Further development of LAP

- Eigenfunction expansions [Weyl [10], Carleman [34], Titchmarsh [46]];
- Krein's method of **directing functionals** [Krein [46], Krein [48]];
- **Gelfand–Kostyuchenko theory** [Povzner [50], Povzner [53]],
  [Gelfand & Kostyuchenko [55], Berezanskii [57], Birman [61]];
- **Rigged (or equipped) spaces** [Gelfand & Vilenkin [61]];
- Limit of the resolvent at the essential spectrum in certain spaces:
  [Rejto [69]], [Agmon [70]], [Yamada [72/73]] (for Dirac operators), [Agmon [75]].

Recent meaning of LAP. Shmuel Agmon and $\mathbf{E} \hookrightarrow \mathbf{X} \hookrightarrow \mathbf{F}$

LAP for $A \in \mathscr{C}(\mathbf{X})$: $\lim(A - zI)^{-1}$ as $z \to \sigma_{\mathrm{ess}}(A)$

While no limit as a map $\mathbf{X} \longrightarrow \mathbf{X}$ when $z \to \sigma(A)$...

[Agmon [70], Agmon [75]]:

$$\exists \lim_{\substack{z\to z_0>0 \\ z\in\mathbb{C}_+}} (-\Delta - zI)^{-1} : L^2_s(\mathbb{R}^d) \to L^2_{-s}(\mathbb{R}^d), \quad \forall s > \tfrac{1}{2}, \quad \forall d \geq 1.$$

$$\|u\|_{L^2_s(\mathbb{R}^d)} := \|(1 + |x|)^s u\|_{L^2(\mathbb{R}^d)}$$

Near the threshold: what if $z \to 0$?

$$d = 3: \quad (-\Delta - zI)^{-1} \sim \frac{e^{-|x-y|\sqrt{-z}}}{4\pi|x-y|}, \quad \exists \text{ limit as } z \to 0$$

$$d = 1: \quad (-\partial_x^2 - zI)^{-1} \sim \frac{e^{-|x-y|\sqrt{-z}}}{2\sqrt{-z}}, \quad \text{no limit as } z \to 0$$

3 Virtual levels

Singularity of the resolvent at threshold:

[Birman [61], Faddeev [63], Vainberg [68], Yafaev [74], Vainberg [75], Simon [76], Rauch [78]]...

Dependence of dispersive estimates on the presence of a virtual level:

Schrödinger operators:

[Jensen & Kato [79], Yafaev [83], Erdoğan & Schlag [04], Yajima [05]];

Dirac operators:

[Boussaïd [06], Boussaïd [08], Erdoğan & Green [17], Erdoğan et al. [19]]...

Virtual levels of selfadjoint Schrödinger operators in $\mathbb{R}^d$

$\boldsymbol{d = 3}$: [Yafaev [75], Jensen & Kato [79]] at most one virtual state "s"

$\boldsymbol{d \geq 4}$: [Jensen [80], Yafaev [83], Jensen [84]] at most one virtual state in $\mathbb{R}^4$;
only eigenstates for $d \geq 5$

$\boldsymbol{d = 1}$: [Bollé et al. [85], Bollé et al. [87]] at most one virtual state

$\boldsymbol{d = 2}$: [Bollé et al. [88]] (if $\int_{\mathbb{R}^2} V(x)\,dx \neq 0$) up to three virtual states: "s^1", "p^2"

$\boldsymbol{d \geq 1}$: [Jensen & Nenciu [01]] boundedness of $|V|^{\frac{1}{2}}(-\Delta + V - zI)^{-1}|V|^{\frac{1}{2}}$
(bad weights)

Equivalent characterizations of virtual levels

$H = -\Delta + V(x),\ x \in \mathbb{R}^d,\ d \geq 1,\ V \in C_{\mathrm{comp}}(\mathbb{R}^d)$

Following properties seem equivalent:

(P1) $H\psi = z_0\psi$ has a nonzero solution in L^2 or a **slightly larger** space;

(P2) $(H - zI)^{-1} : L^2_s(\mathbb{R}^d) \to L^2_{-s'}(\mathbb{R}^d)$ has no limit as $z \to z_0\ \forall s,\ s' \gg 1$;

(P3) Under an arbitrarily small perturbation, an eigenvalue can bifurcate from z_0.

