On the convergence of adaptive approximations for SDEs

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25th July 2024

Outline

Introduction

Consider the following Itô SDE on $[0, T]$:

$$dy_t = W_t \, dW_t. \tag{1}$$

Then, we know the solution is given by $y_t = \int_0^t W_s \, dW_s = \frac{1}{2}\big((W_t)^2 - t\big)$.

We can also approximate (1) using the Euler-Maruyama method:

$$\begin{aligned} Y_{k+1} &:= Y_k + W_{t_k}\big(W_{t_{k+1}} - W_{t_k}\big), \\ Y_0 &:= y_0\,, \end{aligned}$$

where $t_k := kh$ and $h = \frac{T}{K}$ for $k \in \{0, 1, \cdots, K\}$. It is then easy to show

$$\mathbb{E}\Big[\big(Y_K - y(T)\big)^2\Big] = \frac{1}{2}hT,$$

which converges to zero as $h \to 0$ (or, equivalently, as $K \to \infty$).

Introduction

What if we make the step size adaptive?
(which is popular in ODEs numerics)

For example, given a fixed λ, we can consider a condition of the form:

$$\left|W_{t_{k+1}} - W_{t_k}\right| \leq \lambda\sqrt{h}, \tag{2}$$

to help reduce errors when W has large fluctuations. In [1], they define

$$Y_{k+1} := \begin{cases} Y_k + W_{t_k}\big(W_{t_{k+1}} - W_{t_k}\big), & \text{if (2) holds,} \\ Y_k + W_{t_k}\big(W_{t_{k+\frac{1}{2}}} - W_{t_k}\big) + W_{t_{k+\frac{1}{2}}}\big(W_{t_{k+1}} - W_{t_{k+\frac{1}{2}}}\big), & \text{otherwise.} \end{cases}$$

Surprisingly however, it was shown in [1, Section 4.1] that this adaptive Euler method **fails to converge to the Itô solution!** (as $h \to 0$).

Introduction

Consider the following SDE:

$$dy_t = W_t^1 \, dW_t^2 \,, \tag{3}$$

where W^1 and W^2 denote two independent Brownian motions.

We can approximate (3) using Euler-Maruyama or a “trapezium” rule:

$$\begin{aligned} Y_{k+1} &:= Y_k + \frac{1}{2}\big(W_{t_k}^1 + W_{t_{k+1}}^1\big)\big(W_{t_{k+1}}^2 - W_{t_k}^2\big), \\ Y_0 &:= y_0 \,, \end{aligned}$$

where $k \in \{0, 1, \cdots, K\}$. By Itô’s isometry, it is straightforward to show

$$\mathbb{E}\Big[\big(Y_K - y(T)\big)^2\Big] = \begin{cases} \frac{1}{2}hT & \text{if Euler-Maruyama is used} \\ \frac{1}{4}hT & \text{if the trapezium rule is used} \end{cases},$$

where $T = Kh$. Hence, we see that the trapezium rule is more accurate.

Introduction

However, consider the following (less natural) adaptive step size:

We choose either h (i.e. 1 step) or $\frac{1}{2}h$ (i.e. 2 half-steps) to maximise Y.

$$Y_{k+1} = \max\bigg\{Y_k + \frac{1}{2}\big(W^1_{t_k} + W^1_{t_{k+1}}\big)\big(W^2_{t_{k+1}} - W^2_{t_k}\big),$$

$$Y_k + \frac{1}{2}\big(W^1_{t_k} + W^1_{t_{k+\frac{1}{2}}}\big)W^2_{t_k,\, t_{k+\frac{1}{2}}} + \frac{1}{2}\big(W^1_{t_{k+\frac{1}{2}}} + W^1_{t_{k+1}}\big)W^2_{t_{k+\frac{1}{2}},\, t_{k+1}}\bigg\},$$

where $W^i_{s,\,t} := W^i_t - W^i_s$. Then, it can be shown that for any $h > 0$,

