Deformations and gluing of asymptotically cylindrical manifolds with exceptional holonomy

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Abstract

In Berger's classification of Riemannian holonomy groups there are several infinite families and two exceptional cases: the groups $Spin(7)$ and G_2. This thesis is mainly concerned with 7-dimensional manifolds with holonomy G_2. A metric with holonomy contained in G_2 can be defined in terms of a torsion-free G_2-structure, and a G_2-manifold is a 7-dimensional manifold equipped with such a structure.

There are two known constructions of compact manifolds with holonomy exactly G_2. Joyce found examples by resolving singularities of quotients of flat tori. Later Kovalev found different examples by gluing pairs of exponentially asymptotically cylindrical (EAC) G_2-manifolds (not necessarily with holonomy exactly G_2) whose cylinders match. The result of this gluing construction can be regarded as a generalised connected sum of the EAC components, and has a long approximately cylindrical neck region.

We consider the deformation theory of EAC G_2-manifolds and show, generalising from the compact case, that there is a smooth moduli space of torsion-free EAC G_2-structures. As an application we study the deformations of the gluing construction for compact G_2-manifolds, and find that the glued torsion-free G_2-structures form an open subset of the moduli space on the compact connected sum. For a fixed pair of matching EAC G_2-manifolds the gluing construction provides a path of torsion-free G_2-structures on the connected sum with increasing neck length. Intuitively this defines a boundary point for the moduli space on the connected sum, representing a way to 'pull apart' the compact G_2-manifold into a pair of EAC components. We use the deformation theory to make this more precise.

We then consider the problem whether compact G_2-manifolds constructed by Joyce's method can be deformed to the result of a gluing construction. By proving a result for resolving singularities of EAC G_2-manifolds we show that some of Joyce's examples can be pulled apart in the above sense. Some of the EAC G_2-manifolds that arise this way satisfy a necessary and sufficient topological condition for having holonomy exactly G_2.

We prove also deformation results for EAC $Spin(7)$-manifolds, i.e. dimension 8 manifolds with holonomy contained in $Spin(7)$. On such manifolds there is a smooth moduli space of torsion-free EAC $Spin(7)$-structures. Generalising a result of Wang for compact manifolds we show that for EAC G_2-manifolds and $Spin(7)$-manifolds the special holonomy metrics form an open subset of the set of Ricci-flat metrics.

Preface

Originality. This dissertation is my own work and includes nothing which is the outcome of work done in collaboration except where specifically indicated in the text.

Contents

Chapter 1

Introduction

The holonomy group of a Riemannian manifold is the group of isometries of a tangent space generated by parallel transport around closed paths based at a point using the Levi-Civita connection. In the classification of Riemannian holonomy groups due to Berger [4] there are two exceptional cases: G_2 and $Spin(7)$. This thesis is concerned with manifolds of exceptional holonomy. They can be described as Riemannian manifolds with different versions of parallel 'octonionic structures' on the tangent spaces. We will study in particular the deformation theory of manifolds with exceptional holonomy which are asymptotically cylindrical.

The main focus of the thesis is on the G_2 case. We apply the deformation results to study the properties of a gluing construction that uses asymptotically cylindrical G_2-manifolds. This leads to the idea that one can attempt to reverse the gluing construction, and 'pull apart' a compact manifold with holonomy G_2 into a pair of asymptotically cylindrical connected summands. We apply this idea to some of the compact G_2-manifolds that Joyce constructed by a Kummer-type method. In particular we find some examples of asymptotically cylindrical manifolds with holonomy exactly G_2.

G_2 is a compact simple simply-connected Lie group of dimension 14. It is one of the exceptional simple Lie groups. It can be defined as the automorphism group of the normed algebra of octonions $\mathbb{O}$, or equivalently as the group of linear transformations of $\mathbb{R}^7$ that preserve the Euclidean metric and a vector cross product. A G_2-*structure* on a manifold M^7 is a smoothly varying identification of the tangent space at each point of M with $\mathbb{R}^7$ equipped with a metric and vector cross product. A G_2-structure can be defined by a differential 3-form φ, and is said to be *torsion-free* if $\nabla\varphi = 0$. A metric on M^7 has holonomy contained in G_2 if and only if it can be defined by a torsion-free G_2-structure.

G_2-manifolds, i.e. dimension 7 manifolds equipped with a torsion-free G_2-structure, are Ricci-flat.

Similarly, $Spin(7)$ is a compact simple simply-connected Lie group of dimension 21. It is the double cover of $SO(7)$. Its spinor representation is faithful and real of rank 8, so it can be considered as a subgroup of $SO(8)$. This action of $Spin(7)$ on $\mathbb{R}^8$ preserves a certain triple cross product, which can be defined in terms of the algebraic structure of the octonions. A $Spin(7)$-*structure* on a manifold M^8 is an identification of the tangent space of M at each point with $\mathbb{R}^8$ equipped with a metric and triple cross product. A $Spin(7)$-structure can be defined by a differential 4-form ψ, and is said to be *torsion-free* if $\nabla\psi = 0$. A metric on M^8 has holonomy contained in $Spin(7)$ if and only if it can be defined by a torsion-free $Spin(7)$-structure. $Spin(7)$-manifolds too are Ricci-flat.

G_2 is the stabiliser in $Spin(7)$ of a vector in $\mathbb{R}^8$. Similarly the stabiliser in G_2 of a vector in $\mathbb{R}^7$ is isomorphic to $SU(3)$. We will therefore also be interested in 6-dimensional manifolds with holonomy $SU(3)$. These are Calabi-Yau 3-folds. They have an integrable complex structure and a Ricci-flat Kähler metric. They also have a holomorphic volume form (i.e. a nowhere vanishing $(3,0)$-form) so the first Chern class vanishes.

Constructing manifolds with holonomy group exactly G_2 or $Spin(7)$ is complicated. The first local examples were found by Bryant [8], and the first complete examples by Bryant and Salamon [10]. The first compact examples were constructed by Joyce in [26], by desingularising quotients of flat tori.

By contrast it is easy to find compact Calabi-Yau manifolds. A large supply of compact Kähler manifolds with vanishing first Chern class can be found using complex algebraic geometry, and such manifolds have Ricci-flat Kähler metrics by Yau's solution of the Calabi conjecture [56]. Taking the product of a Calabi-Yau 3-fold with a circle gives a G_2-manifold, but a reducible one. Generally any G_2-manifold whose holonomy is a proper subgroup of G_2 is locally reducible, in the sense that any point has a neighbourhood that is a product of two Riemannian manifolds of non-zero dimension, while a 7-dimensional manifold with holonomy exactly G_2 is irreducible.

Another way to obtain irreducible compact G_2-manifolds is by gluing a pair of non-compact G_2-manifolds which are asymptotically cylindrical. A manifold is said to have cylindrical ends if it is homeomorphic to a cylinder outside a compact piece. An asymptotically cylindrical manifold is a Riemannian manifold with cylindrical ends for which the metric is asymptotic to a product metric on the cylindrical ends. Asymptotically cylindrical manifolds are easier to work with than arbitrary non-compact manifolds, e.g. many analysis

results for elliptic operators on compact manifolds can be generalised to statements about asymptotically translation-invariant elliptic operators acting on suitable spaces of sections on an asymptotically cylindrical manifold. Such results are proved in e.g. [39] and [40]. In some arguments it is helpful to impose a stronger condition, requiring the manifold to be *exponentially asymptotically cylindrical* (EAC).

Given a pair of EAC G_2-manifolds whose cylinders match one can form a generalised connected sum by truncating the cylinders after some large but finite length and gluing them together. If the 'neck length' is sufficiently large then the EAC G_2-structures can be glued to form a torsion-free G_2-structure on the connected sum. This is a *gluing construction* for compact G_2-manifolds. Kovalev [34] proves an EAC version of the Calabi conjecture to produce EAC Calabi-Yau 3-folds. By multiplying with circles reducible EAC G_2-manifolds are obtained, which can be glued to form irreducible compact G_2-manifolds of different topological types from those constructed by Joyce.

In chapter 2 we review the basic properties of the exceptional holonomy groups, and of EAC manifolds. Chapter 3 discusses the local deformation problem for compact manifolds with exceptional holonomy, which was solved previously by Joyce [26]. The group of diffeomorphisms isotopic to the identity on a compact G_2-manifold M^7 acts on the space of torsion-free G_2-structures by pull-backs, and the quotient is the moduli space of torsion-free G_2-structures. We prove that the moduli space is a smooth manifold. Since the torsion-free G_2-structures are induced by closed 3-forms there is a natural projection map to the de Rham cohomology $H^3(M)$, and this is a local diffeomorphism. The argument used here is a modification of that used by Hitchin [24], and we adapt it to deal also with deformations of compact $Spin(7)$-manifolds and Calabi-Yau 3-folds (in these cases the projection to de Rham cohomology is an immersion rather than a local diffeomorphism). The deformation problem for Calabi-Yau manifolds (in any dimension ≥ 3) was solved independently by Tian [52] and Todorov [53], using the Kodaira-Spencer theory for deformations of complex manifolds. Goto [17] has also studied the deformations of compact G-manifolds when G is any of the Ricci-flat holonomy groups $SU(n)$, $Sp(n)$, $Spin(7)$ or G_2.

In chapter 4 we discuss the properties of EAC G-manifolds, where G is one of the exceptional holonomy groups G_2 and $Spin(7)$. The main result here is that the moduli spaces of EAC torsion-free G-structures are smooth manifolds. The proofs are generalisations from the compact case. The extra difficulties are largely to do with the analysis and understanding the boundary conditions.

For an EAC G_2-manifold M^7 with cross-section X^6 let $\mathcal{M}_+$ be the moduli space of

torsion-free EAC G_2-structures on M. This is the quotient of the space of EAC torsion-free G_2-structures (with any exponential rate of decay) by the group of EAC diffeomorphisms isotopic to the identity. The cross-section X of M is a compact Calabi-Yau 3-fold. By the results in chapter 3 the moduli space $\mathcal{N}$ of Calabi-Yau structures on X is a smooth manifold. There is a natural boundary map

$$B : \mathcal{M}_+ \to \mathcal{N}. \tag{1.1}$$

In order to obtain an immersion of $\mathcal{M}_+$ it is not enough to project the EAC torsion-free G_2-structures to the de Rham cohomology $H^3(M)$, but we must also use the image in $H^2(X)$ of the Kähler class of the induced Calabi-Yau structure on X. Theorem 4.2.2 states that $\mathcal{M}_+$ is a smooth manifold, and that the map

$$\mathcal{M}_+ \to H^3(M) \times H^2(X)$$

is an immersion. The dimension of the moduli space is given by the formula

$$\dim \mathcal{M}_+ = b^4(M) + \tfrac{1}{2}b^3(X) - b^1(M) - 1.$$

Moreover the boundary map B is a submersion onto its image, which is a submanifold of $\mathcal{N}$ defined locally by a topological condition.

Theorem 4.3.2 is the corresponding result for $Spin(7)$-manifolds. The moduli space $\mathcal{M}_+$ of EAC torsion-free $Spin(7)$-structures on an EAC $Spin(7)$-manifold M^8 is a smooth manifold, and the projection to de Rham cohomology

$$\mathcal{M}_+ \to H^4(M)$$

is an immersion. In this case the cross-section X is a compact G_2-manifold, and we rely on the result that the moduli space $\mathcal{N}$ of torsion-free G_2-structures on X is a smooth manifold. Again there is a natural boundary map $B : \mathcal{M}_+ \to \mathcal{N}$, and it is a submersion onto a submanifold. The dimension is given by

$$\dim \mathcal{M}_+ = b^4(M) - b^4_+(M) + b^1(X) - b^1(M) - 1,$$

where $b^4_+(M)$ denotes the dimension of the positive part of the compactly supported subspace of $H^4(M)$.

We also prove theorems 4.1.11 and 4.1.19, which give necessary and sufficient topological conditions for an asymptotically cylindrical G_2-manifold or $Spin(7)$-manifold M to have a holonomy exactly G_2 or $Spin(7)$, respectively, and not a proper subgroup. In the G_2 case the condition for full holonomy is that the fundamental group $\pi_1(M)$ is finite and that neither M nor any double cover of M is homeomorphic to a cylinder. This is a generalisation of a known result for compact G_2-manifolds. In the $Spin(7)$ case one must additionally assume that the cross-section $\tilde{X}$ of the universal cover has $b^1(\tilde{X}) = 0$ to ensure that the holonomy is exactly $Spin(7)$.

It is usually most effective to study manifolds with exceptional holonomy $G = Spin(7)$ or G_2 in terms of the G-structures, but in chapter 5 we consider the deformations of the metrics themselves. A result due to Wang [55] states that small Ricci-flat deformations of G-metrics still have holonomy contained in G. We give a careful account of this result, and explain how it can be extended to the EAC case by using the deformation theory of EAC G-manifolds from chapter 4. We also show that the moduli space $\mathcal{W}_G$ of G-metrics is smooth, and that the moduli space of G-structures is a locally trivial fibre bundle over $\mathcal{W}_G$ (for manifolds with holonomy exactly G the two moduli spaces are diffeomorphic).

In chapter 6 we discuss how the gluing of EAC G_2-manifolds behaves under deformations. If M^7 is a compact G_2-manifold produced by the gluing construction then the glued G_2-structures form an open subset of the moduli space $\mathcal{M}$ of torsion-free G_2-structures on M. To set up the proof of this claim we first note that the data required for the gluing construction is a pair of torsion-free EAC G_2-structures with matching asymptotic limits together with a 'neck length' parameter. The parameter specifies how far along the infinite ends the EAC G_2-manifolds are truncated before gluing, so it controls the diameter of the glued G_2-structure. We use the results about the boundary map (1.1) to show that there is a smooth moduli space $\mathcal{G}$ of data for the gluing construction. Then the construction gives a smooth gluing map

$$Y : \mathcal{G} \to \mathcal{M}. \tag{1.2}$$

Theorem 6.1.9 states that this is a local diffeomorphism, so in particular its image is open.

Intuitively, a path given by gluing a matching pair of EAC G_2-structures with an increasing neck length defines a boundary point of $\mathcal{M}$, corresponding to 'pulling apart' M into a pair of EAC connected-summands. We can use the gluing map (1.2) to make this precise. Theorem 6.1.10 states that $\mathcal{M}$ can be partially compactified by inclusion in a manifold $\overline{\mathcal{M}}$ with boundary, so that paths of increasing neck length converge to boundary points of $\overline{\mathcal{M}}$.

In chapter 7 we study how some of the examples of compact G_2-manifolds constructed by Joyce's method can be pulled apart in the above sense. We explain how Joyce's method of resolving singularities of G_2-orbifolds can be extended to the EAC setting. We can then pull apart some of the simpler of Joyce's examples, by first performing a resolution of singularities to construct their EAC components. Some of the EAC G_2-manifolds that arise this way are simply-connected, and have holonomy exactly G_2 by the topological condition stated above.

Joyce's examples come with a path of torsion-free G_2-structures degenerating to the orbifold metric the construction started from. When it can be pulled apart we therefore find that a connected component of the moduli space of torsion-free G_2-structures has boundary points of both orbifold and connected-sum type.

Chapter 2

Background material

In this chapter we review some background material. We define the holonomy group of a Riemannian manifold, describe the exceptional holonomy groups G_2 and $Spin(7)$, and explain how a metric with holonomy $G = Spin(7)$, G_2 or $SU(3)$ can be defined in terms of a torsion-free G-structure. We explain what exponentially asymptotically cylindrical (EAC) manifolds are, and give some results about elliptic differential operators and Hodge theory on such manifolds.

2.1 Preliminaries

To begin with we introduce the notion of the holonomy of a Riemannian manifold, and explain how a holonomy reduction gives rise to decompositions of spaces of harmonic forms similar to the Hodge decomposition on a Kähler manifold. We also summarise the elements of spin geometry.

2.1.1 Riemannian holonomy

For a fuller discussion of holonomy see e.g. Joyce [27, §2] or Besse [5, §10].

Definition 2.1.1. Let M^n be a manifold with a Riemannian metric g. If $x \in M$ and γ is a closed piecewise C^1 loop in M based at x then the parallel transport around γ (with respect to the Levi-Civita connection of the metric) defines an orthogonal linear map $P_\gamma : T_x M \to T_x M$. The *holonomy group* $Hol(g, x) \subseteq O(T_x M)$ at x is the group $\{P_\gamma : \gamma$ is a closed loop based at $x\}$. The *restricted holonomy group* $Hol^0(g, x)$ is the subgroup of $Hol(g, x)$ consisting of parallel transport maps around null-homotopic loops.

If $x, y \in M^n$ and τ is a path from x to y then we can define a group isomorphism $Hol(g, x) \to Hol(g, y)$ by $P_\gamma \mapsto P_\tau \circ P_\gamma \circ P_\tau^{-1}$. Provided that M is connected we can therefore identify $Hol(g, x)$ with a subgroup of $O(n)$, independently of x up to conjugacy, and talk simply of the holonomy group of g. Another way to express this is that the holonomy is well-defined up to isomorphism as an orthogonal group representation. The restricted holonomy group is the identity component of the holonomy group, and it is a compact Lie subgroup of $SO(n)$ (see [5, §10.E]).

There is a correspondence between tensors fixed by the holonomy group and parallel tensor fields on the manifold.

Proposition 2.1.2 ([27, Proposition 2.5.2]). *Let M^n be a Riemannian manifold, $x \in M$ and E a vector bundle on M associated to TM. If s is a parallel section of E then $s(x)$ is fixed by $Hol(g, x)$. Conversely if $s_0 \in E_x$ is fixed by $Hol(g, x)$ then there is a parallel section s of E such that $s(x) = s_0$.*

If M is a simply-connected Riemannian manifold then the holonomy group must be connected, so $Hol(M) \subseteq SO(n)$. One can ask which subgroups of $SO(n)$ can occur as the holonomy group of a simply-connected Riemannian manifold. The problem simplifies in two cases. If M is a symmetric space it is a homogeneous space G/H, and the holonomy is the adjoint action of H. If M is a Riemannian product then the holonomy group is the product of the holonomy groups of the factors. The list of possible holonomy groups of non-symmetric irreducible Riemannian manifolds is known as *Berger's list*.

Theorem 2.1.3. *Let (M, g) be a simply-connected non-symmetric irreducible Riemannian manifold. Then one of the following 7 cases holds:*

- $Hol(M) = SO(n)$ *and* $\dim M = n$,

- $Hol(M) = U(n)$ *and* $\dim M = 2n$,

- $Hol(M) = SU(n)$ *and* $\dim M = 2n$,

- $Hol(M) = Sp(n)$ *and* $\dim M = 4n$,

- $Hol(M) = Sp(n)Sp(1)$ *and* $\dim M = 4n$,

- $Hol(M) = G_2$ *and* $\dim M = 7$,

- $Hol(M) = Spin(7)$ *and* $\dim M = 8$.

