MA10207: Exercise sheet 18
Please hand in solutions to homework problems by Monday, 22nd April, 2:15pm.

(Derivatives and extrema & the MVT; Revision)

Warmup problems

Problem T 18.1. Let \( n \in \mathbb{N} \). Prove from the definition that \( f : \mathbb{R} \rightarrow \mathbb{R}, \ x \mapsto f(x) = x^n \) is differentiable with \( f'(x) = nx^{n-1} \). [Hint: use the binomial theorem.]

Problem T 18.2. Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be differentiable. Suppose that \( f \) has \( n + 1 \) distinct zeroes in \( \mathbb{R} \). Prove that \( f' \) has at least \( n \) distinct zeroes in \( \mathbb{R} \).

Homework problems

Problem H 18.1. Let \( k \in \mathbb{Z} \) be an integer and \( f : (0, \infty) \rightarrow \mathbb{R}, \ x \mapsto f(x) := x^k \). Prove that \( f \) is differentiable for every \( x \in (0, \infty) \) with \( f'(x) = kx^{k-1} \).

Problem H 18.2. Find the minimum and maximum values of \( [-1, 1] \ni x \mapsto 9x^4 - 10x^3 + 3x^2 \in \mathbb{R} \).

Problem H 18.3. Use Rolle’s Theorem to prove that a polynomial of degree \( n \geq 1 \) has at most \( n \) zeroes.

Problem H 18.4. Suppose that \( f : (0, \infty) \rightarrow \mathbb{R} \) solves the initial value problem \( f'(x) = \frac{1}{x}; \ f(1) = 0 \). Prove from the differential equation that \( f(xy) = f(x) + f(y) \) for \( x, y \in (0, \infty) \).

Quiz questions

Problem Q 18.1. Let \( f : I \rightarrow \mathbb{R} \) be differentiable on an open interval \( I \subset \mathbb{R} \).

(i) If \( f \) has no critical points then \( \frac{1}{f} \) is differentiable with \( \left(\frac{1}{f}\right)' = \frac{1}{f^2} \). [Hint]

(ii) If \( p \) is a polynomial then \( p \circ f \) is differentiable. [Hint]

(iii) If there exists \( x_0 \in I \) such that \( f'(x_0) = 0 \) then \( f \) has a global extremum on \( I \). [Hint]

(iv) If there exists \( x_0 \in I \) such that \( f'(x_0) = 0 \) then \( f \) has a local extremum on \( I \). [Hint]

(v) If \( f \) has at least two local minima in \( I \) then \( f \) has at least three critical points in \( I \). [Hint]

Evaluate

Problem Q 18.2. Let \( I \subset \mathbb{R} \) be an open interval and \( f : I \rightarrow \mathbb{R} \) be differentiable.

(i) For every \( \xi \in I \) there are \( a, b \in I \) with \( a < \xi < b \) so that \( f'((b-a) = f(b) - f(a) \). [Hint]

(ii) The function \( x \mapsto f'(x) \) is continuous on \( I \). [Hint]

(iii) If \( f'(a) < 0 < f'(b) \) for some \( a, b \in I \) with \( a < b \) then \( f \) has a critical point in \( (a, b) \). [Hint]

(iv) If \( f \) is Lipschitz continuous then \( f' \) is bounded. [Hint]

(v) If \( f' \) is bounded then \( f \) is Lipschitz continuous on \( I \). [Hint]

Evaluate

Revision problems

These problems are meant as ideas of how you could revise during the Easter vacation (or in general). The idea is that R 18.1 be discussed in tutorials after the break, you could hand your graph of R 18.2 in to your tutor to receive some feedback.
**Problem R 18.1.** Make a list of all terms that were defined in Sect 4 of the course. Revise and discuss those that were difficult.

**Problem R 18.2.** List the important theorems of Sect 4 of the course. Revise their statements (clearly distinguishing between assumptions and claims) and proofs (main ideas and flow of arguments) without writing them down.

Create a dependency graph: represent each theorem by a node and connect two nodes by an arrow if one theorem was used to prove the other. Indicate which definitions/notions play a role in each theorem or its proof.

**Problem R 18.3.** Go through all previous quiz questions carefully again, making sure that you not only get the correct answers but also fully understand the reasons for why a statement is true or not.