MA20222 Problem Sheet 7

Do all questions and hand in your answers to the ⋆**starred**⋆ questions as instructed by your tutor.

⋆E7.1. Consider the simple initial-value problem:

$$\frac{dy}{dt} = y, \quad t \geq 0, \quad y(0) = 1.$$

(a) Write down the exact solution $y(h)$ for any $h > 0$.

(b) Taking h as the time step for Euler's method, find Y_1 and write down the error $e_1 = y(h) - Y_1$ as an expansion in powers of h.

(c) Repeat for the Crank–Nicolson method, as defined in Eq. (4.16).

(d) Repeat for the improved Euler method, as defined in Eqs. (4.18) and (4.19).

In each case, you should see that the error after one step of the method converges to 0 with an order one power of h faster than the truncation error.

E7.2. Consider the improved Euler method for the problem

$$\frac{dy}{dt} = f(y), \quad y(0) = y_0.$$

written like $Y_n = Y_{n-1} + hF_h(Y_{n-1})$, for a suitable function F_h (as defined in the printed notes after Eq. (4.19)).

Use Taylor expansions to show that the truncation error for this method satisfies

$$\tau_{n+1} = C_n\, h^2 \;+\; \mathcal{O}(h^3),$$

where C_n is a constant independent of h and n, but depending on $f^{(k)}(y(t_n))$, $k = 0, 1, 2$, which you should specify. You may assume that f has as many derivatives as you need.

⋆E7.3. Complete the proof of Theorem 4.7: Suppose that f is Lipschitz continuous with L denoting the Lipschitz constant. Let $h > 0$ and let $n \in \mathbb{N}$ be such that $t_n = nh \leq T$. If $y(t_n)$ is the solution to Eq. (4.2) at time t_n and Y_n is the nth iterate of the Crank–Nicolson method defined in Eq. (4.16), you may assume that $e_n = y(t_n) - Y_n$ satisfies

$$|e_n| \leq (1 + hL)^2 |e_{n-1}| + h(1 + hL)|\tau_n| \quad \text{for all } n \geq 1.$$

(a) Show by induction that

$$|e_n| \leq h \sum_{j=0}^{n-1} (1 + hL)^{2j+1} |\tau_{n-j}|.$$

(b) Hence deduce that

$$|e_n| \leq \frac{\exp(2TL) - 1}{L} \max_{1 \leq j \leq n} |\tau_j|.$$