In particular we may note that the Riemannian holonomy groups occur in enumerated infinite families, except for G_2 and $Spin(7)$. For a non-symmetric irreducible manifold M that is not simply-connected we can consider its universal cover and deduce that the restricted holonomy $Hol^0(M)$ must be one of the groups on the list.

If a simply-connected neighbourhood U of $x \in M$ is isometric to a product manifold then $Hol(U, x) \subseteq Hol^0(M, x)$ acts reducibly on $T_x M$. A converse also holds.

Proposition 2.1.4 (cf. [5, Theorem 10.38])**.** *Let M be a Riemannian manifold. If $T_x M$ splits as a sum of irreducible representations $V_1, \ldots, V_k$ under the action of $Hol^0(M, x)$ then there are submanifolds $N_i \subseteq M$ with $T_x N_i = V_i$ such that a neighbourhood of x in M is isometric to $N_1 \times \cdots \times N_k$.*

If M is complete and simply-connected and $Hol(M)$ acts reducibly then there is in fact a corresponding global isometry $M \cong N_1 \times \cdots \times N_k$.

The holonomy of M imposes algebraic constraints on the Riemannian curvature. In particular

Theorem 2.1.5. *If M is a Riemannian manifold and $Hol(M)$ is contained in $SU(n)$, $Sp(n)$, G_2 or $Spin(7)$, then M is Ricci-flat.*

2.1.2 Laplacians

For a Riemannian manifold with holonomy H one can define a *Lichnerowicz Laplacian* on vector bundles associated to the tangent bundle. On differential forms this agrees with the usual Hodge Laplacian, as is explained in [5, §1I]. It was originally noted by Chern [13] that this may be used to study how the Hodge Laplacian interacts with the decomposition of $\Lambda^m T^* M$ into H-invariant subbundles.

Suppose that M is a Riemannian manifold with holonomy group $Hol(M) \subseteq H$. Then the frame bundle of M can be reduced to a principal H-bundle, i.e. M has an H-structure. Let $\rho : H \to GL(E)$ be a representation of H, and E_ρ the corresponding associated vector bundle. Let $\mathfrak{h}_{\mathrm{ad}}$ be the vector bundle induced by the adjoint representation of H. $\mathfrak{h}_{\mathrm{ad}}$ can be identified with a subbundle of $\Lambda^2 T^* M$, and because $Hol(M) \subseteq H$ the Riemannian curvature tensor R is a symmetric section of $\mathfrak{h}_{\mathrm{ad}} \otimes \mathfrak{h}_{\mathrm{ad}}$. We use the Lie algebra representation $D\rho : \mathfrak{h} \to End(E)$ to define

$$(D\rho)^2 : \mathfrak{h} \otimes \mathfrak{h} \to End(E), \ \ a \otimes b \mapsto D\rho(a) \circ D\rho(b).$$

This induces a bundle map $\mathfrak{h}_{\mathrm{ad}} \otimes \mathfrak{h}_{\mathrm{ad}} \to End(E_\rho)$. Symmetry of R implies that $(D\rho)^2(R)$ is self-adjoint.

Definition 2.1.6. Let M be a Riemannian manifold with $Hol(M) \subseteq H$ and ρ a representation of H. The *Lichnerowicz Laplacian* on E_ρ is the formally self-adjoint operator

$$\triangle_\rho = \nabla^*\nabla - 2(D\rho)^2(R) \ : \ \Gamma(E_\rho) \to \Gamma(E_\rho),$$

where ∇ is the connection on E_ρ induced by the Levi-Civita connection on M.

Lemma 2.1.7. *Let M^n be a Riemannian manifold. The Lichnerowicz Laplacian corresponding to the standard representation of $O(n)$ on $\Lambda^m(\mathbb{R}^n)^*$ is the usual Hodge Laplacian.*

Proof. See [5, §1I]. $\qquad\square$

The expression of the Hodge Laplacian in terms of the Lichnerowicz Laplacian of the standard representation of $O(n)$ is often called the *Weitzenböck formula*. The formula is particularly simple on 1-forms.

Proposition 2.1.8. *If M is a Riemannian manifold and ϕ is a 1-form then*

$$\triangle\phi = \nabla^*\nabla\phi + Ric(\phi),$$

where the metric is used to interpret the Ricci curvature as a section of $End(TM)$.

Proof. See [5, (1.155)]. $\qquad\square$

Corollary 2.1.9. *Let M a Ricci-flat manifold and ϕ a 1-form. If ϕ is parallel then ϕ is harmonic. If M is compact then the converse also holds.*

Lemma 2.1.10 (cf. [27, Theorem 3.5.3]). *Let M be a Riemannian manifold and suppose that $Hol(M) \subseteq H$. Let $\phi : E \to F$ be an equivariant map of H-representations (E, ρ), (F, σ). Then ϕ induces a bundle map $E_\rho \to F_\sigma$, and the diagram below commutes.*

$$
\begin{array}{ccc}
\Gamma(E_\rho) & \xrightarrow{\ \phi\ } & \Gamma(F_\sigma) \\
{\scriptstyle \triangle_\rho}\big\downarrow & & \big\downarrow{\scriptstyle \triangle_\sigma} \\
\Gamma(E_\rho) & \xrightarrow{\ \phi\ } & \Gamma(F_\sigma)
\end{array}
$$

In particular, if ρ_1 and ρ_2 are H-representations then $\triangle_{\rho_1 \oplus \rho_2} = \triangle_{\rho_1} \oplus \triangle_{\rho_2}$.

Proof. This is obvious from the fact that the Lichnerowicz Laplacian is defined naturally by the representations. $\qquad\square$

When $\Lambda^m \mathbb{R}^n$ decomposes as a direct sum of subrepresentations $\Lambda^m \mathbb{R}^n = \bigoplus \Lambda_d^m \mathbb{R}^n$ under the action of $H \subseteq O(n)$ there is a corresponding H-invariant decomposition of the exterior product bundle

$$\Lambda^m T^* M = \bigoplus \Lambda_d^m T^* M. \tag{2.1}$$

We write $\Omega_d^m(M)$ for the space of sections of $\Lambda_d^m T^* M$, the 'forms of type d'. We can define bundle projections $\pi_d : \Lambda^m T^* M \to \Lambda_d^m T^* M$. These extend to maps of sections $\pi_d : \Omega^m(M) \to \Omega_d^m(M)$, and allow us to decompose forms into type components.

We can combine lemmas 2.1.7 and 2.1.10 to see that the Hodge Laplacian respects the type decompositions.

Corollary 2.1.11. *Let M^n be a Riemannian manifold with $Hol(M) \subseteq H$. If $\phi : \Lambda_d^m \mathbb{R}^n \to \Lambda_e^k \mathbb{R}^n$ is an H-equivariant map of H-subrepresentations of $\Lambda^m \mathbb{R}^n, \Lambda^k \mathbb{R}^n$ then*

$$
\begin{array}{ccc}
\Omega_d^m(M) & \xrightarrow{\ \phi\ } & \Omega_e^k(M) \\
\triangle \downarrow & & \downarrow \triangle \\
\Omega_d^m(M) & \xrightarrow{\ \phi\ } & \Omega_e^k(M)
\end{array}
$$

commutes. In particular, if $\Lambda_d^m T^* M$ is an H-invariant subbundle of $\Lambda^m T^* M$ then $\triangle$ commutes with π_d on $\Omega^m(M)$, and maps $\Omega_d^m(M)$ to itself.

It follows that given an H-invariant decomposition (2.1) of $\Lambda^m T^* M$ into subbundles there is a corresponding decomposition of the harmonic forms

$$\mathcal{H}^m = \bigoplus \mathcal{H}_d^m.$$

If M is compact then by Hodge theory the natural map $\mathcal{H}^m \to H^m(M)$ is an isomorphism. If we let $H_d^m(M)$ be the image of $\mathcal{H}_d^m$ then we obtain a decomposition of the de Rham cohomology

$$H^m(M) = \bigoplus H_d^m(M). \tag{2.2}$$

2.1.3 Spinors

We collect here some facts about spin representations, spinors and the Dirac operator. For background see e.g. Lawson and Michelsohn [37], or Roe [49] for the analysis side.

One reason why this is relevant for the study of the Ricci-flat holonomy groups is that, as we will see, harmonic spinors on a compact scalar-flat manifold are parallel, while the existence of parallel spinors implies a holonomy reduction by proposition 2.1.2. Indeed Wang [54] gives an explicit characterisation of torsion-free Ricci-flat holonomy structures in terms of parallel spinors on a Riemannian manifold. Determining the dimension of the space of harmonic spinors on a manifold in terms of topological invariants is therefore of interest.

Another reason for us to review spinors is that $Spin(7)$ acting on its spin representation is one of the exceptional holonomy representations. We will also use the ellipticity of the Dirac operator in some of the technical slice arguments when constructing moduli spaces.

For $n \geq 3$ the fundamental group of $SO(n)$ is $\mathbb{Z}_2$. Its universal cover is a Lie group, called the spin group $Spin(n)$. We call the action of $Spin(n)$ on $\mathbb{R}^n$ that factors through $SO(n)$ the *vector representation*. $Spin(n)$ also has a natural complex representation σ_n called the *spin representation*. Without going into the details of how these are constructed we summarise some of their properties.

- If $n = 2k + 1$ then σ_n is irreducible of rank 2^k.

- If $n = 2k$ then σ_n is reducible of rank 2^k. It has two distinct irreducible components $\sigma_n^{\pm}$ of rank 2^{k-1} each.

$$\sigma_n = \sigma_n^+ \oplus \sigma_n^- \tag{2.3}$$

σ_n has a natural Hermitian metric, so that it is a unitary representation of $Spin(n)$. There is a $Spin(n)$-equivariant map

$$\mathbb{R}^n \otimes \sigma_n \to \sigma_n.$$

called *Clifford multiplication*. For $v \in \mathbb{R}^n$ the multiplication map $r_v \in End(\sigma_n)$ is skew-Hermitian. If n is even then r_v maps $\sigma_n^{\pm} \to \sigma_n^{\mp}$ (so σ_n is irreducible considered as an $\mathbb{R}^n$-module). Moreover the *Clifford relations* hold.

- $r_v r_w + r_w r_v = -2 <v, w> id_{\sigma_n}$ for any $v, w \in \mathbb{R}^n$.

There is an obvious inclusion $Spin(n) \hookrightarrow Spin(n+1)$ by splitting $\mathbb{R}^{n+1} = \mathbb{R}^n \oplus \mathbb{R}$, and σ_{n+1} can be restricted to a representation of $Spin(n)$.

- If n is even then $\sigma_{n+1} \cong \sigma_n$ as $Spin(n)$-representations.

- If n is odd then $\sigma_{n+1}^{\pm} \cong \sigma_n$ as $Spin(n)$-representations.

If $n \equiv m \mod 8$ then the spin representations of $Spin(n)$ and $Spin(m)$ have similar properties, a phenomenon sometimes referred to as Bott periodicity. For example

- If $n \equiv 0 \mod 8$ then σ_{n+1}, $\sigma_n^{\pm}$ and σ_{n-1} are real representations.

Let M^n be an oriented Riemannian manifold. M is said to be spin if it has a *spin structure*, i.e. a lift of the principal $SO(n)$-bundle defined by the metric to a principal $Spin(n)$-bundle. This condition is independent of the metric; it is equivalent to the vanishing of the second Stiefel-Whitney class $w_2(M) \in H^2(M, \mathbb{Z}_2)$. If M is spin then the spin structures are in bijection with $H^1(M, \mathbb{Z}_2)$.

Given a spin structure on M^n one may for each representation of $Spin(n)$ define an associated vector bundle on M. In particular one can define the *spinor bundle S* associated to the spin representation σ_n. The Levi-Civita connection of M induces a connection on S

$$\nabla : \Gamma(S) \to \Gamma(T^*M \otimes S).$$

Composing this with the bundle map $T^*M \otimes S \to S$ given by fibre-wise Clifford multiplication defines the *Dirac operator*

$$\eth : \Gamma(S) \to \Gamma(S).$$

This is a first-order partial differential operator, which is elliptic and formally self-adjoint. When n is even the spinor bundle splits as $S = S^+ \oplus S^-$ modelled on (2.3), and $\eth$ maps $\Gamma(S^{\pm}) \to \Gamma(S^{\mp})$. In other words it can be written as a sum of two formally adjoint operators $\eth_{\pm} : \Gamma(S^{\pm}) \to \Gamma(S^{\mp})$. If $n \equiv 2 \mod 4$ then $\operatorname{ind} \eth_{\pm} = 0$ because $S^+ \cong (S^-)^*$. When $n \equiv 0 \mod 4$ the Atiyah-Singer index theorem (see [2]) gives that $\operatorname{ind} \eth_+ = \hat{A}(M)$, where $\hat{A}(M)$ is a characteristic class called the *A-hat genus* of M. In particular it is a topological invariant of M.

The second-order elliptic formally self-adjoint operator $\eth^2 : \Gamma(S) \to \Gamma(S)$ is called the *Dirac Laplacian*. Note that if M is closed then $\eth^2 \psi = 0 \Leftrightarrow \eth \psi = 0$. There is an expression for the Dirac Laplacian similar to those discussed in §2.1.2, called the *Lichnerowicz formula*.

$$\eth^2 = \nabla^* \nabla + \tfrac{1}{4} s, \tag{2.4}$$

where s is the scalar curvature of M. If M is closed and scalar-flat then an integration by parts argument like in proposition 2.1.8 implies that any solution of the Dirac equation $\eth\psi = 0$ is parallel.

Remark 2.1.12. In the notation of §2.1.2

$$s\, id_S = 16(D\sigma_n)^2(R)$$

(cf. [5, 1.142]). Therefore when M is scalar-flat the Dirac Laplacian equals $\nabla^*\nabla$, just like the Lichnerowicz Laplacian (proposition 2.1.8).

2.2   The exceptional holonomy groups

We discuss the groups G_2 and $Spin(7)$ which occur as exceptional cases in the classification Riemannian holonomy groups, and how manifolds with such holonomy can be defined in terms of certain closed differential forms. We also consider manifolds with holonomy $SU(3)$, which are Calabi-Yau 3-folds. $SU(3)$ is not an exceptional holonomy group since it belongs to the infinite family $SU(n)$, but the fact that $SU(3)$ is the stabiliser in G_2 of a vector means that it still has a place in the exceptional holonomy story.

For more detail see e.g. Harvey [22], Joyce [27] or Salamon [50].

2.2.1   The group G_2

In this section we recall some elementary properties of the Lie group G_2, and explain some related linear algebra (taken from Hitchin [24, §7.1]).

One way to define G_2 is as the automorphism group of $\mathbb{O}$, the normed algebra of octonions. The automorphisms preserve the splitting $\mathbb{O} = \mathbb{R} \oplus \operatorname{im}\mathbb{O}$ and act trivially on $\mathbb{R}$, so can therefore be identified with a subgroup of $GL(\mathbb{R}^7)$. Since the inner product on $\operatorname{im}\mathbb{O}$ is defined in terms of the normed algebra structure it is preserved by the automorphisms. It is also the case that the automorphisms preserve the orientation, so $G_2 \subseteq SO(7)$.

G_2 is a compact Lie subgroup of $SO(7)$. It is 14-dimensional, simple, connected and simply-connected.

If we identify $\operatorname{im}\mathbb{O}$ with $\mathbb{R}^7$ we can define a vector product on $\mathbb{R}^7$ by

$$a \times b = \operatorname{im} ab.$$

The algebra structure on $\mathbb{R} \oplus \mathrm{im}\,\mathbb{O}$ can be recovered from the vector product $\times$ and the standard inner product g_0 by

$$(x, a)(y, b) = (xy - <a, b>,\ xb + ya + a \times b).$$

An equivalent definition of G_2 is therefore that it is the subgroup of $GL(\mathbb{R}^7)$ that preserves both g_0 and $\times$. From g_0 and $\times$ we can define the trilinear form

$$\varphi_0(a, b, c) = <a \times b, c> . \tag{2.5}$$

In fact this is alternating, so $\varphi_0 \in \Lambda^3(\mathbb{R}^7)^*$. Explicitly in standard coordinates

$$\varphi_0 = dx^{123} + dx^{145} + dx^{167} + dx^{246} - dx^{257} - dx^{347} - dx^{356}. \tag{2.6}$$

If V is a dimension 7 real vector space let $\Lambda^3_+ V^* \subseteq \Lambda^3 V^*$ denote the set of 3-forms φ that are equivalent to φ_0 under some isomorphism $V \cong \mathbb{R}^7$. We will see below that the diffeomorphism orbit $\Lambda^3_+ V^*$ is open in $\Lambda^3 V^*$, and we call such forms *stable*.

For $\varphi \in \Lambda^3_+ V^*$ we can define an inner product g_φ and a volume form vol_φ on V in the following way: For $v, w \in V$ let

$$B_\varphi(v, w) = \frac{1}{6}(v \lrcorner \varphi) \wedge (w \lrcorner \varphi) \wedge \varphi.$$

B_φ is a symmetric bilinear form on V with values in $\Lambda^7 V^*$. B_φ induces a linear map $K_\varphi : V \to V^* \otimes \Lambda^7 V^*$, which has a determinant $\det K_\varphi \in (\Lambda^7 V^*)^9$. Since φ is stable B_φ is non-degenerate, and we can define vol_φ and g_φ by

$$(vol_\varphi)^9 = \det K_\varphi,$$
$$g_\varphi \otimes vol_\varphi = B_\varphi.$$

For φ_0 we can compute that $g_{\varphi_0} = g_0$, so the metric can be recovered from φ_0, and hence so can the vector product $\times$. Thus the stabiliser of φ_0 in $GL(\mathbb{R}^7)$ preserves g_0 and $\times$, and must equal G_2. This gives yet another possible definition of G_2. Since it is in terms of an alternating 3-form it is – as noted by Bryant [8] – a useful one for the purposes of differential geometry.

Since the action of $GL(V)$ (dim 49) on $\Lambda^3 V^*$ (dim 35) has stabiliser G_2 (dim 14) at a stable form φ it follows by dimension-counting that $\Lambda^3_+ V^*$ is open in $\Lambda^3 V^*$. $\Lambda^3_+ V^*$ splits

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into two orbits under the action of $GL_+(V)$. When V is oriented we can label as positive those stable 3-forms whose associated volume form agrees with the orientation.

Definition 2.2.1. If V is oriented we let $\Lambda_{G_2}V^* \subseteq \Lambda^3 V^*$ be the 3-forms equivalent to φ_0 under some oriented linear isomorphism $V \cong \mathbb{R}^7$, and call its elements *positive 3-forms*.

