Problem sheet 3

Please submit the question marked * to your tutor.

E3.1*

In this exercise, you will write a function to compute the error of the piecewise linear interpolant. Your function takes as input arguments $a, b \in \mathbb{R}$ with $a < b$, a positive integer J, and a function f.

Your function should then calculate $h = (b-a)/J$, the points $y_j = a + jh$, $j = 0, \ldots, J$, and evaluate

$$e_h := \max_{j=1,...,J} |f(z_j) - p_{1,J}(z_j)|.$$

where $p_{1,J}$ is the piecewise-linear interpolant of f with respect to the mesh y_j, and z_j is the midpoint of the interval $[y_{j-1}, y_j]$.

For $a = 0, b = 1$ and each of $f(x) = \exp(x)$ and $f(x) = \sin x$, draw up a table of e_h against h for $h = 1/8, 1/16, 1/32, \ldots$.

Experimentally assess the rate of convergence as $h \to 0$. Explain the results (theoretically).

Repeat for $f(x) = |x - 1/2|^{1/4}$ and $[a, b] = [0, 1]$. Explain the results as well as you can also in this case.

In [1]:
```
import numpy as np
def error(a,b,J, f):
    #fill in
    return eh
```

In [2]:
```
#investigating for exp(x)
func = lambda x: np.exp(x)
for i in range(3,10):
    J = 2**i
    # Call error(0,1,J, func) and investigate the error
```

In [3]:
```
#investigating for |x-1/2|^{1/4}
```

E3.2

Let $[a, b] = [0, 1]$ and consider a general mesh $0 = y_0 < y_1 < \cdots < y_J = 1$

Let $p_{1,J}$ denote the piecewise-linear interpolant of f that coincides with f at the end-points of the subintervals. A measure of the error is

$$e_J = \max_{j=1,...,J} |(f - p_{1,J})(z_j)|,$$

where z_j is the mid-point of $[y_{j-1}, y_j]$. I have changed the the notation from e_h to e_J because I want to focus on how the error decreases as J increases. J is a reasonable measure of the amount of computation needed to compute the interpolant.

Consider the function $f(x) = x^{1/4}$.

(a) For the uniform mesh $y_j = j/J$ with $j = 0, \ldots, J$, draw up a table of e_J against J and show experimentally that the rate of convergence of e_J to 0 is about $\mathcal{O}(J^{-1/4})$.

(b) Repeat this experiment with the same $f(x)$ but use instead the adapted mesh $y_j = (j/J)^8$. What rate of convergence do you now observe as J increases?

In [4]:
```
#Code
```

E3.3*

In this question, you will theoretically prove the convergence behaviour observed in E2.

(a) For the uniform mesh, explain the $\mathcal{O}(J^{-1/4})$ convergence by proving an estimate for e_J.

Hint: when estimating the error on the first subinterval, use the triangle inequality to get $\|f - p_{1,J}\|_{\infty,[y_0,y_1]} \leq \|f\|_{\infty,[y_0,y_1]} + \|p_{1,J}\|_{\infty,[y_0,y_1]} \leq 2y_1^{1/4}$. For all other subintervals, use error estimates from lectures.

(b) For the adapted mesh in the previous question, explain your calculations by proving an estimate for e_J that explains the $\mathcal{O}(J^{-2})$ convergence.

In [ ]:
```
```