$$\mathbb{E}\big[Y_K\big] = \frac{1}{8}T,$$

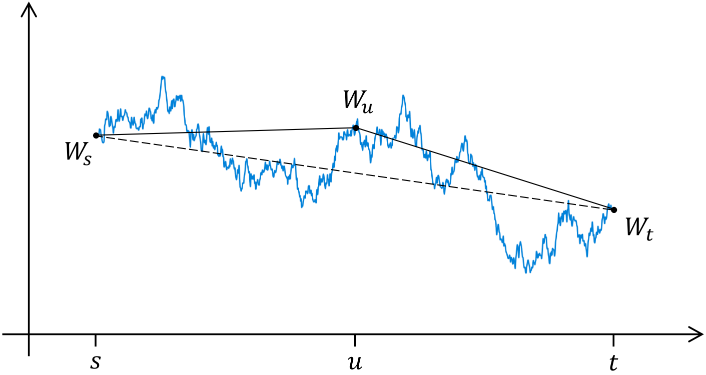
whereas $\mathbb{E}\big[y_T\big] = 0$. So, once again, ***Y* does not converge to the SDE!**

Question

Do adaptive numerical methods for SDEs converge? If so, when?

Lévy's construction of Brownian motion

How can we generate Brownian motion after we halve the step sizes?

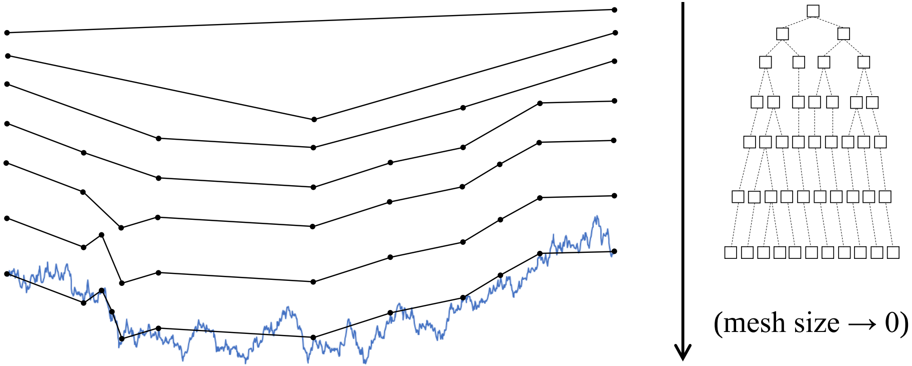


Using the notation $W_{a,b} := W_b - W_a$, we can generate W_u after W_t as

$$W_{s,t} \sim \mathcal{N}\big(0, (t-s)I_d\big), \qquad W_{s,u} \,|\, W_{s,t} \sim \mathcal{N}\bigg(\frac{1}{2}W_{s,t}, \frac{1}{4}(t-s)I_d\bigg).$$

The Brownian tree

By recursively applying Lévy's construction, we can construct a tree:



This is known as the <u>Brownian tree</u> (introduced in [1]) and also gives a natural data structure when generating Brownian sample paths [3, 4].

A "Brownian tree" condition

In our second counterexample, we could "ignore" information about the Brownian path – as the following update is decided using $W_{t_{k+\frac{1}{2}}}$:

$$Y_{k+1} = Y_k + \frac{1}{2}\left(W^1_{t_k} + W^1_{t_{k+1}}\right)\left(W^2_{t_{k+1}} - W^2_{t_k}\right)$$

but then does not use the value of $W_{t_{k+\frac{1}{2}}}$ in the approximation itself.

Hence, this goes against the natural direction of the Brownian tree (indicated by the downwards arrow).

First important condition

If information about the Brownian motion is generated, it <u>must be used</u> "correctly" (to be explained in condition 2). Equivalently, the numerical approximation uses all the information at the lowest level of the tree.

Outline

Stochastic Taylor expansions

Consider the Stratonovich SDE ($y_t \in \mathbb{R}^e$ and $f, g_i : \mathbb{R}^e \to \mathbb{R}^e$ are smooth)

$$dy_t = f(y_t)\,dt + \sum_{i=1}^{d} g_i(y_t) \circ dW_t^i\,, \tag{4}$$

A very useful tool in SDE numerical analysis is the Taylor expansion:

Theorem (Stratonovich-Taylor expansion [5, Thm 5.6.1])

For times $0 \leq s \leq t \leq T$, the solution of the SDE (4) can be expanded as

$$y_t = y_s + f(y_s)h + \sum_{i=1}^{d} g_i(y_s)W_{s,t}^i + \sum_{i,j=1}^{d} g_j'(y_s)g_i(y_s)\int_s^t W_{s,u}^i \circ dW_u^j + R,$$

where $h := t - s$ and there exists $C > 0$ such that $\mathbb{E}\big[\|R\|_2^2\big]^{\frac{1}{2}} \leq Ch^{\frac{3}{2}}$.