Remark 2.2.2. This definition of 'positive' agrees with that in [27], while in [24] 'positive' refers to the elements of $\Lambda^3_+ V^*$.

The natural representation of G_2 on $\mathbb{R}^7$ is irreducible. The splitting of the action of G_2 on $\Lambda^m \mathbb{R}^7$ into irreducible representations is as follows:

$$
\begin{aligned}
\Lambda^2 \mathbb{R}^7 &= \Lambda^2_7 \mathbb{R}^7 \oplus \Lambda^2_{14} \mathbb{R}^7, \\
\Lambda^3 \mathbb{R}^7 &= \Lambda^3_1 \mathbb{R}^7 \oplus \Lambda^3_7 \mathbb{R}^7 \oplus \Lambda^3_{27} \mathbb{R}^7, \\
\Lambda^4 \mathbb{R}^7 &= \Lambda^4_1 \mathbb{R}^7 \oplus \Lambda^4_7 \mathbb{R}^7 \oplus \Lambda^4_{27} \mathbb{R}^7, \\
\Lambda^5 \mathbb{R}^7 &= \Lambda^5_7 \mathbb{R}^7 \oplus \Lambda^5_{14} \mathbb{R}^7.
\end{aligned}
\tag{2.7}
$$

$\Lambda^m_d \mathbb{R}^7$ is an irreducible subrepresentation of $\Lambda^m \mathbb{R}^7$ of rank d. Like in §2.1.2 we denote the projection $\Lambda^m \mathbb{R}^7 \to \Lambda^m_d \mathbb{R}^7$ by π_d. Note that $\Lambda^2_7(\mathbb{R}^7)^* = \mathbb{R}^7 \lrcorner \varphi_0$ and $\Lambda^3_7(\mathbb{R}^7)^* = \mathbb{R}^7 \lrcorner (*\varphi_0)$, where $\lrcorner$ denotes the contraction of a vector with an alternating form. $\Lambda^2_{14}(\mathbb{R}^7)^*$ is isomorphic to the adjoint representation $\mathfrak{g}_2$ of G_2.

Because G_2 is a simply-connected subgroup of $SO(7)$ it can also be considered as a subgroup of $Spin(7)$, so the spin representation σ_7 can be restricted to a representation of G_2. As G_2-representations

$$
\sigma_7 \cong \mathbb{R} \oplus \mathbb{R}^7.
\tag{2.8}
$$

It would be possible to define a G_2-structure in terms of the dual 4-form $*_\varphi \varphi$. Because the Hodge star depends on φ the relation between φ and $*_\varphi \varphi$ is non-linear. At times we will want to use a notation that does less to disguise the non-linearity.

Definition 2.2.3. For $\varphi \in \Lambda_{G_2}V^*$ let $\Theta(\varphi) = *_\varphi \varphi$.

Proposition 2.2.4. *The derivative of* $\Theta : \Lambda_{G_2}V^* \to \Lambda^4 V^*$ *at* $\varphi \in \Lambda_{G_2}V^*$ *is*

$$
D\Theta_\varphi : \Lambda^3 V^* \to \Lambda^4 V^*, \quad \chi \mapsto *\tfrac{4}{3}\pi_1 \chi + *\pi_7 \chi - *\pi_{27}\chi,
\tag{2.9}
$$

where the Hodge star and type decomposition are defined by φ.

Proof. See [26, Lemma 3.1.1] or [24, Lemma 20]. $\qquad\qquad\square$

2.2.2 G_2-manifolds

An effective approach to G_2-structures, due to Bryant, is to define them in terms of positive 3-forms.

Definition 2.2.5. A G_2-*structure* on an oriented manifold M^7 is a section φ of $\Lambda_{G_2} T^* M$.

If $\varphi \in \Gamma(\Lambda_{G_2} T^* M)$ we can define a subbundle Q of the frame bundle of M consisting of all frames $T_p M \to \mathbb{R}^7$ $(p \in M)$ which identify φ_p with φ_0. Q is a principal G_2-bundle, so any positive 3-form defines a G_2-structure on M in the usual sense. Conversely for any principal G_2-bundle Q which induces the correct orientation on M we can find a corresponding positive 3-form.

A G_2-structure φ induces a Riemannian metric g_φ on M, and hence also a Levi-Civita connection ∇_φ, a Hodge star $*_\varphi$ and a codifferential d_φ^*. We may drop the subscripts if the G_2-structure is clear from the context.

Definition 2.2.6. A G_2-structure defined by a positive 3-form φ is *torsion-free* if $\nabla_\varphi \varphi = 0$.

Remark 2.2.7. There is a notion of the *intrinsic torsion* of a G-structure on M for a general structure group G (see e.g. [27, §2.6]). A G_2-structure has zero intrinsic torsion in this sense if and only if it is torsion-free according to definition 2.2.6.

As an immediate application of proposition 2.1.2 we have that metrics with holonomy contained in G_2 are equivalent to torsion-free G_2-structures.

Corollary 2.2.8. *Let (M^7, g) be a Riemannian manifold. Then $Hol(g)$ is a subgroup of G_2 if and only if there is a torsion-free G_2-structure φ on M such that $g = g_\varphi$.*

Definition 2.2.9. A G_2-*manifold* is a manifold M^7 equipped with a torsion-free G_2-structure φ and the associated Riemannian metric g_φ.

A G_2-structure on M induces decompositions of $\Lambda^m T^* M$ into subbundles modelled on (2.7). Considering how $d\varphi$ and $d_\varphi^* \varphi$ are obtained algebraically from $\nabla_\varphi \varphi$ shows that

Theorem 2.2.10 ([50, Lemma 11.5]). *For $\varphi \in \Lambda_{G_2} T^* M$*

(i) $\nabla_\varphi \varphi = 0$ *if and only if $d\varphi = 0$ and $d_\varphi^* \varphi = 0$,*

(ii) $\pi_7 d\varphi = 0$ *if and only if $\pi_7 d_\varphi^* \varphi = 0$.*

Because $G_2 \subseteq Spin(7)$ a G_2-structure on a manifold M^7 induces a natural spin structure. The point-wise isomorphism (2.8) implies that the spinor bundle splits as

$$S \cong \mathbb{R} \oplus T^*M, \qquad (2.10)$$

where $\mathbb{R}$ is the trivial line bundle. Conversely, Gray [18] observed that any 7-dimensional spin manifold admits a G_2-structure. For if M^7 is an oriented Riemannian manifold with a spin structure then, because the rank of the spinor bundle S on M is greater than the dimension of M, there is a global unit norm section of S. Since the stabiliser in $Spin(7)$ of any non-zero element of the spin representation σ_7 is conjugate to G_2 (see (2.13) below), this defines a G_2-structure on M.

If M^7 has a torsion-free G_2-structure then (2.10) identifies parallel spinors with parallel elements of $\Omega^0(M) \oplus \Omega^1(M)$. In particular any G_2-manifold has a non-zero parallel spinor. Conversely, proposition 2.1.2 implies

Theorem 2.2.11. *An oriented Riemannian manifold M^7 has a spin structure with a non-zero parallel spinor if and only if $Hol(M) \subseteq G_2$.*

Wang [54] gives an explicit way to construct a parallel positive 3-form from a parallel spinor.

For compact G_2-manifolds there is a known necessary and sufficient condition for the holonomy group to be exactly G_2.

Theorem 2.2.12 ([27, Proposition 10.2.2])**.** *Let M^7 be a compact G_2-manifold. Then the holonomy $Hol(M) = G_2$ if and only if the fundamental group $\pi_1(M)$ is finite.*

2.2.3 The group $Spin(7)$

$Spin(7)$ is the double cover of $SO(7)$. It is a compact simple simply-connected Lie group of dimension 21. Its spin representation σ_7 is a real irreducible representation of rank 8. It is faithful, so $Spin(7)$ can be regarded as a subgroup of $SO(8)$. $Spin(7)$ is conjugate to the stabiliser in $GL(\mathbb{R}^8)$ of

$$
\begin{aligned}
\psi_0 = {}& dx^{1234} + dx^{1256} + dx^{1278} + dx^{1357} - dx^{1368} - dx^{1458} - dx^{1467} \\
& - dx^{2358} - dx^{2367} - dx^{2457} + dx^{2468} + dx^{3456} + dx^{3478} + dx^{5678} \in \Lambda^4(\mathbb{R}^8)^*. \quad (2.11)
\end{aligned}
$$

To see that $Spin(7)$ leaves ψ_0 invariant consider first $Spin(8)$ and its three real rank 8 representations. As in §2.1.3 we use $\mathbb{R}^8$ to denote the vector representation, i.e. the representation that factors through the natural action of $SO(8)$, and let $\sigma_8^\pm$ be the two irreducible spin representations. $Spin(8)$ has an exceptional triality property – its outer automorphism group is isomorphic to S_3, permuting $\mathbb{R}^8$, σ_8^+ and σ_8^- (cf. [22, §14]). Another manifestation of the triality is that the Clifford multiplication

$$\mathbb{R}^8 \times \sigma_8^\pm \to \sigma_8^\mp$$

is non-degenerate. This implies that each of $\mathbb{R}^8$, σ_8^+ and σ_8^- can be identified (but not equi-variantly) with a division algebra – necessarily the octonions $\mathbb{O}$ – in such a way that the Clifford multiplication $\mathbb{R}^8 \times \sigma_8^+ \to \sigma_8^-$ is identified with the multiplication map $\mathbb{O} \times \mathbb{O} \to \mathbb{O}$, $(x, y) \mapsto xy$ (in order to satisfy the Clifford relations $\mathbb{R}^8 \times \sigma_8^- \to \sigma_8^+$ must then be identified with $\mathbb{O} \times \mathbb{O} \to \mathbb{O}$, $(x, y) \mapsto -\bar{x}y$).

Since $Spin(7)$ is a connected simply-connected subgroup of $SO(8)$ it can be regarded as a subgroup of $Spin(8)$. $\mathbb{R}^8 \cong \sigma_7$ as representations of $Spin(7)$ by our choice of inclusion $Spin(7) \hookrightarrow Spin(8)$. We choose the orientation on $\mathbb{R}^8$ so that as $Spin(7)$-representations

$$\sigma_8^+ \cong \mathbb{R} \oplus \mathbb{R}^7, \tag{2.12a}$$

$$\sigma_8^- \cong \mathbb{R}^8. \tag{2.12b}$$

The double Clifford multiplication $\mathbb{R}^8 \times \mathbb{R}^8 \times \sigma_8^- \to \sigma_8^-$ then gives a $Spin(7)$-equivariant trilinear map $(\mathbb{R}^8)^{\otimes 3} \to \mathbb{R}^8$. We call its alternation the *triple cross product*. Identifying $\mathbb{R}^8$ with the octonions as above, the triple cross product is given by (see Harvey [23, p. 145])

$$\mathbb{O} \times \mathbb{O} \times \mathbb{O} \to \mathbb{O}, \quad (x, y, z) \mapsto x \times y \times z = \tfrac{1}{2}(x(\bar{y}z) - z(\bar{y}x)).$$

The 4-form ψ_0 can be written in terms of the triple cross product and the inner product as

$$\psi_0(x, y, z, w) = <x \times y \times z, w> .$$

Hence ψ_0 is invariant under the action of $Spin(7)$. To show that its stabiliser is exactly $Spin(7)$, note that the stabiliser in $Spin(7)$ of a non-zero vector in $\mathbb{R}^8$ contains G_2 because of the decomposition (2.8). On the other hand, if we let t be the first $\mathbb{R}$-coordinate in

$\mathbb{R}^8 = \mathbb{R} \oplus \mathbb{R}^7$ then comparing the expressions (2.6) and (2.11) shows that up to equivalence

$$\psi_0 \cong *_{\varphi_0} \varphi_0 + dt \wedge \varphi_0. \tag{2.13}$$

Therefore the stabiliser of ψ_0 together with a non-zero vector is contained in (a conjugate of) G_2. Hence the stabiliser of ψ_0 is exactly $Spin(7)$, and the stabiliser in $Spin(7)$ of a non-zero vector in $\mathbb{R}^8$ is conjugate to exactly G_2.

For an oriented vector space V of dimension 8 let $\Lambda_{Spin(7)}V^* \subseteq \Lambda^4 V^*$ be the subset of forms equivalent to ψ_0 under some oriented linear isomorphism $V \cong \mathbb{R}^8$. Since the stabiliser of ψ_0 is $Spin(7)$ we can think of $\psi \in \Lambda_{Spin(7)}V^*$ as defining a $Spin(7)$-structure, including an inner product. Note that the 4-form ψ is self-dual with respect to the inner product it defines.

The action of $Spin(7)$ on $\mathbb{R}^8$ is irreducible. We make a note of the decomposition of $\Lambda^* \mathbb{R}^8$ into irreducible components. We let $\Lambda_d^m \mathbb{R}^8$ denote an irreducible component of rank d, and note first that $\Lambda^4 \mathbb{R}^8$ splits into the self-dual and anti-self-dual parts $\Lambda_\pm^4 \mathbb{R}^8$. Further (cf. [27, Proposition 10.5.4])

$$\begin{aligned}
\Lambda_+^4 \mathbb{R}^8 &= \Lambda_1^4 \mathbb{R}^8 \oplus \Lambda_7^4 \mathbb{R}^8 \oplus \Lambda_{27}^4 \mathbb{R}^8, \\
\Lambda_-^4 \mathbb{R}^8 &= \Lambda_{35}^4 \mathbb{R}^8, \\
\Lambda^3 \mathbb{R}^8 &= \Lambda_8^3 \mathbb{R}^8 \oplus \Lambda_{48}^3 \mathbb{R}^8, \\
\Lambda^2 \mathbb{R}^8 &= \Lambda_7^2 \mathbb{R}^8 \oplus \Lambda_{21}^3 \mathbb{R}^8.
\end{aligned} \tag{2.14}$$

Each of the rank 8 representations is isomorphic to $\mathbb{R}^8$, while each of the rank 7 representations is isomorphic to the vector representation $\mathbb{R}^7$.

$\Lambda_{Spin(7)}V^*$ is not an open subset of $\Lambda^4 V^*$. It is easy to see that

$$T_\psi \Lambda_{Spin(7)}V^* = \Lambda_{1 \oplus 7 \oplus 35}^4 V^*.$$

2.2.4 $Spin(7)$-manifolds

Definition 2.2.13. A $Spin(7)$-*structure* on an oriented manifold M^8 is a section of the subbundle $\Lambda_{Spin(7)}T^*M \subseteq \Lambda^4 T^*M$.

Since $Spin(7) \subset SO(8)$ a $Spin(7)$-structure ψ naturally defines a Riemannian metric g_ψ on M, and hence also a Levi-Civita connection ∇_ψ, a Hodge star $*_\psi$ and a codifferential d_ψ^*.

Definition 2.2.14. A $Spin(7)$-structure ψ is *torsion-free* if $\nabla_\psi \psi = 0$.

Corollary 2.2.15. *Let (M^8, g) be a Riemannian manifold. Then $Hol(g)$ is a subgroup of $Spin(7)$ if and only if there is a torsion-free $Spin(7)$-structure ψ on M such that $g = g_\psi$.*

Definition 2.2.16. A $Spin(7)$-*manifold* is a manifold M^8 equipped with a torsion-free $Spin(7)$-structure ψ and the associated Riemannian metric g_ψ.

Theorem 2.2.17 ([50, Lemma 12.4]). *$\psi \in \Lambda_{Spin(7)} T^* M$ is torsion-free iff $d\psi = 0$.*

Since $Spin(7) \hookrightarrow Spin(8)$ any manifold M^8 with a $Spin(7)$-structure has a natural spin structure, which defines spinor bundles $S^\pm$. The equivalence of representations (2.12b) implies that

$$S^- \cong T^* M. \tag{2.15}$$

Therefore, if M is compact then the dimension of the space of harmonic negative spinors on M is $b^1(M)$. Also, recall that the difference between the dimensions of harmonic positive and negative spinors is given by the topological invariant $\hat{A}(M)$. Since M is scalar-flat any harmonic spinor is parallel. The dimension of the space of parallel positive spinors on a compact $Spin(7)$-manifold M can therefore be written in terms of topological invariants as

$$\dim \mathcal{H}^{S^+} = b^1(M) + \hat{A}(M). \tag{2.16}$$

One use for this expression is to give a topological criterion for when a compact $Spin(7)$-manifold has holonomy exactly $Spin(7)$; this is because $Spin(7)$ fixes a unique line in its action on the positive spin representation σ_8^+, while any proper subgroup of $Spin(7)$ that can occur as the holonomy of a simply-connected Riemannian manifold fixes a subspace of dimension at least 2. We will discuss this in §4.1.5.

Theorem 2.2.18 (cf. [27, Theorem 10.6.1]). *A compact $Spin(7)$-manifold M^8 has holonomy exactly $Spin(7)$ if and only if M is simply-connected and $\hat{A}(M) = 1$.*

2.2.5   The group $SU(3)$

In the context of the exceptional holonomy groups $SU(3)$ appears as the stabiliser in G_2 of a vector. We explain how $SU(3)$-structures can be defined in terms of forms in a similar way to G_2-structures and $Spin(7)$-structures.

Identify $\mathbb{R}^6$ with $\mathbb{C}^3$ by using complex coordinates $z^1 = x^1 + ix^2$, $z^2 = x^3 + ix^4$, $z^3 = x^5 + ix^6$. Then $SU(3)$ can be considered as the stabiliser in $GL_+(\mathbb{R}^6)$ of the pair

$$\Omega_0 = \mathrm{re}(dz^1 \wedge dz^2 \wedge dz^3), \tag{2.17a}$$

$$\omega_0 = -\tfrac{i}{2}(dz^1 \wedge d\bar{z}^1 + dz^2 \wedge d\bar{z}^2 + dz^3 \wedge d\bar{z}^3). \tag{2.17b}$$

These forms are obviously invariant under $SU(3)$. To see that the stabiliser is exactly $SU(3)$ consider first the stabiliser of just Ω_0. The standard complex structure on $\mathbb{C}^3$ can be recovered from Ω_0 and the standard orientation in a similar way to how the standard Euclidean metric was recovered from φ_0 in §2.2.1 (see Hitchin [24, §2.2] for details). The stabiliser of Ω_0 in $GL_+(\mathbb{R}^6)$ must therefore be contained in $SL(\mathbb{C}^3)$. In particular it preserves also the unique form

$$\hat{\Omega}_0 = \mathrm{im}\,(dz^1 \wedge dz^2 \wedge dz^3)$$

such that $\Omega_0 + i\hat{\Omega}_0$ has type $(3,0)$ with respect to the standard complex structure. In turn, $SU(3)$ is the stabiliser of ω_0 in $SL(\mathbb{C}^3)$.