(P1) – (P3) are satisfied for $-\partial_x^2$ on $\mathbb{R}$ near $z_0 = 0$;

(P1) – (P3) are **not** satisfied for $-\Delta$ in $\mathbb{R}^3$ near $z_0 = 0$.

Such equivalence for general exterior elliptic problems: [Vainberg [75]]

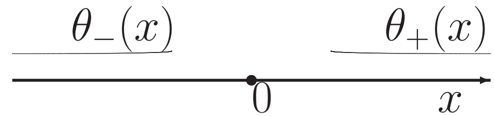
From Schrödinger operators in 1D to general theory

$$(-\partial_x^2 + V - z)u = 0, \qquad u(x) \in \mathbb{C}, \quad x \in \mathbb{R}; \qquad \int_{\mathbb{R}} (1+|x|)|V(x)|\,dx < \infty.$$

$$V = 0 \quad \Rightarrow \quad (-\partial_x^2 - zI)^{-1} \sim \frac{e^{-|x-y|\sqrt{-z}}}{2\sqrt{-z}}$$

Bad at $z \to z_0 = 0$ since Jost solutions $\theta_-(x),\ \theta_+(x)$ are linearly dependent!

$$\begin{cases} \partial_x^2 \theta_-(x) = 0 \\ \theta_-(x) \underset{x\to-\infty}{\approx} 1, \end{cases} \qquad \begin{cases} \partial_x^2 \theta_+(x) = 0 \\ \theta_+(x) \underset{x\to+\infty}{\approx} 1, \end{cases}$$



$$(-\partial_x^2 - 0)^{-1} \sim G(x,y) = \frac{1}{W[\theta_+, \theta_-](y)} \begin{cases} \theta_-(x)\theta_+(y), & x < y \\ \theta_-(y)\theta_+(x), & x > y \end{cases}$$ does not work.

Instead, $\Psi(x) = \theta_-(x) = \theta_+(x) = 1 \in \boldsymbol{L}^\infty(\mathbb{R})$: **virtual state**.

From Schrödinger operators in 1D to general theory

$$(-\partial_x^2 + V - z)u = 0, \qquad u(x) \in \mathbb{C}, \quad x \in \mathbb{R}; \qquad \int_{\mathbb{R}} (1+|x|)|V(x)|\,dx < \infty.$$

- **No virtual level at $z_0 = 0$:**
  - $\exists \lim\limits_{z\to z_0} (-\partial_x^2 + V - zI)^{-1} : L^2_{3/2+0}(\mathbb{R}) \to L^\infty(\mathbb{R})$;
  - $\nexists\, \Psi \in L^\infty(\mathbb{R}), \quad \Psi \ne 0, \quad (-\partial_x^2 + V - z_0)\Psi = 0.$

- **Virtual level at $z_0 = 0$:**
  - $\nexists \lim\limits_{z\to z_0} (-\partial_x^2 + V - zI)^{-1}$;
  - $\exists\, \Psi \in L^\infty(\mathbb{R}), \quad \Psi \ne 0, \quad (-\partial_x^2 + V - z_0)\Psi = 0$;

Corresponding **virtual state**: $(-\partial_x^2 + V + \boldsymbol{W} - z_0)\Psi = \boldsymbol{W}\Psi, \quad \boldsymbol{W} \in C_{\mathrm{comp}}(\mathbb{R})$

$$\Rightarrow \quad \Psi = (-\partial_x^2 + V + \boldsymbol{W} - z_0 I)^{-1}\boldsymbol{W}\Psi.$$

Definition 3.1 (Virtual levels) $\mathbf{X}$: Banach space; $A \in \mathscr{C}(\mathbf{X})$; $\Omega \subset \rho(A)$.

