Warmup problems

Problem T 15.1. Let \( f : \mathbb{R} \to \mathbb{R}, f(x) := (3x + 1)(4x - 1) = 12x^2 + x - 1 \). Approximate the zeroes of \( f \) by following the proof of the IVT. When does the procedure lead to an exact solution?

Problem T 15.2. Prove Lemma 4.13: Show that the restriction \( f|_S \) of a continuous function \( f \in C^0(D) \) to some subset \( S \subseteq D \) is continuous.

Homework problems

Problem H 15.1. Let \( f : \mathbb{R} \to \mathbb{R}, x \mapsto f(x) := 16x^5 - 20x^4 - 168x^3 + 203x^2 + 80x - 30 \).

(\( \alpha \)) Use the IVT to prove that \( f \) has zeroes in \((-1, 0), (0, 1) \) and \((1, 2) \).

(\( \beta \)) Find these zeroes following the proof of the IVT.

(\( \gamma \)) Find five zeroes of \( f \) (these are all: we shall see a slick proof of this later).

[Note: using a calculator defies the point of this problem.]

Problem H 15.2. Let \( p(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 \) be a monic polynomial and set \( M := \max\{1, 2n|a_{n-1}|, \ldots, 2n|a_0|\} \).

(a) Prove that \( |x| > M \Rightarrow \left| \sum_{i=1}^{n} \frac{a_n}{x^n} \right| < \frac{1}{2} \).

(b) Deduce that \( x > M \Rightarrow p(x) > \frac{x^n}{2} \), and hence \( \lim_{x \to \infty} p(x) = \infty \).

(c) Deduce that \( \lim_{x \to -\infty} p(x) = -\infty \) if \( n = 0 \mod 2 \) and \( \lim_{x \to -\infty} p(x) = -\infty \) if \( n = 1 \mod 2 \).

(d) Deduce that every polynomial of odd degree has a real root.

Problem H 15.3. Let \( f : \mathbb{R} \to \mathbb{R} \) be continuous with \( \lim_{x \to \pm \infty} f(x) = \infty \). Use Weierstrass’ theorem to prove that \( f \) has a minimum. Conclude that every polynomial of even degree has either a minimum or a maximum, respectively, depending on the sign of the leading term.

Quiz questions

Problem Q 15.1. In this problem the question is whether or not there is a continuous function \( f : D \to \mathbb{R} \) with the specified “co-domain” (or, “range”) \( f(D) \). If you think there is then tick the corresponding field in the following table, if not then leave the field empty:

<table>
<thead>
<tr>
<th>( f(D) = {0, 1} )</th>
<th>( f(D) = (0, 1) )</th>
<th>( f(D) = [0, 1] )</th>
<th>( f(D) = (0, \infty) )</th>
<th>( f(D) = [0, \infty) )</th>
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<tr>
<td>( D = {0, 1} )</td>
<td>( D = (0, 1) )</td>
<td>( D = [0, 1] )</td>
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</tbody>
</table>

Evaluate
Problem Q 15.2.

(i) Let \( f, g : [a, b] \to \mathbb{R} \) be continuous. If \( f(a) < g(a) \) and \( f(b) > g(b) \), then there exists \( z \in (a, b) \) such that \( f(z) = g(z) \). [Hint]

(ii) The polynomial \( f(x) = x^5 - 5x^4 + 2x + 1 \) has at least two distinct roots in \([-1, 1]\). [Hint]

(iii) Follow the procedure in the proof of the IVT in order to determine the zero of \( f : [-1, 1] \to \mathbb{R}, f(x) := (4x)^3 - 16x + 1 \), up to an accuracy of \( \frac{1}{16} = 2 \times 2^{-5} = 0.0625\); \( f \) has a zero in the interval \((x - \frac{1}{16}, x + \frac{1}{16})\), where \( x = \) (input the centre of the interval in decimal format).

(iv) The zero approximated in (iii) is the only zero of the function in the interval \((-1, 1)\). [Hint]

(v) A number \( m \in \mathbb{R} \) is a maximum of \( f \) if \( \forall x \in I : f(x) \leq m \). [Hint]

(vi) If \( I \) is a closed interval and \( f : I \to \mathbb{R} \) is continuous, then \( f(I) \) is a closed interval. [Hint]

Evaluate