For V an oriented real vector space of dimension 6 let $\Lambda^3_+ V^*$ be the set of $\Omega \in \Lambda^3 V^*$ such that Ω is equivalent to Ω_0 under some linear isomorphism $V \cong \mathbb{R}^6$. By dimension-counting $\Lambda^3_+ V^*$ is open in $\Lambda^3 V^*$, and following [24] we call its elements Ω *stable*. From the above it follows that any $\Omega \in \Lambda^3_+ V^*$ naturally defines a complex structure on V, and a three-form $\hat{\Omega}$ such that $\Omega + i\hat{\Omega}$ has type $(3,0)$. (Reversing the orientation of V changes the sign of both J and $\hat{\Omega}$.)

Furthermore, let $\Lambda_{SU(3)} V^* \subset \Lambda^3 V^* \oplus \Lambda^2 V^*$ be the set of pairs (Ω, ω) equivalent to (Ω_0, ω_0) under some oriented linear isomorphism $V \cong \mathbb{R}^6$. Such a pair (Ω, ω) naturally defines a complex structure on V together with a Hermitian inner product. With respect to this inner product $\hat{\Omega} = *\Omega$.

Any $(\Omega, \omega) \in \Lambda_{SU(3)} V^*$ must satisfy the algebraic relations $\Omega \wedge \omega = 0$ and $\frac{1}{4}\Omega \wedge \hat{\Omega} = \frac{1}{6}\omega^3$. Any tangent (σ, τ) to $\Lambda_{SU(3)} V^*$ at (Ω, ω) must therefore satisfy the linearisation of these conditions. We can use Schur's lemma (and the decomposition (2.21) below) to check that the derivative of the homogenous map $\Omega \mapsto \Omega \wedge \hat{\Omega}$ is $\cdot \wedge 2\hat{\Omega}$ (cf. [24, p. 10]), so the linearisations are

$$L_1(\sigma, \tau) = \sigma \wedge \hat{\Omega} - \tau \wedge \omega^2 = 0, \tag{2.18a}$$

$$L_2(\sigma, \tau) = \sigma \wedge \omega + \Omega \wedge \tau = 0. \tag{2.18b}$$

Indeed the tangent space is precisely

$$T_{(\Omega,\omega)}\Lambda_{SU(3)}V^* = \{(\sigma,\tau) \in \Lambda^{3\oplus2}V^* : L_1(\sigma,\tau) = L_2(\sigma,\tau) = 0\}. \qquad (2.19)$$

Comparing the expressions (2.6) and (2.17) we observe that if we let t denote the first coordinate in $\mathbb{R}^7 = \mathbb{R} \oplus \mathbb{R}^6$ then φ_0 is equivalent to

$$\varphi_0 = \Omega_0 + dt \wedge \omega_0. \qquad (2.20)$$

It follows that the stabiliser in G_2 of a non-zero vector in $\mathbb{R}^7$ is conjugate to $SU(3)$.

Let us also make a note of how the exterior powers of $\mathbb{R}^6$ split into irreducible $SU(3)$-representations.

$$\Lambda^2\mathbb{R}^6 = \Lambda_1^2\mathbb{R}^6 \oplus \Lambda_6^2\mathbb{R}^6 \oplus \Lambda_8^2\mathbb{R}^6,$$
$$\Lambda^3\mathbb{R}^6 = \Lambda_{1\oplus1}^3\mathbb{R}^6 \oplus \Lambda_6^3\mathbb{R}^6 \oplus \Lambda_{12}^3\mathbb{R}^6, \qquad (2.21)$$
$$\Lambda^4\mathbb{R}^6 = \Lambda_1^4\mathbb{R}^6 \oplus \Lambda_6^4\mathbb{R}^6 \oplus \Lambda_8^4\mathbb{R}^6.$$

Each of the subrepresentations $\Lambda_d^m\mathbb{R}^6$ is irreducible, but $\Lambda_{1\oplus1}^3\mathbb{R}^6$ is trivial of rank 2. These decompositions are related to the decompositions of the exterior powers of $\mathbb{C}^3$ by type, e.g. $\Lambda_{1\oplus8}^2(\mathbb{R}^6)^* \otimes \mathbb{C} = \Lambda^{1,1}(\mathbb{C}^3)^*$, while $\Lambda_6^2(\mathbb{R}^6)^* \otimes \mathbb{C} = (\Lambda^{2,0} \oplus \Lambda^{0,2})(\mathbb{C}^3)^*$.

2.2.6 Calabi-Yau manifolds

There are many commonly used definitions of what it means for a manifold X of real dimension $2n$ to be Calabi-Yau. The following alternative definitions are all non-equivalent:

- X is a Riemannian manifold with $Hol(X) \subseteq SU(n)$.

- X is a Riemannian manifold with $Hol(X) = SU(n)$.

- X is a Ricci-flat Kähler manifold.

- X is a Kähler manifold with $c_1(X) = 0$.

To see how they are related first recall that $Hol(X) \subseteq U(n)$ if and only if X is Kähler, i.e. when the metric is Hermitian with respect to a parallel complex structure. Equivalently the Kähler form $\omega = g(J\cdot,\cdot)$ is closed. If $Hol(X) \subseteq SU(n)$ then there is additionally a parallel $(n,0)$-form ϕ. It is elementary to show that there is a linear relation between the Ricci curvature and the curvature of the canonical bundle $\Lambda^{n,0}T^*X$ on a Kähler manifold X.

Therefore if there is a parallel section of $\Lambda^{n,0}T^*X$ then X is Ricci-flat. The linear relation implies also that if X is Ricci-flat Kähler then the restricted holonomy $Hol^0(X) \subseteq SU(n)$, but there need not be a *global* $(n,0)$-form.

A parallel $(n,0)$-form on a Kähler manifold X is holomorphic, so if there is one then the canonical bundle $\Lambda^{n,0}T^*X$ is trivial, and the first Chern class $c_1(X) \in H^2(X,\mathbb{Z})$ vanishes. By Chern-Weil theory $c_1(X) \in H^2(X,\mathbb{R})$ is represented by a multiple of the curvature of the canonical bundle, so for $c_1(X)$ to vanish in the de Rham cohomology it suffices that X is Ricci-flat Kähler. Yau [56] proved the Calabi conjecture, which provides a converse.

Theorem 2.2.19. *Let X be a compact Kähler manifold. If $c_1(X) \in H^2(X,\mathbb{R})$ vanishes then for any Kähler form ω on X there is a unique Kähler form ω' in the cohomology class of ω such that ω' defines a Ricci-flat metric.*

A special feature of the case $n = 3$ – which is the case of interest for us – is that the almost complex structure J on a Calabi-Yau manifold can be recovered from the real part Ω of the holomorphic 3-form together with the orientation. This is a consequence of the point-wise linear algebra discussed in §2.2.5. We therefore find it convenient to define a Calabi-Yau structure on a manifold X of real dimension 6 in terms of a pair of forms $(\Omega, \omega) \in \Omega^3(X) \oplus \Omega^2(X)$.

Definition 2.2.20. An $SU(3)$-*structure* on a manifold X^6 is a section (Ω, ω) of $\Lambda_{SU(3)}T^*X$.

(Ω, ω) is said to be a *Calabi-Yau structure* if $\nabla\Omega = 0, \nabla\omega = 0$ with respect to the metric induced by (Ω, ω). X equipped with the structure (Ω, ω) and the associated Riemannian metric is called a *Calabi-Yau 3-fold*.

Lemma 2.2.21. *Let (Ω, ω) be an $SU(3)$-structure on a manifold X^6. Then (Ω, ω) is a Calabi-Yau structure if and only if $d\Omega = d\hat{\Omega} = 0$, $d\omega = 0$.*

Sketch proof. $d(\Omega + i\hat{\Omega}) = 0$ ensures that the almost complex structure defined by Ω is integrable. ω defines a Kähler metric, so J and ω are parallel. $\Omega + i\hat{\Omega}$ is a holomorphic section of a Hermitian line bundle and has constant norm, so must be parallel with respect to the Chern connection. $\square$

One can also consider the structure defined by the 3-form Ω on its own.

Definition 2.2.22. An $SL(\mathbb{C}^3)$-*structure* on an oriented manifold X^6 is a section Ω of $\Lambda_+^3 T^*X$. Ω is *torsion-free* if $d\Omega = d\hat{\Omega} = 0$.

If Ω is torsion-free then so is the almost complex structure J it defines, and $\Omega + i\hat{\Omega}$ is a global holomorphic $(3,0)$-form. A torsion-free $SL(\mathbb{C}^3)$-structure is therefore equivalent to a complex structure with trivial canonical bundle, together with a choice of trivialisation.

2.2.7 Cylindrical G-structures

Let X^6 be a compact manifold, and denote by t the $\mathbb{R}$-coordinate on the cylinder $X \times \mathbb{R}$.

Definition 2.2.23. A G_2-structure φ on $X \times \mathbb{R}$ is *cylindrical* if it is translation-invariant and the associated metric is a product metric $g_\varphi = g_X + dt^2$.

We find that torsion-free cylindrical G_2-structures on $X \times \mathbb{R}$ correspond to Calabi-Yau structures on X. Looking at the point-wise model (2.20) we see that (Ω, ω) is an $SU(3)$-structure on X with metric g_X if and only if the translation-invariant G_2-structure $\varphi = \Omega + dt \wedge \omega$ on $X \times \mathbb{R}$ defines the product metric $g_X + dt^2$. $Hol(g_X + dt^2) \subseteq G_2$ if and only if $Hol(g_X) \subseteq SU(3)$, so (Ω, ω) is torsion-free if and only if $\Omega + dt \wedge \omega$ is. Hence

Proposition 2.2.24. *Let X be a manifold of dimension 6. (Ω, ω) is a Calabi-Yau structure on X if and only if $\Omega + dt \wedge \omega$ is a torsion-free cylindrical G_2-structure on $X \times \mathbb{R}$.*

Remark 2.2.25. If $\varphi = \Omega + dt \wedge \omega$ is a cylindrical G_2-structure then

$$*_\varphi \varphi = \tfrac{1}{2}\omega^2 - dt \wedge \hat{\Omega}.$$

Similarly contemplation of the expression (2.13) leads to

Proposition 2.2.26. *Let X be a manifold of dimension 7. φ is a torsion-free G_2-structure on X if and only if $*_\varphi \varphi + dt \wedge \varphi$ is a torsion-free cylindrical $Spin(7)$-structure on $X \times \mathbb{R}$.*

2.3 Asymptotically cylindrical manifolds

In this section we define what an asymptotically cylindrical manifold is and collect some results about analysis and Hodge theory on such manifolds.

2.3.1 Manifolds with cylindrical ends

We define manifolds with cylindrical ends and exponentially asymptotically cylindrical (EAC) metrics.

Definition 2.3.1. A manifold M is said to have *cylindrical ends* if it is a union of two pieces M_0 and M_∞ with common boundary X, where M_0 is compact, and M_∞ is identified with $X \times \mathbb{R}^+$ by a diffeomorphism (identifying ∂M_∞ with $X \times \{0\}$). X is called the *cross-section* of M.

A *cylindrical coordinate* on M is a smooth function $t : M \to \mathbb{R}$ which is equal to the $\mathbb{R}^+$-coordinate on M_∞ and is negative in the interior of M_0.

If M has cylindrical ends it can be compactified by inclusion in $\overline{M} = M_0 \cup (X \times [0, \infty])$, i.e. by 'adding a copy of X at infinity'. Conversely the interior of any compact manifold with boundary can be considered as a manifold with cylindrical ends by the collar neighbourhood theorem. The choice of diffeomorphism $M_\infty \to X \times \mathbb{R}^+$ can be regarded to determine a cylindrical-end structure on M.

Cylindrical ends allow us to define a notion of asymptotic translation-invariance.

Definition 2.3.2. A tensor field or differential operator on $X \times \mathbb{R}$ is called *translation-invariant* if it is invariant under the obvious $\mathbb{R}$-action on $X \times \mathbb{R}$.

Definition 2.3.3. Let M be a manifold with cylindrical ends. Call a smooth function $\rho : M \to \mathbb{R}$ a *cut-off function for the cylinder* if it is 0 on the compact piece M_0 and 1 outside a compact subset of M.

If s_∞ is a section of a vector bundle associated to the tangent bundle on the cylinder $X \times \mathbb{R}$ and ρ is a cut-off function for the cylinder of M then ρs_∞ can be considered to be a section of the corresponding vector bundle over M.

Definition 2.3.4. Let M be a manifold with cylindrical ends and cross-section X. Pick an arbitrary product metric $g_X + dt^2$ on $X \times \mathbb{R}$, and a cut-off function ρ for the cylinder. A section s of a vector bundle associated to TM is said to be *decaying* if $\|\nabla^k s\| \to 0$ uniformly on X as $t \to \infty$ for all $k \geq 0$. s is said to be *asymptotic* to a translation-invariant section s_∞ of the corresponding bundle on $X \times \mathbb{R}$ if $s - \rho s_\infty$ decays.

Similarly s is said to be *exponentially decaying* with rate $\delta > 0$ if $e^{\delta t} \|\nabla^k s\|$ is bounded on M_∞ for all $k \geq 0$, and *exponentially asymptotic* to a translation-invariant section s_∞ if $s - \rho s_\infty$ decays exponentially. Denote by $C^\infty_\delta(E)$ the space of sections of E which decay exponentially with rate δ.

To develop the analysis for elliptic operators on a manifold M with cylindrical ends in subsection 2.3.2 it will suffice to equip M with an asymptotically translation-invariant metric, but for the Hodge theory developed in subsection 2.3.4 we will need to assume that M is EAC (exponentially asymptotically cylindrical).

Definition 2.3.5. A metric g on a manifold M with cylindrical ends is said to be *EAC* if it is exponentially asymptotic to a product metric $g_X + dt^2$ on $X \times \mathbb{R}^+$. An EAC manifold is a manifold with cylindrical ends equipped with an EAC metric.

Definition 2.3.6. Let M be a manifold with cylindrical ends and cross-section X. A diffeomorphism Ψ_∞ of the cylinder $X \times \mathbb{R}$ is said to be cylindrical if it is of the form

$$\Psi_\infty(x, t) = (\Xi(x), t + h),$$

where Ξ is a diffeomorphism of X and $h \in \mathbb{R}$. A diffeomorphism Ψ of M is said to be *EAC* with rate $\delta > 0$ if there is a cylindrical diffeomorphism Ψ_∞ of $X \times \mathbb{R}$, a real $T > 0$ and an exponentially decaying vector field V on M such that on $X \times (T, \infty)$

$$\Psi = (\exp V) \circ \Psi_\infty.$$

We can think of two different choices of diffeomorphism $M_\infty \to X \times \mathbb{R}^+$ in definition 2.3.1 as defining 'δ-equivalent' cylindrical-end structures if they differ by a rate δ EAC diffeomorphism – then they define equivalent notions of exponential translation-invariance. Nevertheless it is convenient to fix the identification $M_\infty \to X \times \mathbb{R}^+$.

Theorem 3.1.5 states that any isometry of smooth Riemannian manifolds is smooth. In a sense this means that the smooth structure can be recovered from the metric (cf. Palais [48]). We wish to generalise this, and show that isometries of EAC metrics are EAC.

Proposition 2.3.7. *Any isometry of EAC (rate $\delta > 0$) manifolds is C^∞ and EAC with rate δ.*

Sketch proof. By [46, Theorem 8] the isometries are C^∞, so we just need to prove that they are also EAC.

Let M be a manifold with a Riemannian metric g. We need to show that if for $i = 1, 2$, $M_{i,\infty} \subseteq M$ have compact complements, and $\Psi_i : M_{i,\infty} \to X_i \times \mathbb{R}^+$ are diffeomorphisms defining cylindrical-end structures with respect to which g is EAC, then $\Psi_1 \circ \Psi_2^{-1}$ is EAC.

Consider the space R of half-lines in M, i.e. equivalence classes of unit speed globally distance-minimising geodesic rays $\gamma : [0, \infty) \to M$, where two rays are equivalent if one is a subset of the other. We can define a distance function on R by

$$d([\gamma], [\sigma]) = \lim_{u \to \infty} \inf_v d(\gamma(u), \sigma(v)). \tag{2.22}$$

g is pushed forward to an EAC metric on $X_i \times \mathbb{R}^+$ by Ψ_i. It is straight-forward to solve the geodesic equation in local coordinates to show that for each $x \in X_i$ there is a unique half-line $[\gamma]$ such that the X-component of $\gamma(u)$ approaches x as $u \to \infty$. Ψ_i therefore induce isometries $\Xi_i : R \to X_i$. Then $\Xi_1 \circ \Xi_2^{-1}$ is smooth by [46, Theorem 8]. If t_i is the $\mathbb{R}^+$-coordinate on $X_i \times \mathbb{R}^+$ then $grad(t_i) - \frac{d\gamma}{du}$ decays exponentially as $u \to \infty$ for each half-line $[\gamma]$, so $\Psi_1 \circ \Psi_2^{-1}$ is exponentially asymptotic to $(x,t) \mapsto (\Xi_1 \circ \Xi_2^{-1}(x), t + h)$ for some $h \in \mathbb{R}$. $\qquad \square$

It is a classical result that the isometry group of a compact Riemannian manifold is compact (cf. Myers and Steenrod [46]). While we will not use it in the main arguments, it is worth noting the following generalisation.

Lemma 2.3.8 (see e.g. [35, Lemma 3.6]). *The isometry group of an EAC manifold with a single end is compact.*

2.3.2 Analysis on manifolds with cylindrical ends

Here we collect some results that we will need about analysis on manifolds M with cylindrical ends. The results are mainly taken from Lockhart and McOwen [39] and Maz'ya and Plamenevskiĭ [40].

Definition 2.3.9. Let M^n be a manifold with cylindrical ends. If g is an asymptotically translation-invariant metric g on M, E is a vector bundle associated to TM, δ is a real number and s is a section of E, we define the *Hölder norm with weight δ* (or $C^{k,\alpha}_\delta$-norm) of s in terms of the Hölder norm by

$$\|s\|_{C^{k,\alpha}_\delta(g)} = \|e^{\delta t} s\|_{C^{k,\alpha}(g)}, \tag{2.23}$$

where t is the cylindrical coordinate on M. Denote the space of sections of E with finite $C^{k,\alpha}_\delta$-norm by $C^{k,\alpha}_\delta(E)$.

Up to Lipschitz equivalence the weighted norms are independent of the choice of asymptotically translation-invariant metric g, and of the choice of t on the compact piece M_0. In particular, the topological vector spaces $C^{k,\alpha}_\delta(E)$ are independent of these choices.

$$\bigcap_{k>0} C^{k,\alpha}_\delta(E) = C^\infty_\delta(E),$$

and the natural choice of topology on $C^\infty_\delta(E)$ is the inverse limit topology.