$\mathbf{E} \hookrightarrow \mathbf{X} \hookrightarrow \mathbf{F}$ Banach spaces, continuous embeddings

- Spaces are not necessarily reflexive
- Embeddings are not necessarily dense

Assume that A has a closable extension onto $\mathbf{F}$

► $z_0 \in \sigma_{\mathrm{ess}}(A) \cap \partial\Omega$ **regular point of** σ_{ess} **relative to** $\Omega, \mathbf{E}, \mathbf{F}$ if there is LAP:

$$\exists \ (A - z_0 I)^{-1}_{\Omega,\mathbf{E},\mathbf{F}} := \operatorname*{w-lim}_{z \to z_0,\, z \in \Omega} (A - zI)^{-1} : \ \mathbf{E} \to \mathbf{F}$$

► $z_0 \in \sigma_{\mathrm{ess}}(A)$ **virtual level of rank** $\boldsymbol{r} \in \mathbb{N}$ **relative to** $\Omega, \mathbf{E}, \mathbf{F}$ if it is the smallest r such that $\exists B \in \mathscr{B}_{00}(\mathbf{F}, \mathbf{E})$, $\operatorname{rank} B = r$, so that

$$\exists \ (A + B - z_0 I)^{-1}_{\Omega,\mathbf{E},\mathbf{F}} := \operatorname*{w-lim}_{z \to z_0,\, z \in \Omega} (A + B - zI)^{-1} : \ \mathbf{E} \to \mathbf{F}$$

Remark 3.2 For $T \in \mathbb{M}_{n\times n}$,

$$\dim \mathbf{ker}(T) = \min \left\{ \operatorname{rank} B;\ B \in \mathbb{M}_{n\times n},\ \det(T+B) \neq 0 \right\}.$$

For $T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ **we take** $B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$; **indeed,** $\dim \mathbf{ker}(T) = 1.$

Example 3.3 $A = -\Delta$ in $L^2(\mathbb{R}^d)$, $\mathfrak{D}(A) = H^2(\mathbb{R}^d)$.

- $z_0 > 0$ is **regular** point of $\sigma_{\mathrm{ess}}(-\Delta)$ relative to $\Omega = \mathbb{C}_+$, L^2_s, L^2_{-s} if $s > \frac{1}{2}$:

$$\exists \underset{z \to z_0 > 0,\, z \in \mathbb{C}_+}{\text{w-lim}} (-\Delta - z_0 I)^{-1} : \ L^2_s(\mathbb{R}^d) \to L^2_{-s}(\mathbb{R}^d), \qquad s > \frac{1}{2}, \quad d \geq 1$$

- $z_0 = 0$ is a **regular** point of $\sigma_{\mathrm{ess}}(-\Delta)$ relative to $\mathbb{C} \setminus \overline{\mathbb{R}_+}$

  if $d \geq 3$, $s,\ s' > \frac{1}{2}$, $s + s' > 2$

Example 3.4 $V \in L^1(\mathbb{R})$

$(\partial_x + V - z)u = f$, $A = \partial_x + V : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, $\mathfrak{D}(A) = H^1(\mathbb{R})$.

If $\operatorname{Re} z < 0$,

$$(A - zI)^{-1} : f \mapsto u(x) = \int_{-\infty}^{x} e^{z(x-y)+W(y)-W(x)} f(y)\, dy,$$

$$W(x) := \int_{-\infty}^{x} V(y)\, dy$$

$\forall\, z_0 \in \sigma_{\mathrm{ess}}(A) = \mathrm{i}\mathbb{R}$ is **regular** relative to $\{\operatorname{Re} z < 0\}$, $L^1(\mathbb{R})$, $L^\infty(\mathbb{R})$:

$$\exists\ (A - z_0 I)^{-1}_{\operatorname{Re} z<0} := \operatorname*{w-lim}_{\substack{z \to z_0 \\ \operatorname{Re} z<0}} (A - zI)^{-1} : L^1(\mathbb{R}) \to L^\infty(\mathbb{R})$$

Example 3.5 (Zero operator)

$\dim \mathbf{X} = \infty$, $\quad N : \mathbf{X} \to 0 \in \mathbf{X}$, $\quad \sigma(N) = \sigma_{\mathrm{ess}}(N) = \{0\}$.

