MA10207: Exercise sheet 13
Please hand in solutions to homework problems by Monday, 4th March, 2:15pm.

(Continuity)

Warmup problems

Problem T 13.1. Let \( f : D \rightarrow \mathbb{R} \); suppose that \( \forall \delta > 0 : (D \setminus \{x\}) \cap (x-\delta, x+\delta) \neq \emptyset \), i.e., \( x \in D \) is not “isolated”. Prove that \( f \) is continuous at \( x \) iff \( \lim_{y \to x} f(y) = f(x) \). What can be said at “isolated” points \( x \in D \)?

Problem T 13.2. Formulate and prove an Inertia Property for continuous functions.

Homework problems

Problem H 13.1. Let \( f \in C^0(\mathbb{R}) \) and \( A \subset \mathbb{R} \) open. Prove that \( f^{-1}(A) := \{x \in \mathbb{R} \mid f(x) \in A\} \) is open.

Problem H 13.2. Let \( (x_n)_{n \in \mathbb{N}} \) be a sequence and \( l \in \mathbb{R} \); let \( D := \{\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{0\} \) and define \( f : D \rightarrow \mathbb{R}, \ x \mapsto f(x) := \begin{cases} l & \text{if } x = 0 \\ x_n & \text{if } \frac{1}{x} = n \in \mathbb{N}. \end{cases} \)

Prove that \( f \in C^0(D) \) if and only if \( x_n \to l \) as \( n \to \infty \).

Problem H 13.3. Prove from the definition that \( \exp : \mathbb{R} \rightarrow \mathbb{R} \) is continuous at \( x = 0 \).

Use the functional equality \( e^{x+y} = e^x e^y \) to conclude that \( \exp \) is continuous at every \( x \in \mathbb{R} \).

Quiz questions

Problem Q 13.1.
(i) Let \( f : \mathbb{R} \rightarrow \mathbb{R} \). If there is a sequence \( x_n \to 0 \) so that \( f(x_n) \to f(0) \) then \( f \) is continuous at 0.

(ii) There is a function \( f : \mathbb{R} \rightarrow \mathbb{R} \) that is nowhere continuous. [Hint]

(iii) A function \( f : \mathbb{Z} \rightarrow \mathbb{R} \) is everywhere continuous. [Hint]

(iv) If \( f \) is a polynomial then \( f \in C^0(\mathbb{R}) \). [Hint] 

Evaluate

Problem Q 13.2.
(i) If \( f, g : I \rightarrow \mathbb{R} \) and \( f \) and \( f \cdot g \) are continuous at \( x \in I \) then so is \( g \). [Hint]

(ii) If \( f : I \rightarrow J \) is continuous at \( x \in I \) and \( g : J \rightarrow \mathbb{R} \) is continuous at \( y = f(x) \) then \( g \circ f : I \rightarrow \mathbb{R} \) is continuous at \( x \). [Hint]

(iii) If \( f : I \rightarrow J \) is continuous at \( x \in I \) and \( g : J \rightarrow \mathbb{R} \) so that \( g \circ f \) is continuous then \( g \) is continuous at \( y = f(x) \). [Hint]

(iv) If \( f : I \rightarrow J \) and \( g : J \rightarrow \mathbb{R} \) are functions so that \( g \circ f \) is continuous at \( x \in I \) and \( g \) is continuous at \( y = f(x) \) then \( f \) is continuous at \( x \). [Hint] 

Evaluate