Remark 2.3.10. There is a similar definition of weighted Sobolev norms $L^2_{k,\delta}$. Aubin [3, Theorem 2.21] shows that Sobolev embedding holds on any complete manifold which has bounded curvature and injectivity radius bounded away from 0, conditions that are clearly satisfied for a manifold with cylindrical ends and an asymptotically translation-invariant metric. From that can deduce that on a manifold M^n with cylindrical ends $L^2_{l,\delta}$ embeds continuously in $C^{k,\alpha}_\delta$ for any $l > k + \alpha + \frac{n}{2}$. Because of this Sobolev embedding result one could use weighted Hölder norms and weighted Sobolev norms interchangeably in many arguments. However $\bigcap_k C^{k,\alpha} = C^\infty$, while the same is not true for Sobolev spaces on non-compact manifolds. We choose to use Hölder spaces for convenience in the regularity arguments of §4.2.6.

Definition 2.3.11. Let M be a manifold with cylindrical ends, E and F vector bundles associated to TM, A a smooth linear differential operator $\Gamma(E) \to \Gamma(F)$ of order m, and A_∞ a translation-invariant operator on the corresponding bundles over $X \times \mathbb{R}$.

The restriction of A to the cylindrical end M_∞ can be written in terms of the Levi-Civita connection of an arbitrary product metric on $X \times \mathbb{R}$ as $A = \sum_{i=0}^m a_i \nabla^i$, with coefficients $a_i \in C^\infty((TM)^i \otimes E^* \otimes F)$. A is said to be {*exponentially*} *asymptotic* to A_∞ if the coefficients in the expression for A are {exponentially} asymptotic to those of A_∞.

One of the main results of [39] is Theorem 6.2, which states

Theorem 2.3.12. *Let M be Riemannian manifold with cylindrical ends and asymptotically translation-invariant metric. Let $A : \Gamma(E) \to \Gamma(F)$ be a smooth linear elliptic differential operator of order r asymptotic to a translation-invariant operator A_∞.*

For $\lambda \in \mathbb{C}$ let $S_A(\lambda)$ be the space of solutions s to $A_\infty s = 0$ on the cylinder $X \times \mathbb{R}$ of the form $s = e^{i\lambda t} p(x, t)$, where p is a section of E over $X \times \mathbb{R}$ that is smooth in x and polynomial in t. Let $d_A(\lambda) = \dim_\mathbb{C} S_A(\lambda)$, and

$$\mathcal{B}_A = \{\operatorname{im}\lambda : \lambda \in \mathbb{C}, d_A(\lambda) \neq 0\}. \tag{2.24}$$

Then A extends to a bounded map $C^{k+r,\alpha}_\delta(E) \to C^{k,\alpha}_\delta(F)$ for all k and δ, which is Fredholm if $\delta \notin \mathcal{B}_A$. In the latter case let $i_\delta(A) = \operatorname{ind}\left(A : C^{k+r,\alpha}_\delta(E) \to C^{k,\alpha}_\delta(F)\right)$. If $\delta_1, \delta_2 \notin \mathcal{B}_A$ and $\delta_1 \leq \delta_2$ then

$$i_{\delta_1}(A) - i_{\delta_2}(A) = \sum_{\delta_1 < \operatorname{im}\lambda < \delta_2} d_A(\lambda).$$

Remark 2.3.13. Strictly speaking, the results in [39] use weighted Sobolev spaces rather than weighted Hölder spaces, but the arguments are the same in both cases. See also [40, Theorem 6.4].

Proposition 2.3.14 ([39, Lemma 7.1]). *If the interval $[\delta_1, \delta_2]$ contains no elements of $\mathcal{B}_A$ then the kernels of A in $C_{\delta_1}^{k,\alpha}(E)$ and $C_{\delta_2}^{k,\alpha}(E)$ are equal.*

We can use integration by parts arguments, provided that the rates of decay are fast enough to ensure the integrals converge.

Lemma 2.3.15. *Let M be a Riemannian manifold with cylindrical ends and asymptotically translation-invariant metric. Let $A : \Gamma(E) \to \Gamma(F)$ be a smooth linear asymptotically translation-invariant differential operator of order r with formal adjoint $A^* : \Gamma(F) \to \Gamma(E)$. Suppose that $\alpha \in C_{\delta_1}^{r,\alpha}(E)$, $\beta \in C_{\delta_2}^{r,\alpha}(F)$ with $\delta_1 + \delta_2 > 0$. Then*

$$<A\alpha, \beta>_{L^2} = <\alpha, A^*\beta>_{L^2} .$$

Proof. This holds for compactly supported α, β by definition. The condition $\delta_1 + \delta_2 > 0$ ensures that
$$C_{\delta_1}^{r,\alpha}(E) \times C_{\delta_2}^{r,\alpha}(F), \quad (\alpha, \beta) \mapsto <A\alpha, \beta>_{L^2} - <\alpha, A^*\beta>_{L^2}$$
is continuous. As the compactly supported forms are dense in the Hölder spaces the result follows. $\square$

Once we know that the maps $A : C_{\delta}^{k+r,\alpha}(E) \to C_{\delta}^{k,\alpha}(F)$ are Fredholm we can phrase the 'Fredholm alternative' in the following way.

Proposition 2.3.16. *Let M be a manifold with cylindrical ends and an asymptotically translation-invariant metric. Let $A : \Gamma(E) \to \Gamma(F)$ be a smooth linear elliptic asymptotically translation-invariant differential operator of order r, and $A^* : \Gamma(F) \to \Gamma(E)$ its formal adjoint.*

If δ is a real number not in $\mathcal{B}_A$ then the image of $A : C_{\delta}^{k+r,\alpha}(E) \to C_{\delta}^{k,\alpha}(F)$ is precisely the L^2-orthogonal complement to the kernel of $A^* : C_{-\delta}^{k+r,\alpha}(F) \to C_{-\delta}^{k+r,\alpha}(E)$ in $C_{\delta}^{k,\alpha}(F)$.

Proof. Let I be the image of A in $C_{\delta}^{k+r,\alpha}(E)$, and K the kernel of A^* in $C_{-\delta}^{k+r,\alpha}(F)$. Then K is contained in $C_{-\delta'}^{k+r,\alpha}(F)$ for some $\delta' > \delta$. Integration by parts shows that K contains the L^2-orthogonal complement to I in $C_{-\delta'}^{k+r,\alpha}(F)$, and that I is contained in the L^2-orthogonal complement to K in $C_{\delta}^{k+r,\alpha}(E)$. Since I is closed the result follows. $\square$

[39, Theorem 7.4] gives a method for computing the index of a formally self-adjoint elliptic asymptotically translation-invariant differential operator acting on a weighted Hölder space with small weight.

Theorem 2.3.17. *Let M be a manifold with cylindrical ends and with an asymptotically translation-invariant metric. Let $A : \Gamma(E) \to \Gamma(E)$ be a smooth linear elliptic formally self-adjoint asymptotically translation-invariant differential operator of order r.*

Let $\epsilon > 0$ be the smallest positive element of $\mathcal{B}_A$. Then for $0 < \delta < \epsilon$ the index of

$$A : C^{k+r,\alpha}_{\pm\delta}(E) \to C^{k,\alpha}_{\pm\delta}(E) \tag{2.25}$$

(which is Fredholm by theorem 2.3.12) is

$$\mp \tfrac{1}{2} \sum_{\lambda \in \mathbb{R}} d(\lambda).$$

Proof. It follows from self-adjointness and proposition 2.3.16 that $i_\delta(A) = -i_{-\delta}(A)$, so using theorem 2.3.12 we obtain

$$-2i_\delta(A) = i_{-\delta}(A) - i_\delta(A) = \sum_{-\delta < \mathrm{im}\,\lambda < \delta} d_A(\lambda) = \sum_{\lambda \in \mathbb{R}} d_A(\lambda). \qquad \square$$

2.3.3 The Laplacian on EAC manifolds

In this subsection we use results from the previous subsection to study the Hodge Laplacian and its kernel on an EAC (exponentially asymptotically cylindrical) manifold. The condition that the metric is EAC (rather than just asymptotically translation-invariant as in §2.3.2) allows us to prove a Hodge decomposition result for exponentially decaying forms, and is needed to develop the Hodge theory in the next subsection.

Theorem 2.3.12 about elliptic operators being Fredholm on the weighted Hölder spaces for all but a discrete set of weights applies in particular to the Hodge Laplacian $\triangle$ of an asymptotically translation-invariant metric, and we can define

Definition 2.3.18. Given an asymptotically cylindrical manifold M^n let $\epsilon_1 > 0$ be the smallest positive element of $\mathcal{B}_\triangle$ (where $\mathcal{B}_\triangle$ is defined as in (2.24), and $\triangle$ is considered to act on sections of $\bigoplus_{m=0}^n \Lambda^m$).

Throughout this subsection we will use weights δ such that $0 < \delta < \epsilon_1$, so that the Laplacian is Fredholm on weighted spaces for all positive weights less than or equal to δ. It

is obvious from the definition of $\mathcal{B}_\triangle$ that ϵ_1 depends only on the asymptotic model $g_X + dt^2$ for the metric on M.

Lemma 2.3.19. *ϵ_1 is a lower semi-continuous function of g_X with respect to the C^1-norm.*

Proof. ϵ_1^2 is in fact the smallest positive eigenvalue λ_1 of the Hodge Laplacian $\triangle_X$ defined by g_X on $\Omega^*(X)$ (cf. (2.27)). To prove the proposition it therefore suffices to show that λ_1 is lower semi-continuous in g_X.

Let g, g' be smooth Riemannian metrics on X, $\triangle, \triangle'$ their Laplacians and λ_1, λ_1' the smallest positive eigenvalues of the Laplacians. Let T be the $L^2(g)$-orthogonal complement to $\ker \triangle$ in $C^{2,\alpha}(\Lambda^* T^* X)$. Then for any $\beta \in T$ with unit $L^2(g)$-norm

$$\lambda_1 \leq <\triangle\beta, \beta>_{L^2(g)} = \|d\beta\|_{L^2(g)}^2 + \|d^*\beta\|_{L^2(g)}^2.$$

Since $d + d^*$ is an elliptic operator it gives a Fredholm map $L_1^2(\Lambda^* T^* X) \to L^2(\Lambda^* T^* X)$, so it is bounded below transverse to its kernel. In other words, there is a constant C_1 such that $\|\beta\|_{L_1^2(g)}^2 \leq C_1 \left(\|d\beta\|_{L^2(g)}^2 + \|d^*\beta\|_{L^2(g)}^2 \right)$ for any $\beta \in T \cap L_1^2(\Lambda^* T^* X)$.

Let e_1 be an eigenvector of $\triangle'$ with eigenvalue λ_1'. By Hodge theory for compact manifolds $\ker \triangle$ and $\ker \triangle'$ have the same dimension, so $(\ker \triangle' \oplus \mathbb{R} e_1) \cap T$ is non-trivial. Hence

$$\lambda_1' \geq \frac{<\triangle'\beta, \beta>_{L^2(g')}}{<\beta, \beta>_{L^2(g')}} = \frac{\|d\beta\|_{L^2(g')}^2 + \|d^{*'}\beta\|_{L^2(g')}^2}{\|\beta\|_{L^2(g')}^2}$$

for some $\beta \in T$ with unit $L^2(g)$-norm. The RHS depends differentiably on g' (with respect to the $C^1(g)$-norm) and the derivative at $g' = g$ can be estimated in terms of $\|\beta\|_{L_1^2(g)}^2$. Therefore there is a constant C_2 (independent of β) such that for any g' close to g

$$\lambda_1' \geq \frac{\|d\beta\|_{L^2(g')}^2 + \|d^{*'}\beta\|_{L^2(g')}^2}{\|\beta\|_{L^2(g')}^2}$$
$$\geq \|d\beta\|_{L^2(g)}^2 + \|d^*\beta\|_{L^2(g)}^2 - C_2\|g' - g\|_{C^1(g)}\|\beta\|_{L_1^2(g)}^2 \geq \left(1 - \|g' - g\|_{C^1(g)}C_1 C_2\right)\lambda_1. \quad \square$$

Now let M be an EAC manifold with rate δ_0 and cross-section X, and assume that $0 < \delta < \min\{\epsilon_1, \delta_0\}$. We fix some notation for various spaces of harmonic forms.

Definition 2.3.20. Denote by

  (i) $\mathcal{H}_\pm^m$ the space of harmonic m-forms in $C_{\pm\delta}^{k,\alpha}(\Lambda^m T^* M)$,

  (ii) $\mathcal{H}_0^m$ the space of bounded harmonic m-forms on M,

(iii) $\mathcal{H}^m_\infty$ the space of translation-invariant harmonic m-forms on $X \times \mathbb{R}$,

(iv) $\mathcal{H}^m_X$ the space of harmonic m-forms on X.

By elliptic regularity $\mathcal{H}^m_\pm$ consists of smooth forms, and is independent of k for $k \geq 2$.

We can use theorem 2.3.17 to compute the index of the Laplacian. The result can also be found in [38, §3] and [42, §6.4].

Proposition 2.3.21. *Let M be an asymptotically cylindrical manifold. For $0 < \delta < \epsilon_1$*

$$\mathrm{ind}\left(\triangle : C^{k+2,\alpha}_{\pm\delta}(\Lambda^m) \to C^{k,\alpha}_{\pm\delta}(\Lambda^m)\right) = \mp(b^{m-1}(X) + b^m(X)).$$

Proof. $\triangle$ is asymptotic to $\triangle_\infty$, the Laplacian of the product metric $g_\infty = g_X + dt^2$ on the cylinder $X \times \mathbb{R}$. According to theorem 2.3.17 we can find the desired index by computing the dimensions $d_\triangle(\lambda)$ of the spaces of solutions $S_\triangle(\lambda)$ for $\lambda \in \mathbb{R}$.

Any section of $\Lambda^m T^*(X \times \mathbb{R})$ on $X \times \mathbb{R}$ can be written uniquely as $\psi + dt \wedge \tau$, where ψ and τ are sections of $\Lambda^m T^*X$ and $\Lambda^{m-1}T^*X$ respectively. We can express $\triangle_\infty$ in terms of the Laplacian $\triangle_X$ on X by

$$\triangle_\infty(\psi + dt \wedge \tau) = -\frac{\partial^2\psi}{\partial t^2} + \triangle_X\psi + \left(-\frac{\partial^2\tau}{\partial t^2} + \triangle_X\tau\right) \wedge dt. \qquad (2.26)$$

Now suppose that $\triangle_\infty\left(e^{i\lambda t}(\psi + dt \wedge \tau)\right) = 0$, where $\psi + \tau \wedge dt$ is a nontrivial polynomial of degree r in t, and $\lambda \in \mathbb{R}$. Denote the degree s coefficients by ψ_s and τ_s (these are independent of t, so can be considered as forms on X). Using (2.26) the condition that the degree r component of $\triangle_\infty\left(e^{i\lambda t}(\psi + dt \wedge \tau)\right)$ vanishes becomes

$$\lambda^2\psi_r + \triangle_X\psi_r = 0, \; \lambda^2\tau_r + \triangle_X\tau_r = 0. \qquad (2.27)$$

Since $\triangle_X$ is semi-positive-definite it follows that $\lambda = 0$ (in particular $d_\triangle(\lambda) = 0$ for $\lambda \in \mathbb{R} \setminus \{0\}$). It follows from (2.27) that $\psi_r \in \mathcal{H}^m_X, \tau_r \in \mathcal{H}^{m-1}_X$.

Suppose that the degree $r \geq 2$. Then, since $\lambda = 0$, the condition that the degree $r - 2$ component of $\triangle_\infty(e^{i\lambda t}(\psi + dt \wedge \tau))$ vanishes becomes

$$-r(r-1)\psi_r + \triangle_X\psi_{r-2} = 0, \; -r(r-1)\tau_r + \triangle_X\tau_{r-2} = 0.$$

This is impossible since ψ_r and τ_r are harmonic (not both zero) forms on X, and the image of $\triangle_X$ is L^2-orthogonal to the kernel. Hence any solution of $\triangle_\infty(\psi + dt \wedge \tau) = 0$ that is

polynomial in t is actually linear. The space of such solutions is precisely

$$S_\triangle(0) = \{t(\psi_1 + \tau_1 \wedge dt) + (\psi_0 + \tau_0 \wedge dt) : \psi_s \in \mathcal{H}_X^m, \tau_s \in \mathcal{H}_X^{m-1}\}, \qquad (2.28)$$

and its dimension is

$$d_\triangle(0) = 2b^m(X) + 2b^{m-1}(X).$$

The result now follows from theorem 2.3.17. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 2.3.22. It follows from (2.26) that

$$\mathcal{H}_\infty^m = \{\psi + dt \wedge \tau : \psi \in \mathcal{H}_X^m, \tau \in \mathcal{H}_X^{m-1}\}.$$

In particular the dimension of $\mathcal{H}_\infty^m$ is $-\mathrm{ind}\left(\triangle : C_\delta^{k+2,\alpha}(\Lambda^m) \to C_\delta^{k,\alpha}(\Lambda^m)\right)$ for small $\delta > 0$.

Now that we have computed the index of the Laplacian on weighted Hölder spaces we use an index-counting argument to deduce results about the kernel and a Hodge decomposition theorem. Let M be an EAC manifold with rate δ_0, $k \geq 0$ and $0 < \delta < \min\{\epsilon_1, \delta_0\}$.

Let $i = b^m(X) + b^{m-1}(X)$. $\mathcal{H}_\infty^m$ has dimension i and the index of

$$\triangle : C_\delta^{k+2,\alpha}(\Lambda^m) \to C_\delta^{k,\alpha}(\Lambda^m)$$

is $-i$. $\triangle(\rho\mathcal{H}_\infty^m)$ and $\triangle(\rho t\mathcal{H}_\infty^m)$ consist of exponentially decaying forms. Therefore

$$\triangle : C_\delta^{k+2,\alpha}(\Lambda^m) \oplus \rho\mathcal{H}_\infty^m \oplus \rho t\mathcal{H}_\infty^m \to C_\delta^{k,\alpha}(\Lambda^m) \qquad (2.29)$$

is well-defined, and its index is $+i$. The map

$$\triangle : C_{-\delta}^{k+2,\alpha}(\Lambda^m) \to C_{-\delta}^{k,\alpha}(\Lambda^m) \qquad (2.30)$$

also has index $+i$, and its kernel $\mathcal{H}_-^m$ contains that of (2.29). By proposition 2.3.16 the image of (2.30) is exactly the orthogonal complement of $\mathcal{H}_+^m$ in $C_{-\delta}^{k,\alpha}(\Lambda^m)$. The image of (2.29) is a subset, so it is contained in the orthogonal complement of $\mathcal{H}_+^m$ in $C_\delta^{k,\alpha}(\Lambda^m)$. Thus the cokernel of (2.29) is at least a large as the cokernel of (2.30).