Let $\mathbf{E} \hookrightarrow \mathbf{X} \hookrightarrow \mathbf{F}$ (continuously), $\dim \mathbf{E} = \infty$. Let $B \in \mathscr{B}_{00}(\mathbf{F}, \mathbf{E})$.

Projection onto $\mathbf{ker}(B) \subset \mathbf{E}$: $P_0 := -\frac{1}{2\pi \mathrm{i}} \oint\limits_{|\zeta|=\epsilon} (B - \zeta I)^{-1}\, d\zeta$.

Then

$$(N + B - zI)^{-1} P_0 = -z^{-1} P_0 : \mathbf{E} \to \mathbf{F}, \qquad z \neq 0,$$

cannot be bounded uniformly in $z \in \mathbb{C} \setminus \{0\}$.

Hence, $z_0 = 0$ is not an exceptional point of finite rank relative to $\mathbb{C} \setminus \{0\}, \mathbf{E}, \mathbf{F}$.

Example 3.6 Left shift:

$$L:\ \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N}), \qquad (x_1, x_2, \dots) \mapsto (x_2, x_3, \dots), \qquad \sigma(L) = \overline{\mathbb{D}_1}.$$

$$L - zI = \begin{bmatrix} -z & 1 & 0 & \cdots \\ 0 & -z & 1 & \cdots \\ 0 & 0 & -z & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \qquad (L - zI)^{-1} = -\begin{bmatrix} z^{-1} & z^{-2} & z^{-3} & \cdots \\ 0 & z^{-1} & z^{-2} & \cdots \\ 0 & 0 & z^{-1} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

$$|((L - zI)^{-1}x)_i| \le |z^{-1}x_i| + |z^{-1}x_{i+1}| + \cdots \le \|x\|_{\ell^1};$$

$$\Rightarrow \qquad (L - zI)^{-1}:\ \ell^1(\mathbb{N}) \to \ell^\infty(\mathbb{N}), \quad \textbf{uniformly in } |z| > 1.$$

$\forall\, |z_0| = 1$ is a **regular** point of $\sigma_{\mathrm{ess}}(L)$: $\exists \lim\limits_{\substack{z \to z_0 \\ |z|>1}} (L - zI)^{-1}:\ \ell^1(\mathbb{N}) \to \ell^\infty(\mathbb{N})$.

To construct $A \in \mathscr{B}(\ell^2(\mathbb{N}))$ with a **virtual level** at $z_0 \in \mathbb{C},\ |z_0| = 1$:

Fix $\quad \phi \in \ell^1(\mathbb{N}), \quad K = \frac{1}{\langle\phi,\phi\rangle}\phi \otimes \langle\phi,\ \cdot\rangle \in \mathscr{B}_{00}\big(\ell^\infty(\mathbb{N}), \ell^1(\mathbb{N})\big); \quad K\phi = \phi.$

$A = L - K(L - z_0)$ has a **virtual level** at $z_0 \in \sigma_{\mathrm{ess}}(A)$ relative to

$$\Omega = \mathbb{C} \setminus \overline{\mathbb{D}_1},\ \ell^1,\ \ell^\infty$$

since z_0 is **regular** point of $\sigma_{\mathrm{ess}}(\underbrace{A + B}_{L}),\ B := K(L - z_0) \in \mathscr{B}_{00}\big(\ell^\infty(\mathbb{N}), \ell^1(\mathbb{N})\big).$

The corresponding virtual state: $\Psi = (L - z_0 I)^{-1}_{\Omega,\ell^1,\ell^\infty}\phi \in \ell^\infty(\mathbb{N}).$

Note:

$$(A - z_0)\Psi = \big(\underbrace{L - K(L - z_0)}_{A} - z_0\big)(L - z_0 I)^{-1}_{\Omega,\ell^1,\ell^\infty}\phi = (I - K)\phi = 0.$$

Similar concepts:

1. Spectral singularities [Naimark [54], Schwartz [60], Pavlov [66]],

[Ljance [67], Konotop et al. [19]]

Absent for selfadjoint operators.