Non-Gaussian integrals involving Brownian motion

The stochastic integrals $\left\{\int_s^t W_{s,u}^i \circ dW_u^j\right\}_{1\leq i,j\leq d}$ are non-Gaussian and an algorithm for exact simulation has only been found when $d = 2$ [6].

However, this does not have a "Lévy's construction", so cannot be used adaptively. Therefore, we shall approximate these Brownian integrals.

$$\mathbb{E}\left[\int_s^t W_{s,u}^i \circ dW_u^j \,\bigg|\, W_{s,t}\right] = \frac{1}{2} W_{s,t}^i W_{s,t}^j . \tag{5}$$

Among the $W_{s,t}$-measurable estimators, this minimises the $L^2(\mathbb{P})$ error.

We can also approximate Lévy area using increments and <u>integrals</u> of W,

$$\mathbb{E}\left[\int_s^t W_{s,u}^i \circ dW_u^j \,\bigg|\, W_{s,t}\ \int_s^t W_{s,u}\,, du\right] = \frac{1}{2} W_{s,t}^i W_{s,t}^j + W_{s,t}^i \int_s^t \frac{u-s}{h} dW_u^j - W_{s,t}^j \int_s^t \frac{u-s}{h} dW_u^i .$$

An "integral" condition

Second important condition

The numerical method for the Stratonovich SDE (4) should satisfy

$$Y_{k+1} = Y_k + f(Y_k)h + \sum_{i=1}^{d} g_i(Y_k)W_k^i + \frac{1}{2}\sum_{i,j=1}^{d} g_j'(Y_k)g_i(Y_k)W_k^i W_k^j + R,$$

where $h := t_{k+1} - t_k$, $W_k := W_{t_{k+1}} - W_{t_k}$ and $R \sim o(h)$ almost surely.

More generally, if the numerical approximation uses certain Gaussian integrals $\mathcal{W}_k$ generated over the interval $[t_k, t_{k+1}]$, then we require:

$$\begin{aligned} Y_{k+1} = Y_k + f(Y_k)h &+ \sum_{i=1}^{d} g_i(Y_k)W_k^i \\ &+ \sum_{i,j=1}^{d} g_j'(Y_k)g_i(Y_k)\mathbb{E}\left[\int_{t_k}^{t_{k+1}} W_{t_k,t}^i \circ dW_t^j \,\middle|\, \mathcal{W}_k\right] + o(h). \end{aligned}$$

Examples of methods satisfying the integral condition

Milstein's method*

*using $q+1$ integrals of Brownian motion (which are Gaussian [8, 9])

$$Y_{k+1} := Y_k + f(Y_k)h + \sum_{i=1}^{d} g_i(Y_k)W_k^i$$

$$+ \sum_{i,j=1}^{d} g_j'(Y_k)g_i(Y_k)\mathbb{E}\left[\int_{t_k}^{t_{k+1}} W_{t_k,t}^i \circ dW_t^j \,\Big|\, \left\{\int_{t_k}^{t_{k+1}} \left(\tfrac{t-t_k}{h}\right)^m dW_t\right\}_{0\leq m\leq q}\right].$$

Heun's method (expanding will give $\frac{1}{2}W_{s,t}^i W_{s,t}^j$ instead of $\int_s^t W_{s,u}^i \circ dW_u^j$)

$$\widetilde{Y}_{k+1} = Y_k + f(Y_k)h + \sum_{i=1}^{d} g_i(Y_k)W_k^i\,,$$

$$Y_{k+1} = Y_k + \frac{1}{2}\big(f(Y_k) + f(\widetilde{Y}_{k+1})\big)h + \frac{1}{2}\sum_{i=1}^{d}\big(g_i(Y_k) + g_i(\widetilde{Y}_{k+1})\big)W_k^i\,.$$