It follows by dimension-counting that the kernels of (2.29) and (2.30) are equal and that the cokernels have the same dimension. In particular

Proposition 2.3.23. *Let M be EAC with rate δ_0, $k \geq 0$ and $0 < \delta < \min\{\epsilon_1, \delta_0\}$. Then any harmonic form in $C_{-\delta}^{2,\alpha}(\Lambda^m)$ lies in*

$$C_{\delta}^{k,\alpha}(\Lambda^m) \oplus \rho\mathcal{H}_{\infty}^m \oplus \rho t\mathcal{H}_{\infty}^m.$$

Since $\mathcal{H}_-^m$ contains both the L^2 harmonic forms and the bounded harmonic forms $\mathcal{H}_0^m$ it follows that

Proposition 2.3.24. *Let M be EAC with rate δ_0, $k \geq 0$ and $0 < \delta < \min\{\epsilon_1, \delta_0\}$. Then the space of L^2-integrable harmonic m-forms on M is exactly $\mathcal{H}_+^m$, and*

$$\mathcal{H}_0^m \subset C_{\delta}^{k,\alpha}(\Lambda^m) \oplus \rho\mathcal{H}_{\infty}^m.$$

Remark 2.3.25. Results like propositions 2.3.23 and 2.3.24 about the asymptotic behaviour of solutions of elliptic equations are less a consequence of theorem 2.3.12 than a part of its proof. The ordering here is for presentational convenience.

We can also use the index-counting argument to prove a Hodge decomposition result. We need to distinguish $dC_{\delta}^{k,\alpha}(\Lambda^{m-1})$, the space of exterior derivatives of decaying $(m-1)$-forms, from the space of decaying exact m-forms.

Definition 2.3.26. For $0 < \delta < \epsilon_1$ let

$$C_{\delta}^{k,\alpha}[d\Lambda^{m-1}] = d\Omega^{m-1}(M) \cap C_{\delta}^{k,\alpha}(\Lambda^m),$$
$$C_{\delta}^{k,\alpha}[d^*\Lambda^{m+1}] = d^*\Omega^{m+1}(M) \cap C_{\delta}^{k,\alpha}(\Lambda^m).$$

Theorem 2.3.27. *Let M^n be EAC with rate δ_0, $k \geq 0$, and $0 < \delta < \min\{\epsilon_1, \delta_0\}$. Then there is an L^2-orthogonal direct sum decomposition*

$$C_{\delta}^{k,\alpha}(\Lambda^m) = \mathcal{H}_+^m \oplus C_{\delta}^{k,\alpha}[d\Lambda^{m-1}] \oplus C_{\delta}^{k,\alpha}[d^*\Lambda^{m+1}], \tag{2.31}$$

and the projections onto these summands are bounded. Furthermore, if ρ is a cut-off function for the cylindrical ends of M then any element of $C_{\delta}^{k,\alpha}[d\Lambda^{m-1}]$ can be written as $d\phi$ for some coexact $\phi \in C_{\delta}^{k+1,\alpha}(\Lambda^{m-1}) \oplus \rho\mathcal{H}_{\infty}^{m-1}$, and any element of $C_{\delta}^{k,\alpha}[d^*\Lambda^{m-1}]$ can be written as $d^*\phi$ for some exact $\phi \in C_{\delta}^{k+1,\alpha}(\Lambda^{m+1}) \oplus \rho\mathcal{H}_{\infty}^{m+1}$.

Proof. Theorem 2.3.33 below implies that $\mathcal{H}_+^m \cap C_{\delta}^{k,\alpha}[d\Lambda^{m-1}] = 0$. Applying the Hodge star gives $\mathcal{H}_+^m \cap C_{\delta}^{k,\alpha}[d^*\Lambda^{m+1}] = 0$. Since $C_{\delta}^{k,\alpha}[d\Lambda^{m-1}] \cap C_{\delta}^{k,\alpha}[d^*\Lambda^{m+1}] \subseteq \mathcal{H}_+^m$ it follows that the sum is direct.

Since the cokernels of (2.29) and (2.30) have equal dimension the image of (2.29) is precisely the orthogonal complement of $\mathcal{H}_+^m$ in $C_\delta^{k,\alpha}(\Lambda^m)$, i.e.

$$C_\delta^{k,\alpha}(\Lambda^m) = \mathcal{H}_+^m \oplus \triangle \left(C_\delta^{k+2,\alpha}(\Lambda^m) \oplus \rho\mathcal{H}_\infty^m \oplus \rho t\mathcal{H}_\infty^m \right).$$

Since $\mathcal{H}_+^m$ is finite-dimensional the projections are bounded.

Now let G be a Fredholm inverse for (2.29) and for $\beta \in C_\delta^{k,\alpha}(\Lambda^m)$ define

$$P_E\beta = dd^*G\beta,$$
$$P_{E^*}\beta = d^*dG\beta.$$

P_E and P_{E^*} map into $C_\delta^{k,\alpha}[d\Lambda^{m-1}]$ and $C_\delta^{k,\alpha}[d^*\Lambda^{m+1}]$ respectively, and $\beta = P_E(\beta) + P_{E^*}(\beta)$ for any $\beta \in \triangle \left(C_\delta^{k,\alpha}(\Lambda^m) \oplus \rho\mathcal{H}_\infty^m \oplus \rho t\mathcal{H}_\infty^m \right)$. Therefore

$$\triangle \left(C_\delta^{k+2,\alpha}(\Lambda^m) \oplus \rho\mathcal{H}_\infty^m \oplus \rho t\mathcal{H}_\infty^m \right) = C_\delta^{k,\alpha}[d\Lambda^{m-1}] \oplus C_\delta^{k,\alpha}[d^*\Lambda^{m+1}]$$

with projections P_E and P_{E^*}. The result follows. $\qquad\square$

2.3.4 Hodge theory on EAC manifolds

In this subsection we study the correspondence between harmonic forms and the de Rham cohomology on an EAC Riemannian manifold M^n with cross-section X^{n-1}.

We first describe what form the long exact sequence for relative cohomology takes for an EAC manifold. Following Melrose [42, §6.4] we then define subspaces $\mathcal{H}_{abs}^m, \mathcal{H}_{rel}^m$ of the bounded harmonic forms $\mathcal{H}_0^m$ and describe their relation to the long exact sequence. Hodge theory on classes of non-compact manifolds, including asymptotically cylindrical ones, is also discussed in Lockhart [38].

Remark 2.3.28. While we will usually apply the Hodge theory results to orientable EAC manifolds M, we can still prove them in the non-orientable case. Some arguments rely on using the Hodge star on M. These results we prove first for M orientable. In the case when M is not orientable let $\tilde{M}$ be the orientable double cover of M. $\tilde{M}$ is EAC, and M is a quotient of $\tilde{M}$ by an orientation-reversing involution a. The harmonic forms of M can be identified with the $+1$ eigenspace of a on the harmonic forms of $\tilde{M}$, and the result for M can be deduced from that for $\tilde{M}$.

Definition 2.3.29. Let $H^*_{cpt}(M)$ be the compactly supported cohomology of M, i.e. the cohomology of the cochain complex

$$\Omega^0_{cpt}(M) \xrightarrow{d} \Omega^1_{cpt}(M) \xrightarrow{d} \Omega^2_{cpt}(M) \xrightarrow{d} \cdots$$

of compactly supported smooth forms on M.

Recall that when M has cylindrical ends then we can compactify M by 'adding a copy of X at infinity', i.e. by including it in $\overline{M} = M_0 \cup (X \times [0, \infty])$. The cohomology (with coefficients $\mathbb{R}$) of $\overline{M}$ relative to its boundary can be identified with $H^*_{cpt}(M)$. The long exact sequence for relative cohomology of $\overline{M}$ can be written as

$$\cdots \longrightarrow H^{m-1}(X) \xrightarrow{\partial} H^m_{cpt}(M) \xrightarrow{e} H^m(M) \xrightarrow{j^*} H^m(X) \longrightarrow \cdots \qquad (2.32)$$

$e : H^m_{cpt}(M) \to H^m(M)$ is induced by the inclusion $\Omega^*_{cpt}(M) \hookrightarrow \Omega^*(M)$. The image of e is the subspace of de Rham cohomology classes with compact representatives.

Definition 2.3.30. Let $H^m_0(M) = \mathrm{im}\,\big(e : H^m_{cpt}(M) \to H^m(M)\big)$.

In the asymptotically cylindrical setting the map $j^* : H^m(M) \to H^m(X)$ can be described as follows: For $s \in \mathbb{R}^+$ let $j_s : X \hookrightarrow M$ be the inclusion $x \mapsto (x, s) \in M_\infty$. The maps j_s are homotopic, so they all give the same map $j^* : H^m(M) \to H^m(X)$.

If α is an asymptotically translation-invariant m-form let $B(\alpha)$ denote its asymptotic limit, which is a translation-invariant form on the cylinder $X \times \mathbb{R}$. $B(\alpha)$ can be written as $B_a(\alpha) + dt \wedge B_e(\alpha)$, where $B_a(\alpha)$, $B_e(\alpha)$ are forms on X of degree m and $m-1$ respectively.

If α is a closed asymptotically translation-invariant m-form then for any m-cycle C in X

$$\int_C j_s^*([\alpha]) \to \int_C [B_a(\alpha)]$$

as $s \to \infty$, so

$$j^*([\alpha]) = [B_a(\alpha)]. \qquad (2.33)$$

By proposition 2.3.24 any $\alpha \in \mathcal{H}^m_0$ is exponentially asymptotically translation-invariant, and $B(\alpha) \in \mathcal{H}^m_\infty$. By remark 2.3.22

$$\mathcal{H}^m_\infty \cong \mathcal{H}^m_X \oplus dt \wedge \mathcal{H}^{m-1}_X,$$

so we get maps $B_a : \mathcal{H}^m_0 \to \mathcal{H}^m_X$, $B_e : \mathcal{H}^m_0 \to \mathcal{H}^{m-1}_X$.

Definition 2.3.31. Let

$$\mathcal{H}^m_{abs} = \ker B_e \subseteq \mathcal{H}^m_0,$$
$$\mathcal{H}^m_{rel} = \ker B_a \subseteq \mathcal{H}^m_0.$$

Further let $\mathcal{H}^m_E, \mathcal{H}^m_{E^*} \subseteq \mathcal{H}^m_0$ denote the spaces of bounded exact and coexact harmonic forms respectively.

It follows immediately from (2.33) that $\mathcal{H}^m_E \subseteq \mathcal{H}^m_{rel}$. It follows by applying the Hodge star that $\mathcal{H}^m_{E^*} \subseteq \mathcal{H}^m_{abs}$ (using remark 2.3.28 if M is not orientable).

Let $\Omega^*_{-\delta}(M)$ be the cochain complex of forms β such that $e^{-\delta t}\beta$ is uniformly bounded with all derivatives, and let $H^*_{-\delta}(M)$ be its cohomology.

Lemma 2.3.32. *For $\delta > 0$ the natural map $H^*_{-\delta}(M) \to H^*(M)$ is an isomorphism.*

Proof. The inclusion $i : \Omega^*_{-\delta}(M) \hookrightarrow \Omega^*(M)$ is a chain map, so induces a well-defined map $i : H^*_{-\delta}(M) \to H^*(M)$.

Let $\rho : \mathbb{R} \to [0,1]$ be a smooth function with $\rho(t) = 0$ for $t \leq 0$ and $\rho(t) = 1$ for $t \geq 1$. Define a map $c : M \to M$ which is the identity on the compact piece M_0 and $(x,t) \mapsto (x, (1-\rho(t))t + \rho(t)\arctan t)$ on the cylindrical part. c is smooth, and c^* maps $\Omega^*(M)$ to $\Omega^*_{-\delta}(M)$. It is a chain map, so induces a map $c^* : H^*(M) \to H^*_{-\delta}(M)$. We deduce that i and c^* are inverses from the fact that c is homotopic to the identity. $\square$

The next theorem is part of [42, Theorem 6.18].

Theorem 2.3.33. *Let M be an EAC manifold. The natural map $\mathcal{H}^m_{abs} \to H^m(M)$ is an isomorphism.*

Proof. Note that by integration by parts the elements of $\mathcal{H}^m_0$ are closed, so represent cohomology classes. Pick some $0 < \delta < \epsilon_1$. By the previous lemma it suffices to check that $\mathcal{H}^m_{abs} \to H^m_{-\delta}(M)$ is an isomorphism.

We first prove injectivity. If $\alpha \in \mathcal{H}^m_{abs}$ and $\alpha = d\beta$ for some $\beta \in \Omega^m_{-\delta}(M)$ then $\alpha \in \mathcal{H}^m_+$, since $\mathcal{H}^m_E \subseteq \mathcal{H}^m_{rel}$ and $\mathcal{H}^m_{abs} \cap \mathcal{H}^m_{rel} = \mathcal{H}^m_+$. As α is exponentially decaying with rate δ we can integrate by parts to deduce $\alpha = 0$. Thus $\mathcal{H}^m_{abs} \to H^m_{-\delta}(M)$ is an injection.

It remains to show that if $\gamma \in \Omega^m_{-\delta}(M)$ is closed then γ is cohomologous to an element of $\mathcal{H}^m_{abs}$. $C^{0,\alpha}_{-\delta}(\Lambda^m) = \mathcal{H}^m_+ \oplus \triangle C^{2,\alpha}_{-\delta}(\Lambda^m)$ by proposition 2.3.16. Thus γ can be written as

$$\gamma = \phi + \triangle\psi$$

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with $\phi \in \mathcal{H}_+^m$ and $\psi \in C_{-\delta}^{2,\alpha}(\Lambda^m)$. Let $\chi = d\psi$. Then

$$d\gamma = 0 \Rightarrow dd^*\chi = 0 \Rightarrow \triangle\chi = 0.$$

By proposition 2.3.23, $\chi \in C_\delta^{2,\alpha}(\Lambda^{m+1}) \oplus \rho\mathcal{H}_\infty^{m+1} \oplus \rho t\mathcal{H}_\infty^{m+1}$ so $d^*\chi \in \mathcal{H}_{E^*}^m \subseteq \mathcal{H}_{abs}^m$. Therefore

$$\gamma - dd^*\psi = \phi + d^*\chi \in \mathcal{H}_{abs}^m \tag{2.34}$$

represents the same cohomology class as γ. Hence $\mathcal{H}_{abs}^m \to H_{-\delta}^m(M)$ is surjective. $\quad\square$

Proposition 2.3.34. $\mathcal{H}_{abs}^m = \mathcal{H}_+^m \oplus \mathcal{H}_{E^*}^m$, $\mathcal{H}_{rel}^m = \mathcal{H}_+^m \oplus \mathcal{H}_E^m$.
   Furthermore $\mathcal{H}_E^m = d\mathcal{H}_-^{m-1}$, $\mathcal{H}_{E^*}^m = d^*\mathcal{H}_-^{m+1}$.

Proof. Theorem 2.3.33 implies that $\mathcal{H}_E^m \cap \mathcal{H}_+^m = 0$. Similarly $\mathcal{H}_{E^*}^m \cap \mathcal{H}_+^m = 0$, so the sums are direct. If $\gamma \in \mathcal{H}_0^m$ then, since γ is closed, (2.34) gives

$$\gamma = \phi + d^*\chi + dd^*\psi,$$

where $\phi \in \mathcal{H}_+^m$, $d^*\chi \in d^*\mathcal{H}_-^{m+1}$. Analogously, since γ is also coclosed, $dd^*\psi \in d\mathcal{H}_-^{m-1}$. Hence

$$\mathcal{H}_0^m = \mathcal{H}_+^m \oplus d^*\mathcal{H}_-^{m+1} \oplus d\mathcal{H}_-^{m-1}.$$

Since $d\mathcal{H}_-^{m-1} \subseteq \mathcal{H}_E^m \subseteq \mathcal{H}_{rel}^m$ and $d^*\mathcal{H}_-^{m+1} \subseteq \mathcal{H}_{E^*}^m \subseteq \mathcal{H}_{abs}^m$ the result follows. $\quad\square$

As a corollary of theorem 2.3.33 we can determine that the image of the space $\mathcal{H}_+^m$ of L^2 harmonic forms in the de Rham cohomology $H^m(M)$ is precisely the subspace $H_0^m(M)$ of compactly supported classes. Similar results (with more general hypotheses) appears in e.g. [1, Proposition 4.9] and [38, Theorems 7.6 and 7.9].

Theorem 2.3.35. *Let M be an EAC manifold. The natural map $\mathcal{H}_+^m \to H^m(M)$ is an isomorphism onto $H_0^m(M)$.*

Proof. $\mathcal{H}_+^m$ is $\ker B_a$ in $\mathcal{H}_{abs}^m$, and it follows from theorem 2.3.33 it is mapped isomorphically to $H_0^m(M) = \ker j^* \subseteq H^m(M)$. $\quad\square$

The fact that the image of $\mathcal{H}_+^m$ is contained in $H_0^m(M)$ could be seen more explicitly by applying the following lemma, which is useful for other purposes.

If α is a closed exponentially asymptotically translation-invariant m-form on M we can write it as $\alpha_\infty + \beta_t + dt \wedge \gamma_t$ on the cylindrical part $X \times \mathbb{R}^+$, where α_∞ is translation-invariant, and β_t, γ_t are sections of $\Lambda^m T^*X$, $\Lambda^{m-1}T^*X$ respectively, and are exponentially

decaying in t. Let

$$\eta(\alpha) = \rho \int_t^\infty \gamma_s ds, \tag{2.35}$$

where ρ is a smooth cut-off function for the cylindrical end of M, equal to 1 for $t > t_0$. $\eta(\alpha)$ is a well-defined exponentially decaying $(m-1)$-form on M.

Lemma 2.3.36. *Let M be a manifold with cylindrical ends, and α a closed exponentially asymptotically translation-invariant m-form on M. Then $\alpha + d\eta(\alpha)$ is translation-invariant on $\{y \in M : t > t_0\}$.*

Proof. Closure of α means that $\frac{\partial}{\partial t}\beta_t + d_X\gamma_t = 0$ where d_X is the exterior derivative on X. Thus $\beta_t = -\int_t^\infty d_X\gamma_s ds$. Since X is compact the dominated convergence theorem ensures that for $t > t_0$

$$d\eta = d_X\eta + dt \wedge \tfrac{\partial}{\partial t}\eta = -\beta_t - dt \wedge \gamma_t. \qquad \square$$

Definition 2.3.37. Let $\mathcal{A}^m = B_a(\mathcal{H}_0^m) \subseteq \mathcal{H}_X^m$, $\mathcal{E}^m = B_e(\mathcal{H}_0^{m+1}) \subseteq \mathcal{H}_X^m$, and let A^m, E^m be the subspaces of $H^m(X)$ that they represent (A^m is of course just $j^*(H^m(M)) \subseteq H^m(X)$).