2. Birman's approach [Birman [61], §1.7] for semibounded selfadjoint operators.

E.g., if $H = -\Delta + V$, consider the closure of $\mathbf{X}$ with respect to

$$\mathbf{a}[\varphi] := \int (|\nabla\varphi|^2 + V|\varphi|).$$

Related: subcritical/critical Schrödinger operators [Simon [81], Murata [86]],

[Gesztesy & Zhao [91], Weidl [99], Pinchover & Tintarev [06], Lucia & Prashanth [18]].

Key lemma (abstract version of [Jensen & Kato [79], Lemma 2.4]):

Lemma 3.7 (Left and right inverse of $A - z_0 I$)

Let $z_0 \in \sigma_{\mathrm{ess}}(A)$ be regular relative to $\Omega \subset \rho(A)$:

$$\exists \quad (A - z_0 I)^{-1}_{\Omega} := \operatorname*{w-lim}_{z \to z_0,\, z \in \Omega} (A - zI)^{-1} : \mathbf{E} \to \mathbf{F}.$$

Then $(A - z_0 I)^{-1}_{\Omega}$ is both the left and the right inverse of

$$\hat{A} - z_0 I : \mathbf{Range}\left((A - z_0 I)^{-1}_{\Omega}\right) \to \mathbf{E}.$$

Above, $\hat{A}$ is closed extension of A onto $\mathbf{F}$.

Note: $\quad (-\partial_x^2 + V - z_0)u = \phi \in C^\infty_{\mathrm{comp}}(\mathbb{R}), \quad V \in C^\infty_{\mathrm{comp}}(\mathbb{R}), \quad z_0 = 0,$

has a unique L^∞-solution if there is no virtual level;

has no L^∞-solution if there is a virtual level.

Space of virtual states, $\mathfrak{M}_{\Omega,\mathbf{E},\mathbf{F}}(A - z_0 I)$

[Jensen & Kato [79]], [Birman [61], §1.7]

If $z_0 \in \sigma_{\mathrm{ess}}(A)$ is of rank $r \in \mathbb{N}_0$ relative to $\Omega \subset \rho(A)$:

$$\mathfrak{M}_{\Omega,\mathbf{E},\mathbf{F}}(A - z_0 I) \;:=\; \left\{ \Psi \in \mathbf{Range}\left((A + B - z_0 I)^{-1}_{\Omega}\right) \;;\; (\hat{A} - z_0)\Psi = 0 \right\} \subset \mathbf{F},$$

with some $B \in \mathscr{B}_{00}(\mathbf{F}, \mathbf{E})$.

Theorem 3.8 1. $\mathfrak{M}_{\Omega,\mathbf{E},\mathbf{F}}(A - z_0 I)$ does not depend on the choice of B;

2. $\mathbf{E} \cap \mathbf{ker}(A - z_0) \subset \mathfrak{M}_{\Omega,\mathbf{E},\mathbf{F}}(A - z_0 I)$;

3. $\dim \mathfrak{M}_{\Omega,\mathbf{E},\mathbf{F}}(A - z_0 I) = r$.

If $\mathfrak{M}_{\Omega,\mathbf{E},\mathbf{F}}(A - z_0 I) \not\subset \mathbf{X}$: z_0 is a **genuine virtual level**;

$\Psi \in \mathfrak{M}_{\Omega,\mathbf{E},\mathbf{F}}(A - z_0 I) \setminus \mathbf{X}$ is a **virtual state**.

Independence on the choice of "regularizing" spaces E and F

This is similar to [Agmon [98]] (in the context of resonances)

Theorem 3.9 $A \in \mathscr{C}(\mathbf{X})$, $\Omega \subset \rho(A)$ connected open set, $z_0 \in \partial\Omega \cap \sigma_{\mathrm{ess}}(A)$.

$\mathbf{E}_i \hookrightarrow \mathbf{X} \hookrightarrow \mathbf{F}_i$, $i = 1, 2$, Banach spaces with dense continuous embeddings.

