MA20222 Problem Sheet 5

Do all questions and hand in your answers to the ⋆**starred**⋆ questions as instructed by your tutor.

E5.1. i) Consider the following quadrature rule for approximating $\int_{-1}^{1} f(x)\mathrm{d}x$:

$$Q(f) = w_0 f(-1) + w_1 f(1) + w_2 f'(-1) + w_3 f'(1).$$

Here, the quadrature nodes are $-1, 1$ and Q evaluates f and also its derivative f' at these nodes. Determine the weights w_i so that Q has degree of precision 3.

ii) By applying a change of variable, write this quadrature rule over the general interval $[a, b]$.

⋆E5.2. Consider the 3-point Gauss quadrature rule:

$$Q^{[-1,1]}_{\text{Gauss},2}(f) = \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right).$$

Integrate the polynomial

$$\int_{-1}^{1} 2x^5 + 16x^4 - 3x^2 + 9x - 2\,dx$$

analytically, and subsequently, show that $Q^{[-1,1]}_{\text{Gauss},2}$ integrates this exactly.

Compute $E^{[-1,1]}(x^6)$ and the degree of precision for this rule.

⋆E5.3. When deriving the Gauss quadrature rules in Section §3.3, we have full freedom of the weights and nodes. Suppose we are required to choose the end points i.e. $x_0 = -1$, and $x_N = 1$.

What is the maximum degree of precision achievable using this quadrature rule with $N + 1$ points?

Consider the quadrature rule

$$Q^{[-1,1]}_{\text{R},3}(f) = w_0 f(-1) + w_1 f(x_1) + w_2 f(x_2) + w_3 f(1).$$

Write down equations which must be satisfied for the unknowns and solve them.

HINT: Use symmetry as in Section §3.3.