When M^n is oriented the Hodge star on M identifies $\mathcal{H}_{abs}^m$ and $\mathcal{H}_{rel}^{m-n}$. If $\beta \in \mathcal{H}_0^m$ then $B_e(*\beta) = *B_a(\beta)$, so the Hodge star on X identifies $\mathcal{A}^m$ with $\mathcal{E}^{n-m-1}$ (and hence A^m with E^{n-m-1}).

By proposition 2.3.23 any $\psi \in \mathcal{H}_-^m$ can be written as $\chi + \alpha + t\beta + dt \wedge \gamma + tdt \wedge \delta$ with χ exponentially decaying, $\alpha, \beta \in \mathcal{H}_X^m$ and $\gamma, \delta \in \mathcal{H}_X^{m-1}$. Thus we can define a 'boundary data' map

$$BD : \mathcal{H}_-^m \to (\mathcal{H}_X^m)^2 \oplus (\mathcal{H}_X^{m-1})^2, \quad \psi \mapsto \begin{pmatrix} \beta & \delta \\ \alpha & \gamma \end{pmatrix}.$$

Let BD_a, BD_e be the composition of BD with the projection to $(\mathcal{H}_X^m)^2$ and $(\mathcal{H}_X^{m-1})^2$ respectively. Let $\tilde{\mathcal{A}}^m = BD_a(\mathcal{H}_-^m)$ and $\tilde{\mathcal{E}}^{m-1} = BD_e(\mathcal{H}_-^m)$. The following proposition is a refinement of [42, Lemma 6.15].

Proposition 2.3.38. *Let M be an EAC manifold.*

(i) *If $\psi_1, \psi_2 \in \mathcal{H}_-^m$ with $BD(\psi_i) = \begin{pmatrix} \beta_i & \delta_i \\ \alpha_i & \gamma_i \end{pmatrix}$ then in the L^2 inner product on X*

$$<\alpha_1, \beta_2>_{L^2} = <\alpha_2, \beta_1>_{L^2}, \tag{2.36a}$$

$$<\gamma_1, \delta_2>_{L^2} = <\gamma_2, \delta_1>_{L^2}. \tag{2.36b}$$

(ii) $BD(\mathcal{H}_-^m) = \tilde{\mathcal{A}}^m \oplus \tilde{\mathcal{E}}^m$, and $\tilde{\mathcal{A}}^m \subset (\mathcal{H}_X^m)^2$ and $\tilde{\mathcal{E}}^{m-1} \subset (\mathcal{H}_X^{m-1})^2$ are Lagrangian subspaces.

Proof. First assume that M is oriented. Then $\partial : H^m(X) \to H_{cpt}^{m+1}(M)$ is the Poincaré dual of $j^* : H^{n-m+1}(M) \to H^{n-m}(X)$, so

$$<\alpha_i, \beta_j>_{L^2} = ([\alpha_i], [*\beta_j])_X = ([\alpha_i], j^*[*d\psi_j])_X = (\partial[\alpha_i], [*d\psi_j])_M.$$

Note that $\partial[\alpha_i] = [d(\psi_i + \eta(d\psi_i) - \rho t \beta_i)] \in H_{cpt}^{m+1}(M)$. Hence

$$<\alpha_1, \beta_2>_{L^2} - <\alpha_2, \beta_1>_{L^2} = \int_M (d\psi_1 - d(\rho t \beta_1)) \wedge *d\psi_2 - (d\psi_2 - d(\rho t \beta_2)) \wedge *d\psi_1$$

$$= \int_M d(\rho t(\beta_2 \wedge *d\psi_1 - \beta_1 \wedge *d\psi_2)). \quad (2.37)$$

Since
$$\beta_2 \wedge B(*d\psi_1) - \beta_1 \wedge B(*d\psi_2) = \beta_2 \wedge *\beta_1 - \beta_1 \wedge *\beta_2 = 0$$

the integrand in the RHS of (2.37) is the exterior derivative of an exponentially decaying form. The vanishing of the integral proves (2.36a), and (2.36b) follows by applying $*$.

This proves (i) in the oriented case. When M is not orientable we use remark 2.3.28.

(i) implies that $\tilde{\mathcal{A}}^m \subset (\mathcal{H}_X^m)^2$ and $\tilde{\mathcal{E}}^{m-1} \subset (\mathcal{H}_X^{m-1})^2$ are null spaces. In particular

$$\dim \tilde{\mathcal{A}}^m \leq b^m(X), \ \dim \tilde{\mathcal{E}}^{m-1} \leq b^{m-1}(X).$$

Since $BD(\mathcal{H}_-^m) \subseteq \tilde{\mathcal{A}}^m \oplus \tilde{\mathcal{E}}^{m-1}$ and has dimension $b^{m-1}(X) + b^m(X)$ equality must hold, so $\tilde{\mathcal{A}}^m$ and $\tilde{\mathcal{E}}^{m-1}$ are Lagrangian. $\qquad\square$

Proposition 2.3.39. *If M^n is an EAC manifold with cross-section X then*

$$\mathcal{H}_X^m = \mathcal{A}^m \oplus \mathcal{E}^m$$

is an orthogonal direct sum.

Proof. If $\psi \in \mathcal{H}_-^m$ with $BD_a(\psi) = \begin{pmatrix} \beta_1 \\ \alpha_1 \end{pmatrix} \in \tilde{\mathcal{A}}^m$ then $\beta_1 = B_e(d\psi) \in \mathcal{E}^m$. Thus the second projection $\tilde{\mathcal{A}}^m \to \mathcal{H}_X^m$ has image $\mathcal{E}^m$, and kernel $\mathcal{A}^m$, so

$$\dim \mathcal{A}^m + \dim \mathcal{E}^m = \dim \tilde{\mathcal{A}}^m = \dim \mathcal{H}_X^m$$

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Furthermore if $\alpha_2 \in \mathcal{A}^m$ then $\begin{pmatrix} 0 \\ \alpha_2 \end{pmatrix} \in \tilde{\mathcal{A}}^m$, and (2.36b) implies that $<\beta_1, \alpha_2> = 0$, so $\mathcal{A}^m$ and $\mathcal{E}^m$ are orthogonal. $\qquad\square$

It follows from proposition 2.3.39 that we can define an isomorphism

$$\mathcal{H}^m_{rel} \to H^m_{cpt}(M), \ \alpha \mapsto \begin{cases} [\alpha + d\eta(\alpha)] & \text{for } \alpha \in \mathcal{H}^m_+ \\ \partial([B_e(\alpha)]) & \text{for } \alpha \in \mathcal{H}^m_E \end{cases}.$$

Corollary 2.3.40. *Let M^n be an asymptotically cylindrical manifold which has a single end (i.e. the cross-section X is connected). Then $e : H^1_{cpt}(M) \to H^1(M)$ is injective.*

In particular $\mathcal{H}^1_E = 0$, and $\mathcal{H}^1_0 \to H^1(M)$ is an isomorphism.

Proof. Consider the start of the long exact sequence for relative cohomology

$$H^0_{cpt}(M) \to H^0(M) \to H^0(X) \xrightarrow{\partial} H^1_{cpt}(M) \xrightarrow{e} H^1(M).$$

The dimensions of the first three terms are $0, 1$, and 1, so $\partial = 0$, and thus e is injective. $\mathcal{H}^1_{rel} \hookrightarrow H^1_{cpt}(M)$ identifies $\mathcal{H}^1_E$ with $\ker e$, so the result follows. $\qquad\square$

Finally we make two simple observations for the case when M is Ricci-flat, as it is when the holonomy is G_2 or $Spin(7)$.

Proposition 2.3.41. *If M is a Ricci-flat EAC manifold then $\mathcal{H}^1_0$ is the space of parallel 1-forms on M. In particular $\mathcal{H}^1_+ = 0$, and $j^* : H^1(M) \to H^1(X)$ is injective.*

Proof. This is proved by the same standard 'Bochner argument' as corollary 2.1.9, adapted to the EAC setting. If ϕ is a 1-form then by proposition 2.1.8

$$\triangle\phi = \nabla^*\nabla\phi. \tag{2.38}$$

It follows that any parallel 1-form ϕ is harmonic, and parallel forms are of course bounded.

To show that any bounded harmonic form is parallel we use (2.38) together with an integration by parts argument, justified by lemma 2.3.15.

$\mathcal{H}^1_+ = 0$ since it consists of parallel decaying forms. By theorem 2.3.35, the kernel $H^1_0(M)$ of $j^* : H^1(M) \to H^1(X)$ is represented by $\mathcal{H}^1_+$. $\qquad\square$

Corollary 2.3.42. *If M is a Ricci-flat EAC manifold with a single end then $H^1_{cpt}(M) = 0$.*

Proof. Follows from proposition 2.3.41 and corollary 2.3.40. $\qquad\square$

2.3.5 The Dirac operator on EAC manifolds

We apply the ideas of §2.3.3 to study the Dirac operator on an EAC manifold. This will later be used in slice constructions on EAC G_2-manifolds and $Spin(7)$-manifolds.

Let M^n be an EAC spin manifold with cross-section X, and (complex) spinor bundle S. Let $\mathcal{H}_0^S$ and $\mathcal{H}_+^S$ denote the spaces of bounded and decaying harmonic spinors on M with respect to the Dirac Laplacian $\eth^2$, and let $\mathcal{H}_\infty^S$ be the translation-invariant spinors on the cylinder $X \times \mathbb{R}$. Let S_X be the bundle of spinors on the cross-section, and $\mathcal{H}_X^S$ the harmonic spinors on X.

The Dirac operator on M^n is asymptotically translation-invariant. If n is even then each of the positive and negative spinor bundles on the cylinder $X \times \mathbb{R}$ are isomorphic to the pull-back of S_X from X, and the asymptotic limit can be written as

$$\eth_\infty = \left(\begin{array}{c|c} 0 & \frac{\partial}{\partial t} + \eth_X \\ \hline -\frac{\partial}{\partial t} + \eth_X & 0 \end{array} \right) \ : \ \Gamma(S_X \oplus S_X) \to \Gamma(S_X \oplus S_X).$$

Similarly, if n is odd then the spinor bundle on the cylinder is isomorphic to S_X, and the asymptotic limit is

$$\eth_\infty = \left(\begin{array}{c|c} i\frac{\partial}{\partial t} & 0 \\ \hline 0 & -i\frac{\partial}{\partial t} \end{array} \right) + \eth_X \ : \ \Gamma(S_X^+ \oplus S_X^-) \to \Gamma(S_X^+ \oplus S_X^-).$$

Since the Dirac operator $\eth$ is formally self-adjoint the index of

$$\eth : C_{\pm\delta}^{k+1,\alpha}(S) \to C_{\pm\delta}^{k,\alpha}(S)$$

is $\mp\frac{1}{2} \dim \mathcal{H}_\infty^S$ by theorem 2.3.17.

Integration by parts shows that harmonic asymptotically translation-invariant spinors are solutions of the Dirac equation $\eth\psi = 0$. If ρ is a cut-off function for the cylindrical end on M then $\rho\mathcal{H}_\infty^S$ can be identified with a space of asymptotically translation-invariant spinors on M. It is easy to use index-counting arguments as in §2.3.3 to show that

$$\eth : C_\delta^{k+1,\alpha}(S) \oplus \rho\mathcal{H}_\infty^S \to C_\delta^{k,\alpha}(S) \tag{2.39}$$

has kernel precisely $\mathcal{H}_0^S$ and that the image is the L^2-orthogonal complement to $\mathcal{H}_+^S$. In

particular, spinors in $\mathcal{H}_0^S$ are asymptotic to elements of $\mathcal{H}_\infty^S$, which gives a boundary map

$$B : \mathcal{H}_0^S \to \mathcal{H}_\infty^S$$

By the above the image has dimension $\frac{1}{2} \dim \mathcal{H}_\infty^S$.

Remark 2.3.43. We can check case by case that $\dim \mathcal{H}_\infty^S$ is even *a priori*. If n is even then $\mathcal{H}_\infty^S \cong 2\mathcal{H}_X^S$. If n is odd then $\mathcal{H}_\infty^S \cong \mathcal{H}_X^S$ splits as $\mathcal{H}_X^{S^+} \oplus \mathcal{H}_X^{S^-}$. If $n - 1 \equiv 2 \mod 4$ then $\mathcal{H}_X^{S^+} \cong \overline{\mathcal{H}_X^{S^-}}$ for any spin manifold X of dimension $n-1$. If $n - 1 \equiv 0 \mod 4$ then the index $\hat{A}(X)$ of the Dirac operator $\eth_+$ on X must be 0 because it is a spin cobordism invariant (see [37, (7.9) and Theorem 7.10]), so $\dim \mathcal{H}_X^{S^+} = \mathcal{H}_X^{S^-}$.

If the dimension of M is even then $\mathcal{H}_0^S$ and $\mathcal{H}_\infty^S$ split by chirality and $\mathcal{H}_\infty^{S^\pm} \cong \mathcal{H}_X^S$ because $S^\pm|_X \cong S_X$. Therefore the boundary maps on $\mathcal{H}_0^{S^\pm}$ can be considered as mapping

$$B : \mathcal{H}_0^{S^\pm} \to \mathcal{H}_X^S.$$

Proposition 2.3.44. *Let M^{2n} be an EAC spin manifold with cross-section X^{2n-1}. Then $\mathcal{H}_X^S$ can be written as an L^2-orthogonal sum*

$$\mathcal{H}_X^S = B(\mathcal{H}_0^{S^+}) \oplus B(\mathcal{H}_0^{S^-}).$$

Proof. We saw above that $\dim B(\mathcal{H}_0^S) = \frac{1}{2} \dim \mathcal{H}_\infty^S = \dim \mathcal{H}_X^S$, so we just need to check L^2-orthogonality. Define a map

$$S^+ \times S^- \to TM, \quad (\psi_+, \psi_-) \mapsto \langle \psi_+, \psi_- \rangle$$

by taking $\langle \psi_+, \psi_- \rangle$ to be the metric dual of the 1-form $v \mapsto <v\psi_+, \psi_->$, where $v\psi_+$ denotes the Clifford product. Then

$$<\langle \psi_+, \psi_- \rangle, v> = <v\psi_+, \psi_-> = - <\psi_+, v\psi_->$$

for all $v \in TM$, $\psi_\pm \in S^\pm$. When we identify both of $S^\pm$ with S_X on the cylinder, Clifford

multiplication by $\frac{\partial}{\partial t}$ is identified with $\pm id_{S_X}$. Hence for $\psi_\pm \in \mathcal{H}_0^{S^\pm}$

$$<B(\psi_+), B(\psi_-)>_{L^2} = \int_X dt(\langle \psi_+, \psi_- \rangle) \mathrm{vol}_X = \int_M \mathrm{div} \langle \psi_+, \psi_- \rangle \mathrm{vol}_M$$

$$= \int_M <\eth \psi_+, \psi_-> - <\psi_+, \eth \psi_-> \mathrm{vol}_M = 0. \quad \square$$

Remark 2.3.45. When $\dim M = 8$ the map $\langle, \rangle$ is non-degenerate by the triality discussed in §2.2.3.

Chapter 3

Deformations of compact G-manifolds

In this chapter we will consider the deformation theory of compact G-manifolds, where $G = Spin(7)$, G_2 or $SU(3)$. We will prove in each case that the moduli space of torsion-free G-structures is a smooth manifold. These results were known previously, due to Joyce in the cases $Spin(7)$ and G_2 and Tian and Todorov for $SU(3)$ (and more generally $SU(n)$). Goto [17] also proved smoothness of the moduli space on compact G-manifolds for any of the Ricci-flat holonomy groups $G = SU(n)$, $Sp(n)$, $Spin(7)$ and G_2 in a fairly uniform way. The arguments used here are geared to make it easier to generalise to the asymptotically cylindrical case in chapter 4.

One of the key ingredients for the proofs is to describe a neighbourhood in the moduli space by using a local 'slice' in the space of torsion-free G-structures, containing representatives for all the diffeomorphism classes in the neighbourhood. Before going into the details of the deformation problems we will discuss an elementary slice argument.

§3.2 is a detailed account of an elementary construction of the moduli space of torsion-free G_2-structures on a compact 7-manifold based on an outline by Hitchin [24]. In §3.3 and §3.4 these arguments are adapted to construct moduli spaces of torsion-free $Spin(7)$-structures and $SU(3)$-structures.

3.1 Moduli space constructions

A standard technique in constructing moduli spaces is to use slice arguments. A model example where slice arguments are explained very carefully is Ebin [15], which studies the moduli space of Riemannian metrics on a compact manifold. In the deformation problems considered in this thesis the moduli spaces are – unlike the moduli space of Riemannian

metrics – finite-dimensional. There is therefore some scope for simplifying Ebin's arguments. We prove a slice theorem whose hypotheses are less general than Ebin's, but easier to verify in the problems to which we want to apply it. This will also make it easier to generalise the slice arguments to the asymptotically cylindrical case.

3.1.1 Functional analysis

In the technical arguments we will use spaces of Hölder sections rather than smooth sections. One reason is that this allows application of Fredholm results for elliptic operators, as used in §2.3.

The space of $C^{k,\alpha}$ Hölder sections of a vector bundle is a complete normed vector space, i.e. a Banach space. The inverse function theorem holds for Banach spaces and we will make use of two of its corollaries: the implicit function theorem and the submersion theorem.

Theorem 3.1.1 (see e.g. [36, Theorem XIV 2.1])**.** *Let X, Y and Z be Banach spaces, and U, V open neighbourhoods of 0 in X and Y. Suppose that $F : U \times V \to Z$ is a smooth function, $F(0,0) = 0$, and that $DF_{(0,0)}|_Y : Y \to Z$ is an isomorphism of Y, Z as Banach spaces. Then there exist connected open neighbourhoods $U' \subseteq U$ of 0 in X and $V' \subseteq V$ of 0 in Y such that*

- *there is a smooth map $G : U' \to V'$ such that the set of zeros of F in $U' \times V'$ is precisely the graph $\{(x, G(x)) : x \in U'\}$,*

- *F is an open map on a neighbourhood of 0 in $X \times Y$.*

To ensure that the functions to which we wish to apply the implicit function theorem are really smooth we use a chain rule result: composing $C^{k,\alpha}$-sections with a smooth fibre-wise map gives a smooth mapping of Hölder spaces, with the obvious derivative.