Assume: $\mathbf{E}_1 \cap \mathbf{E}_2$ is dense in both $\mathbf{E}_1$ and $\mathbf{E}_2$;

$\mathbf{F}_1$ and $\mathbf{F}_2$ are dense in $\mathbf{F}_1 + \mathbf{F}_2$;

$(\mathbf{F}_1 + \mathbf{F}_2)^*$ is dense in $\mathbf{F}_1^*$ and in $\mathbf{F}_2^*$;

A has closable extension onto $\mathbf{F}_1 + \mathbf{F}_2$.

Let z_0 be a virtual level with respect to both $\mathbf{E}_1 \to \mathbf{F}_1$ and $\mathbf{E}_2 \to \mathbf{F}_2$.

Then $\quad r_1 = r_2 \quad$ and $\quad \mathfrak{M}_{\Omega,\mathbf{E}_1,\mathbf{F}_1}(A - z_0 I) = \mathfrak{M}_{\Omega,\mathbf{E}_2,\mathbf{F}_2}(A - z_0 I)$.

Dependence on the choice of "regularizing" spaces E and F

We know: $(-\Delta - zI)^{-1} \;:\; \boldsymbol{L^2_s(\mathbb{R}) \to L^2_{-s'}(\mathbb{R})}$, $z \in \mathbb{C}\setminus\overline{\mathbb{R}_+}$,

$$K\big((-\Delta - zI)^{-1}\big)(x,y) = \frac{e^{-|x-y|\sqrt{-z}}}{2\sqrt{-z}},$$

has no limit as $z \to z_0 = 0$. Yet,

Example 3.10 (Roman Romanov) Let $s,\ s' > 1/2,\quad \tau > 1$;

$$\mathbf{E} := \Big\{u \in L^2_s(\mathbb{R}) \;;\; |\hat{u}(\xi)| = O(|\xi|^\tau)\Big\}, \qquad \|u\|_{\mathbf{E}} = \|u\|_{L^2_s(\mathbb{R})} + \limsup_{\substack{\xi\to 0\\ \xi\in\mathbb{R}\setminus\{0\}}} \frac{|\hat{u}(\xi)|}{|\xi|^\tau}$$

Then $z_0 = 0$ is **regular** relative to $\big(\mathbb{C}\setminus\overline{\mathbb{R}_+},\ \mathbf{E},\ L^2_{-s'}(\mathbb{R})\big)$:

$$\exists \operatorname*{w-lim}_{z\to z_0}(-\Delta - zI)^{-1} : \mathbf{E} \to \boldsymbol{L^2_{-s'}(\mathbb{R})}$$

Note: $\mathbf{E}$ and L^2_s are **not mutually dense**, although both are dense in L^2

LAP vs. bifurcations from $\sigma_{\rm ess}(A)$

Let $z_0 \in \sigma_{\rm ess}(A)$ be of rank $r \geq 0$, $r < \infty$ relative to $\Omega \subset \rho(A)$.

$\exists$ a (desired) bifurcation of a family of eigenvalues from z_0 into Ω **iff** $\boldsymbol{r \geq 1}$!!

Theorem 3.11 Let $z_0 \in \sigma_{\rm ess}(A)$ be of rank $r \geq 0$, $r < \infty$ relative to $\Omega \subset \rho(A)$.

1. If $\exists\, V_j \in \mathscr{B}(\mathbf{F}, \mathbf{E})$, $\quad \lim\limits_{j\to\infty} \|V_j\|_{\mathbf{F}\to\mathbf{E}} = 0, \quad z_j \in \sigma_{\rm d}(A+V_j)\cap\Omega,\ z_j \to z_0,$

   then $r \geq 1$; i.e. $\nexists \underset{z\to z_0,\, z\in\Omega}{\text{w-lim}} (A - zI)^{-1} : \mathbf{E} \to \mathbf{F}$

2. If $z_0 \in \sigma_{\rm ess}(A)$ is of rank $r \geq 1$ relative to Ω, then $\forall\, z_j \in \Omega, \quad z_j \to z_0,$

   $\exists\ V_j \in \mathscr{B}_{00}(\mathbf{F}, \mathbf{E})$ such that $\|V_j\|_{\mathbf{F}\to\mathbf{E}} \to 0, \quad z_j \in \sigma_{\rm d}(A + V_j), \quad j \in \mathbb{N}.$

   One can choose $V_j = \zeta_j V$, $\zeta_j \to 0$.