Splitting Path Runge-Kutta (SPaRK is based on the "$q=1$" estimator)

Outline

Main convergence theorem

Theorem (Convergence of adaptive methods [2, Theorem 2.19])

Let $\{Y^n\}$ be a sequence of numerical solutions to (4) computed at times $D_n = \{0 = t_0^n < t_1^n < \cdots < t_{K_n}^n = T\}$ so that D_{n+1} is determined by D_n and

$$\mathcal{W}_k^n := \left\{ \int_{t_k^n}^{t_{k+1}^n} \left(\frac{t - t_k^n}{h_k^n} \right)^m dW_t \right\}_{0 \leq m \leq q}.$$

Suppose $D_{n+1} \subseteq D_n$ and $\mathrm{mesh}(D_n) \to 0$ almost surely (condition 1) and

$$\left\| Y_{k+1}^n - \widetilde{Y}_{k+1}^n \right\|_2 \sim o(h_k^n),$$

where $h_k^n := t_{k+1}^n - t_k^n$ and

$$\widetilde{Y}_{k+1} := Y_k^n + f(Y_k^n) h_k^n + \sum_{i=1}^{d} g_i(Y_k^n) W_{t_k^n, t_{k+1}^n}^i \qquad \text{(condition 2)}$$
$$+ \sum_{i,j=1}^{d} g_j'(Y_k^n) g_i(Y_k^n) \mathbb{E}\left[\int_{t_k^n}^{t_{k+1}^n} W_{t_k^n, t}^i \circ dW_t^j \,\middle|\, \mathcal{W}_k^n \right].$$

Main convergence theorem

Theorem (Convergence of adaptive methods [2], continued)

We assume $Y_0^n = y_0$ and $f, \{g_i\}$ are bounded twice differentiable vector fields with α-Hölder continuous second derivatives for some $\alpha \in (0,1)$.

More precisely, we assume that

$$\left\| Y_{k+1}^n - \widetilde{Y}_{k+1}^n \right\|_2 \leq w(t_k^n, t_{k+1}^n),$$

where

$$\sum_{k=0}^{K_n - 1} w(t_k^n, t_{k+1}^n) \to 0,$$

almost surely. Then the approximations $\{Y^n\}$ converge pathwise. That is

$$\sup_{0 \leq k \leq K_n} \left\| Y_k^n - y_{t_k^n} \right\|_2 \to 0,$$

as $n \to \infty$ almost surely.

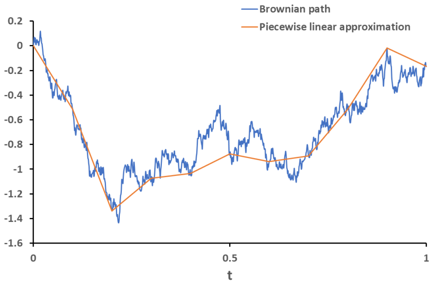
Ideas in the proof

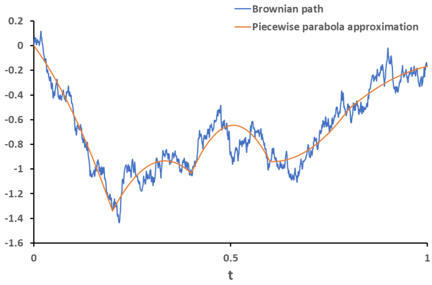
Using the main results of [10, 11], we note that on the interval $[t_k^n, t_{k+1}^n]$,

$$\int_{t_k^n}^{t_{k+1}^n} \left(\frac{t - t_k^n}{h_k^n}\right)^m d\widetilde{W_t}^n = \int_{t_k^n}^{t_{k+1}^n} \left(\frac{t - t_k^n}{h_k^n}\right)^m dW_t,$$

for $0 \leq m \leq q$, where $\widetilde{W_t}^n$ is the degree $n+1$ polynomial defined as

$$\widetilde{W_t}^n := \mathbb{E}\big[W_t \,|\, \{\mathcal{W}_k^n\}_{0 \leq k \leq K_n - 1}\big].$$