Virtual levels of adjoint operators

Let $\mathbf{E} \hookrightarrow \mathbf{X} \hookrightarrow \mathbf{F}$ (continuous embeddings).

Let A have closable extension onto $\mathbf{F}$, A^* have closable extension onto $\mathbf{E}^*$.

Lemma 3.12 If $\bar{z}_0 \in \sigma_{\mathrm{ess}}(A^*)$ of rank $s \geq 0$ relative to $\Omega^* := \left\{\bar{\zeta} \in \Omega\right\}$, $\mathbf{F}^*$, $\mathbf{E}^*$, then $z_0 \in \sigma_{\mathrm{ess}}(A)$ is of rank $r \leq s$ relative to $\Omega, \mathbf{E}, \mathbf{F}$.

If, additionally, $\mathbf{E}$ is reflexive, then $r = s$.

If $\mathbf{E}$, $\mathbf{F}$ were reflexive, just notice that

$$\exists \underset{\substack{z\to z_0\\ z\in\Omega}}{\text{w-lim}}(A+B-zI)^{-1} : \mathbf{E} \to \mathbf{F} \iff \exists \underset{\substack{\bar{z}\to \bar{z}_0\\ \bar{z}\in\Omega^*}}{\text{w-lim}}(A^*+B^*-\bar{z}I)^{-1} : \mathbf{F}^* \to \mathbf{E}^*.$$

The Fredholm alternative

$A \in \mathscr{C}(\mathbf{X})$, $\mathfrak{D}(A) \subset \mathbf{X}$; $\mathbf{E} \hookrightarrow \mathbf{X} \hookrightarrow \mathbf{F}$ (continuously)

Assume that A has closable extension onto $\mathbf{F}$

Lemma 3.13 (Fredholm alternative)

Assume: $z_0 \in \sigma_{\mathrm{ess}}(A)$ of rank $r \in \mathbb{N}_0$ relative to $\Omega \subset \rho(A)$.

Then: $\exists\, P \in \mathrm{End}\,(\mathbf{E})$, $P^2 = P$, $\mathrm{rank}\, P = r$, such that

$$(\hat{A} - z_0)u = \phi, \qquad \phi \in \mathbf{E},$$

has a solution $u \in \textbf{Range}\left((A + B - z_0 I)^{-1}_{\Omega,\mathbf{E},\mathbf{F}}\right) \subset \mathbf{F}$ iff $P\phi = 0$.

This solution is unique under extra constraint $Qu = 0$,

where $Q \in \mathrm{End}\,(\mathbf{F})$ is any projection onto $\mathfrak{M}_{\Omega,\mathbf{E},\mathbf{F}}(A - z_0 I)$

4 Application to Schrödinger operators

Uniform resolvent bounds for selfadjoint Schrödinger operators: [Kenig et al. 87], [Gutiérrez 04], [Frank 11], [Frank & Simon 17], [Bouclet & Mizutani 18], [Ren et al. 18], [Mizutani 19].

To approach general nonselfadjoint Schrödinger operators in all dimensions:

Derive estimates for $A = -\Delta + V$ with e.g. $V(x) = \varepsilon \mathbb{1}_{|x|\le 1}$ for $d \le 2$!!

Prior to [Boussaïd & Comech 21], nonselfadjoint case has not been considered; even in the selfadjoint case, the LAP in dimension $d = 2$ was not available.

Theorem 4.1 $A = -\Delta + V$ in $L^2(\mathbb{R}^d)$, $d \in \mathbb{N}$, $\mathfrak{D}(A) = H^2(\mathbb{R}^d)$;

$$|V(x)| \le C\langle x\rangle^{-\rho}, \ \rho > 2, \ \rho > s + s'.$$

- If $z_0 = 0 \in \sigma_{\mathrm{ess}}(A)$ is **regular** relative to $\Omega = \mathbb{C} \setminus \overline{\mathbb{R}_+}$, then:

$$(A - z_0 I)^{-1}_{\Omega} : L^2_s \to L^2_{-s'}, \begin{cases} s + s' \ge 2, \ s, \ s' > \frac{1}{2}, & d = 1; \\ \boldsymbol{s + s' \ge 2, \ \ s, s' > 2 - \frac{d}{2}, \ \ s, s' \ge 0, \ \ d \ge 2;} \end{cases}$$