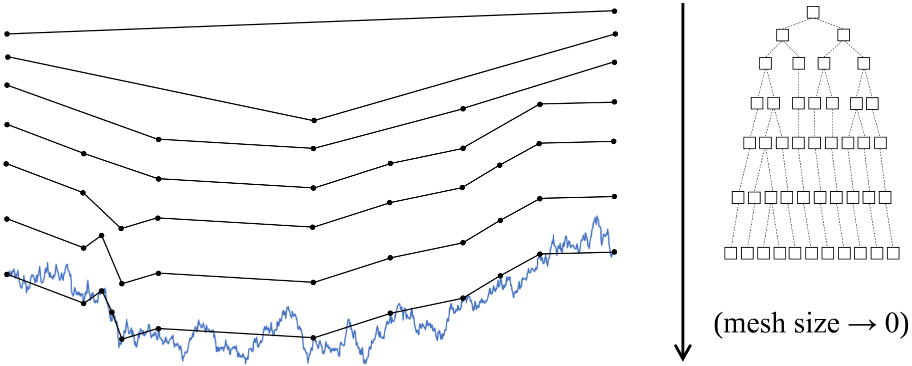




Ideas in the proof

Lemma

Define a sequence of σ*-algebras* $\{\mathcal{F}_n\}_{n\geq 0}$, *by* $\mathcal{F}_0 := \sigma\big(\{\mathcal{W}_0^n\} \cup D_0\big)$ *and* $\mathcal{F}_{n+1} := \sigma\big(\mathcal{F}_n \cup \{\mathcal{W}_k^n\} \cup D_n\big)$. *By the assumptions in the theorem,* $\{\mathcal{F}_n\}$ *is a filtration and* $\widetilde{W}^n = \mathbb{E}\big[W \,|\, \mathcal{F}_n\big]$ *is a square-integrable martingale.*



Ideas in the proof

Using Doob's martingale convergence theorem and maximal inequality, we can show that

$$d_{p\text{-}var;[0,T]}\big(\widetilde{\boldsymbol{W}}^{n}, \boldsymbol{W}\big) \to 0,$$

as $n \to \infty$ almost surely, where $p \in (2,3)$ and

- $\widetilde{\boldsymbol{W}}^{n}$ is the piecewise polynomial "lifted" to a "p-rough path"
- $\boldsymbol{W}$ is "Stratonovich enhanced" Brownian motion (p-rough path)
- $d_{p\text{-}var;[0,T]}(\boldsymbol{X}, \boldsymbol{Y})$ is the p-variation between p-rough paths $\boldsymbol{X}$ and $\boldsymbol{Y}$

It is not clear how to show "rough path" convergence for $\{\widetilde{W}^{n}\}$ without using the martingale property coming from the nested property of $\{D_n\}$.

Ideas in the proof

By the well-known Universal Limit Theorem [12], it now follows that

$$d_{p\text{-}var;[0,T]}\big(\widetilde{\boldsymbol{y}}^{n}, \boldsymbol{y}\big) \to 0,$$

as $n \to \infty$ almost surely, where

- $\widetilde{\boldsymbol{y}}^{n}$ is the solution of the rough differential equation (RDE):

  $$d\widetilde{\boldsymbol{y}}_t^{n} = f\big(\widetilde{\boldsymbol{y}}_t^{n}\big)dt + g\big(\widetilde{\boldsymbol{y}}_t^{n}\big)\, d\widetilde{\boldsymbol{W}}_t^{n},$$

  with $g(y) := (g_1(y), \cdots, g_d(y))$ and initial condition $y_0 \in \mathbb{R}^e$.

- $\boldsymbol{y}$ is the solution of the rough differential equation (RDE):

  $$d\boldsymbol{y}_t = f\big(\boldsymbol{y}_t\big)dt + g\big(\boldsymbol{y}_t\big)\, d\boldsymbol{W}_t\,,$$

  with initial condition $y_0 \in \mathbb{R}^e$.