Also, $\boldsymbol{(A - z_0 I)^{-1}_{\Omega} : L^2_s(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d), \quad \forall s > 2 - \frac{d}{2}, \quad d \le 3.}$

- If $z_0 = 0 \in \sigma_{\mathrm{ess}}(A)$ is a **virtual level**, then $\exists\, \Psi \ne 0$, $(A - z_0)\Psi = 0$,

$$\Psi \in \begin{cases} L^\infty(\mathbb{R}^d), & d \le 2; \\ L^2_{-\frac{1}{2}-0}(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3), & d = 3; \end{cases} \qquad \Psi \in \begin{cases} L^2_{-0}(\mathbb{R}^4), & d = 4; \\ L^2(\mathbb{R}^d), & d \ge 5. \end{cases}$$

Theorem 4.2 (LAP / virtual levels in $\mathbb{R}^2$) $\qquad A = -\Delta + V(x)$ in $L^2(\mathbb{R}^2)$

- If $z_0 = 0 \in \sigma_{\mathrm{ess}}(A)$ is **regular**:

for any $s > 1$ and $|V(x)| \le C\langle x\rangle^{-\rho}, \quad \rho > \max(2, 2s)$,

$$(A - zI)^{-1} : \ L^2_s(\mathbb{R}^2) \to L^2_{-s}(\mathbb{R}^2),$$

$$(A - zI)^{-1} : \ L^1(\mathbb{R}^2) \to L^2_{-s}(\mathbb{R}^2),$$

$$(A - zI)^{-1} : \ L^2_s(\mathbb{R}^2) \to L^\infty(\mathbb{R}^2),$$

uniformly in $z \in \mathbb{C} \setminus \overline{\mathbb{R}_+}$.

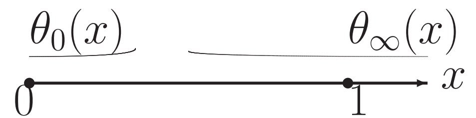
- $z_0 = 0$ is a **virtual level**: $\qquad \exists\, \Psi \in L^\infty(\mathbb{R}^2), \quad (-\Delta + V)\Psi = 0.$

Regularized Laplacian in $\mathbb{R}^2$

Lemma 4.3 For $s,\ s' > 1,\ \mathbf{g} > 0,\ \exists\ \underset{z\to 0}{\text{w-lim}}\,(-\Delta + \mathbf{g}\mathbb{1}_{\mathbb{B}_1^2} - zI)^{-1} : \ L^2_s \to L^2_{-s'}$.

1. $(-\Delta + \mathbf{g}\mathbb{1}_{\mathbb{B}_1^2} - zI)^{-1} : \ L^2_{s,\text{radial}}(\mathbb{R}^2) \to L^2_{-s'}(\mathbb{R}^2)$ has a limit as $z \to 0$.

Consider $\left(-\partial_r^2 - \frac{1}{r}\partial_r + \mathbf{g}\mathbb{1}_{(0,1)} - z\right)\theta(r,z) = 0$; let $\begin{cases} \theta_0(r,z) \approx 1, & 0 < r \ll 1; \\ \theta_\infty(r,z) \approx 1, & r \gg 1. \end{cases}$



Define $G(r,\rho,z) = \dfrac{1}{rW[\theta_0,\theta_\infty](r)} \begin{cases} \theta_0(r,z)\theta_\infty(\rho,z), & 0 < r \le \rho; \\ \theta_\infty(r,z)\theta_0(\rho,z), & 0 < \rho \le r. \end{cases}$

2. $(-\Delta - zI)^{-1} : \ \left(L^2_{s,\text{radial}}(\mathbb{R}^2)\right)^{\perp} \to L^2_{-s'}(\mathbb{R}^2)$ has a limit as $z \to 0$

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