Ideas in the proof

Finally, we need to compare the solution of the CDE

$$d\widetilde{y}_t^n = f(\widetilde{y}_t^n)\,dt + g(\widetilde{y}_t^n)\,d\widetilde{W}_t^n,$$

to our numerical method

$$\widetilde{Y}_{k+1} := Y_k^n + f(Y_k^n)h_k^n + \sum_{i=1}^d g_i(Y_k^n)W^i_{t_k^n, t_{k+1}^n} \qquad \text{(condition 2)}$$
$$+ \sum_{i,j=1}^d g_j'(Y_k^n)g_i(Y_k^n)\mathbb{E}\Bigg[\int_{t_k^n}^{t_{k+1}^n} W^i_{t_k^n, t} \circ dW_t^j \,\Bigg|\, \mathcal{W}_k^n\Bigg],$$

and show the difference is $o(h_n)$. However, this is straightforward as

$$W^n_{t_k^n, t_{k+1}} = \widetilde{W}^n_{t_k^n, t_{k+1}},$$

$$\mathbb{E}\Bigg[\int_{t_k^n}^{t_{k+1}^n} W^i_{t_k^n, t} \circ dW_t^j \,\Bigg|\, \mathcal{W}_k^n\Bigg] = \int_{t_k^n}^{t_{k+1}^n} \big(\widetilde{W}^n_{t_k^n, t}\big)^i \circ d\big(\widetilde{W}_t^n\big)^j.$$

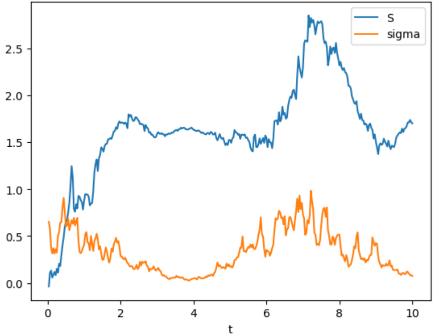
Outline

Numerical example

We consider the SABR stochastic volatility model used in finance [13]:

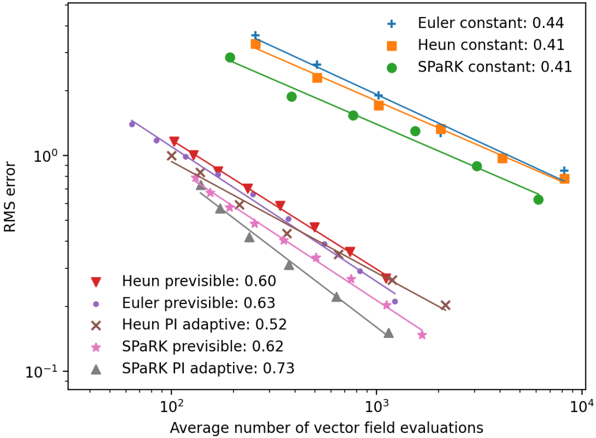
$$
\begin{aligned}
dS_t &= \sqrt{1-\rho^2}\,\sigma_t (S_t)^{\beta} dW_t^1 + \rho\sigma_t (S_t)^{\beta} dW_t^2, \\
d\sigma_t &= \alpha\sigma_t dW_t^2,
\end{aligned} \tag{6}
$$

where $(S_0, \sigma_0) = (0, 1)$ and $(\alpha, \beta, \rho) = (1, 0, 0)$. For each method and step size control, we estimate the $L^2(\mathbb{P})$ error over $[0, T]$ with $T = 10$.



Numerical example

Methods implemented in the JAX-based ODE/SDE library Diffrax [3, 4].



Code for reproducing these results can be found at github.com/andyElking/Adaptive_SABR.

Outline

Conclusion and future work

Conclusion

- Numerical methods that use adaptive step sizes are popular for ODEs, but can experience convergence issues in the SDE setting.
- Using rough paths, we showed that convergence occurs for a large class of adaptive methods (including Milstein and Heun schemes).
- The main idea is that whenever information about W is generated, it must be used (condition 1) in a "correct way" (condition 2).

Future work

- Can we establish explicit convergence rates for adaptive methods?
- Does our convergence analysis extend to high order weak solvers?
- Applications? (e.g. adaptive SDE-based MCMC algorithms, see [4])

Thank you
for your attention!

and the preprint can be found at:

J. Foster and A. Jelinčič.
On the convergence of adaptive approximations for stochastic differential equations, arxiv:2311.14201, 2024.

Outline

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