Proposition 3.1.2. *Let E, F be normed vector bundles over a compact manifold M^n, U an open subbundle of E, and $\Psi : U \to F$ a smooth map that preserves fibres. Let $C^{k,\alpha}(U) \subseteq C^{k,\alpha}(E)$ denote the set of sections of E which are bounded in $C^{k,\alpha}$ Hölder norm and take values in U. Then for $k \geq 0$, Ψ defines a map $\Psi_k : C^{k,\alpha}(U) \to C^{k,\alpha}(F)$ by composition: if $s \in C^{k,\alpha}(U)$ and $p \in M$ then $(\Psi_k(s))(p) = \Psi(s(p))$.*

Moreover, Ψ_k is smooth and its derivative at $r \in C^{k,\alpha}(U)$ is obtained by applying the 'vertical differential' of Ψ at r. In symbols

$$(D\Psi_k)_r s(p) = (D\Psi_p)_{r(p)}(s(p)), \tag{3.1}$$

where $r \in U_k$, $s \in E_k$ and $\Psi_p : U_p \to F_p$ is the restriction of Ψ to the fibre of at p.

We can define a Banach manifold to be a space with coordinate charts (mapping to neighbourhoods in a Banach space) which have smooth transition functions. Since the inverse function theorem, implicit function theorem and submersion theorem are local results they have versions for smooth maps between Banach manifolds. The space of Hölder sections of a smooth fibre bundle E is a Banach manifold and the tangent space at a section is the space of Hölder sections of the bundle of fibre-wise tangents at the section.

We will also need to work with the is the $C^{k+1,\alpha}$ completion $\mathcal{D}_{k+1}$ of the group of diffeomorphisms on a compact manifold. $\mathcal{D}_{k+1}$ is generated by exponentiated $C^{k+1,\alpha}$ vector fields. It is a Banach manifold, with $T_{id}\mathcal{D}_{k+1} \cong C^{k+1,\alpha}(TM)$. The product on $\mathcal{D}_{k+1}$ is continuous, but not differentiable.

If E is a smooth fibre bundle on a compact manifold, then the action of $\mathcal{D}_{k+1}$ on $C^{k,\alpha}(E)$ is C^0 but not C^1. The linearisation of the map

$$\mathcal{D}_{k+1} \times C^{k,\alpha}(E) \to C^{k,\alpha}(E), \ \ (\phi, s) \mapsto \phi^* s$$

depends on the first derivatives of s; the map is therefore C^1 considered as a map into $C^{k-1,\alpha}(E)$, but not into $C^{k,\alpha}(E)$. However

Theorem 3.1.3 (cf. [15, p. 17]). *Let E be a smooth fibre bundle on a compact manifold. If s is a smooth section of E then*

$$\mathcal{D}_{k+1} \to C^{k,\alpha}(E), \ \ \phi \mapsto \phi^* s$$

is smooth.

3.1.2 Slices

The basic setup in this section will be as follows. M^n is a smooth compact manifold. $\mathcal{D}$ is the identity component of the group of diffeomorphisms, i.e. the diffeomorphisms isotopic to the identity. For concreteness we take $\mathcal{C}$ to be a space of G-structures on M, which we consider to be smooth sections of a smooth subbundle $\Lambda_G T^*M \subseteq T^*M$, and let $\mathcal{X} \subseteq \mathcal{C}$ be the closed subspace of torsion-free G-structures. $\mathcal{C}$ and $\mathcal{X}$ are invariant under the action of $\mathcal{D}$ by pull-backs. We wish to describe the moduli space $\mathcal{M} = \mathcal{X}/\mathcal{D}$.

Let $x \in \mathcal{X}$. The tangent space $T_x\mathcal{C}$ is a space of smooth sections of the vector bundle of point-wise tangents to $\Lambda_G T^*M$ at the section x. In principle we try to describe a neigh-

bourhood of $x\mathcal{D}$ in $\mathcal{M}$ in the following way: Identify the tangent space T of the $\mathcal{D}$-orbit at x, and find a direct complement K for T in $T_x\mathcal{C}$. We consider a small manifold $\mathcal{S} \subseteq \mathcal{C}$ with $T_x\mathcal{S} = K$ to be a *slice* for the $\mathcal{D}$-action at x. The *pre-moduli space* near x is $\mathcal{R} = \mathcal{X} \cap \mathcal{S}$. We then try to show that $\mathcal{R}$ contains a unique representative for each $\mathcal{D}$-orbit near x, and that the natural map $\mathcal{R} \to \mathcal{M}$ is a homeomorphism onto a neighbourhood of $x\mathcal{D}$. If so, the problem of describing $\mathcal{M}$ is reduced to describing $\mathcal{R}$.

In practice it is hard to implement the argument outlined above working with just spaces of smooth sections. The space of C^∞ sections of a vector bundle over a compact manifold is a Fréchet space with respect to the sequence of norms C^k. $\mathcal{C}$ can therefore be regarded as a Fréchet manifold. The Nash-Moser theorem (see Hamilton [21]) is a form of the inverse function theorem in the Fréchet setting, but the hypotheses are too restrictive for it to be useful to us. Moreover, it would be unwieldy to carry out the technical work of finding the slice working only in the smooth setting.

We work instead with spaces of $C^{k,\alpha}$ Hölder sections (for some $k \geq 1$, $\alpha \in (0,1)$). In the cases we consider the Riemannian metric used to define the $C^{k,\alpha}$-norm is defined naturally by the G-structure $x \in \mathcal{X}$, but it could involve some arbitrary choice.

So given $x \in \mathcal{X}$ let $\mathcal{C}_k$, $\mathcal{X}_k$ and T_k be the $C^{k,\alpha}$ completions of $\mathcal{C}$, $\mathcal{X}$ and T, respectively. $\mathcal{C}_k$ is a Banach manifold. Note that T_k is the tangent space to the $\mathcal{D}_{k+1}$ orbit at x, where $\mathcal{D}_{k+1}$ is the $C^{k+1,\alpha}$ completion of $\mathcal{D}$. Usually we take K_k to be the L^2-orthogonal subspace to T_k with respect to a metric and prove that $T_x\mathcal{C}_k = T_k \oplus K_k$ with projections bounded in the $C^{k,\alpha}$-norm.

If we let $\mathcal{S}_k$ be a neighbourhood of x in the affine space $x + K_k$, then the linearisation of

$$\mathcal{D}_{k+1} \times \mathcal{S}_k \to \mathcal{C}_k, \ \ (\phi, s) \mapsto \phi^* s$$

at (id, x) is surjective. It would be therefore be convenient to claim that the submersion theorem ensures that the image of $\mathcal{S}_k$ in $\mathcal{C}_k/\mathcal{D}_{k+1}$ is open. The map is, however, continuous but not differentiable near (id, x). Ebin [15] shows carefully, in the case when $\mathcal{C}_k$ is the space of Riemannian metrics, that the conclusion that $\mathcal{S}_k \to \mathcal{C}_k/\mathcal{D}_{k+1}$ is open does hold if one replaces $\mathcal{S}_k$ by another submanifold of $\mathcal{C}_k$ with $T_x\mathcal{S}_k = K_k$, defined as a natural exponentiation of K_k. Ebin then deduces that $\mathcal{S}$, the smooth elements of this $\mathcal{S}_k$, has open image in $\mathcal{C}/\mathcal{D}$. The argument carries over to any situation where the elements of $\mathcal{C}$ define Riemannian metrics.

Since the stabiliser $\mathcal{I}_x \subset \mathcal{D}$ of x is compact it is easy to arrange for the slice $\mathcal{S}_k$ at x to be invariant under $\mathcal{I}_x$. One key property of the slice is that if $\phi^*(\mathcal{S}_k) \cap \mathcal{S}_k \neq \emptyset$ for any

$\phi \in \mathcal{D}_{k+1}$ then $\phi \in \mathcal{I}_x$ (see [15, Theorem 7.1]). Therefore $\mathcal{S}_k/\mathcal{I}_x$ is in fact homeomorphic to an open set in $\mathcal{C}_k/\mathcal{D}_{k+1}$, and $\mathcal{S}/\mathcal{I}_x$ to an open set in $\mathcal{C}/\mathcal{D}$.

Given $x \in \mathcal{X}$ let $\mathcal{R} = \mathcal{S} \cap \mathcal{X}$. This is the pre-moduli space near x. It follows immediately that the image of $\mathcal{R}$ in $\mathcal{M}$ is open, and that $\mathcal{R}/\mathcal{I}_x$ is homeomorphic to a neighbourhood of $x\mathcal{D}$ in $\mathcal{M}$. If one can show that $\mathcal{R}$ is a manifold and that the action of $\mathcal{I}_x$ on $\mathcal{R}$ factors through a suitable finite group then one may be able to use $\mathcal{R}/\mathcal{I}_x$ as a coordinate chart for $\mathcal{M}$, and deduce that $\mathcal{M}$ is an orbifold.

3.1.3 Simplifications

For the deformation problems considered in this text there are some simplifying circumstances.

Firstly, the moduli spaces are in a sense 'unobstructed'. In many deformation problems the pre-moduli space $\mathcal{R}_k$ at some $x \in \mathcal{X}$ is defined as the zero-set in the slice $\mathcal{S}_k$ of some smooth function F. One can then try to apply the implicit function theorem to show that $\mathcal{R}_k$ is a manifold, with tangent space $\ker DF_x$ at x. $\ker DF_x$ can be interpreted as 'infinitesimal deformations' of x. The implicit function theorem requires the derivative DF_x to be surjective but in general there may be a cokernel. If there is a cokernel then there could be elements of $\ker DF_x$ which are not tangent to any path in $\mathcal{R}_k$. The cokernel of DF_x can therefore be interpreted as obstructions to the integrability of the infinitesimal deformations.

For the main deformation problems considered in this thesis we are able to pick the defining function F in such a way that DF_x is surjective, and thus show by direct application of the implicit function theorem that $\mathcal{R}_k$, as well as a neighbourhood of x in $\mathcal{X}_k$, is a manifold (a problem where this does not happen is the deformations of Ricci-flat metrics considered briefly in §5.2.2). Finding such F can involve some technical work but provides an essentially elementary way of showing that the moduli space is unobstructed.

Another simplification is that we will be able to choose the slice $\mathcal{S}$ so that the pre-moduli space $\mathcal{R}_k$ consists of smooth elements. In view of theorem 3.1.3 this allows us to prove a core slice result by a naive application of the submersion theorem, without the extra details in Ebin's setup.

Theorem 3.1.4. *Let $x \in \mathcal{X}$. Choose a Riemannian metric on M and $k \geq 1$.*

Suppose that $T_x\mathcal{C}_k = T_k \oplus K_k$ with T_k and K_k both closed and that $\mathcal{S}_k \subseteq \mathcal{C}_k$ is a submanifold with $T_x\mathcal{S}_k = K_k$. Suppose that x has a neighbourhood $\mathcal{U}_k$ in $\mathcal{X}_k$ that is a

submanifold of $\mathcal{C}_k$, and that $\mathcal{R}_k = \mathcal{U}_k \cap \mathcal{S}_k$ consists of smooth elements. If $\mathcal{S}_k$ is chosen small enough then

$$\mathcal{D}_{k+1} \times \mathcal{R}_k \to \mathcal{X}_k, \ \ (\phi, s) \mapsto \phi^* s \qquad (3.2)$$

is a submersion onto a neighbourhood of x in $\mathcal{U}_k$.

Proof. Note that $T_x \mathcal{U}_k$ must obviously contain T_k. Therefore $\mathcal{U}_k$ and $\mathcal{S}_k$ intersect transversely, so $\mathcal{R}_k$ is a manifold, and $T_x \mathcal{U}_k = T_x \mathcal{R}_k \oplus T_k$. Because the elements of $\mathcal{R}_k$ are smooth, theorem 3.1.3 ensures that (3.2) is a smooth map. The derivative of (3.2) at (x, id) is

$$C^{k+1, \alpha}(TM) \times T_x \mathcal{R}_k, \to T_x \mathcal{C}_k, \ \ (V, y) \mapsto y + \mathcal{L}_V x$$

which is surjective. The result follows from the submersion theorem for Banach spaces. $\square$

We now add the hypothesis that *isomorphisms of elements of $\mathcal{X}$ are smooth*, i.e. that if $x \in \mathcal{X}$ and $\phi^* x \in \mathcal{X}$ for a C^1 diffeomorphism ϕ of M then ϕ is C^∞. This implies in particular that the stabiliser $\mathcal{I}_x$ of $x \in \mathcal{X}$ in $\mathcal{D}_{k+1}$ consists of smooth maps, i.e. that $\mathcal{I}_x \subseteq \mathcal{D}$. The assumption is true if elements of $\mathcal{X}$ naturally define a smooth Riemannian metric.

Theorem 3.1.5 ([46, Theorem 8]). *Any isometry of smooth Riemannian manifolds is smooth.*

The regularity assumption also holds if elements of $\mathcal{X}$ define an almost complex structure, since the Cauchy-Riemann equations are over-determined elliptic.

Corollary 3.1.6. *Let $x \in \mathcal{X}$. If isomorphisms of elements of $\mathcal{X}$ are smooth then the natural map $\mathcal{R}_k \to \mathcal{M}$ is open.*

Proof. If $U \subseteq \mathcal{R}_k$ is open then it follows from theorem 3.1.4 that $U\mathcal{D}_{k+1} \subseteq \mathcal{X}_k$ is open. The assumption that isomorphisms of elements of $\mathcal{X}$ are smooth implies that $U\mathcal{D}_{k+1} \cap \mathcal{X} = U\mathcal{D}$. Hence $U\mathcal{D}$ is open in $\mathcal{X}$, i.e. the image of U in $\mathcal{M}$ is open. $\square$

Like in the previous subsection, if the pre-moduli space $\mathcal{R}_k$ is $\mathcal{I}_x$-invariant then $\mathcal{R}_k/\mathcal{I}_x$ is homeomorphic to a neighbourhood of x in $\mathcal{M}$.

Theorem 3.1.7. *Assume the hypotheses of theorem 3.1.4, and that each element of $\mathcal{X}_k$ naturally defines a Riemannian metric. If $\mathcal{R}_k$ is $\mathcal{I}_x$-invariant and sufficiently small then for any $\phi \in \mathcal{D}_{k+1}$*

$$\phi^*(\mathcal{R}_k) \cap \mathcal{R}_k \neq \emptyset \ \Rightarrow \ \phi \in \mathcal{I}_x.$$

Sketch proof. See Theorem 7.1(2) in Ebin [15] for details. The key point is to show that the orbit of x under $\mathcal{D}_k$, which is an immersed submanifold of $\mathcal{C}_k$ by theorem 3.1.3, is in fact embedded. This follows from a sequential compactness property: if g is a Riemannian metric on a compact manifold and ϕ_i is a sequence of diffeomorphisms such that $\phi_i^* g \to g$ then there is a subsequence which converges to an isometry of g. □

Remark 3.1.8. If the orbit of x is not embedded then the moduli space $\mathcal{M}$ is not Hausdorff.

If the action of $\mathcal{I}_x$ on $\mathcal{R}_k$ factors through a finite group then one can deduce that $\mathcal{M}$ is an orbifold (smooth local right inverses of (3.2) can be used to define the transition functions).

Another simplification in the problems considered is that $\mathcal{X}$ consists of closed forms. There is therefore a natural projection map $\pi_H : \mathcal{X} \to H$ to de Rham cohomology, which is invariant under the action of $\mathcal{D}$. In the problems considered it turns out that the restrictions $\pi_H : \mathcal{R}_k \to H$ are embeddings, so the elements of $\mathcal{R}_k$ represent distinct points in the moduli space. We will therefore not need to worry about ensuring that the slice at x is invariant under the stabiliser $\mathcal{I}_x$, or about how $\mathcal{I}_x$ acts on $\mathcal{R}_k$ (indeed, if the slice is $\mathcal{I}_x$-invariant then $\mathcal{I}_x$ acts trivially on $\mathcal{R}_k$).

Theorem 3.1.9. *Suppose that $\pi_H : \mathcal{X} \to H$ is a smooth $\mathcal{D}$-invariant map to a finite-dimensional vector space H. Suppose also for each $x \in \mathcal{X}$ that the hypotheses of theorem 3.1.4 hold, that $\mathcal{R}_k$ is a manifold consisting of smooth elements and that $\pi_H : \mathcal{R}_k \to H$ is an embedding if $\mathcal{R}_k$ is small enough.*

Then $\mathcal{M}$ has a unique smooth structure such that $\pi_H : \mathcal{M} \to H$ is an immersion.

Proof. Take $x \in \mathcal{X}$. By corollary 3.1.6 the continuous natural map $i : \mathcal{R}_k \to \mathcal{M}$ is open. Since $\pi_H : \mathcal{R}_k \to H$ is injective so is i. Therefore i is in fact a homeomorphism onto a neighbourhood U of $x\mathcal{D}$ in $\mathcal{M}$, and $\pi_H : U \to H$ is a homeomorphism onto an embedded submanifold.

Since the pre-moduli spaces $\mathcal{R}_k$ are manifolds the maps i can be taken as coordinate charts for $\mathcal{M}$, provided that they agree on overlaps. But on an overlap $U_1 \cap U_2$ both charts define the unique smooth structure that makes $\pi_H : U_1 \cap U_2 \to H$ into an embedding, so they agree.

Hence the maps $\mathcal{R}_k \to \mathcal{M}$ can be used to give $\mathcal{M}$ the structure of a smooth manifold, and it is immediate that $\pi_H : \mathcal{M} \to H$ is an immersion. □

Finally, it will be convenient to state the following result about the stabilisers of points in a $\mathcal{I}_x$-invariant slice at x.

Proposition 3.1.10. *Assume the hypotheses of theorem 3.1.9 and that each element of $\mathcal{X}$ naturally defines a Riemannian metric. Let $x \in \mathcal{X}$. If $\mathcal{R}$ is $\mathcal{I}_x$-invariant and small enough then $\mathcal{I}_x$ acts trivially on $\mathcal{R}$ and $\mathcal{I}_y = \mathcal{I}_x$ for all $y \in \mathcal{R}$.*

Proof. Since π_H is injective on $\mathcal{R}$ it is fixed by $\mathcal{I}_x$. The reverse inclusion $\mathcal{I}_y \subseteq \mathcal{I}_x$ follows from theorem 3.1.7. □

3.2 Deformations of compact G_2-manifolds

In this section we show that the moduli space of torsion-free G_2-structures on a compact manifold M^7 is smooth. The result was first published by Joyce [26]. The argument given here is based on Hitchin [24], although we provide some more details. The argument is adjusted to make it easier to generalise to the case when M is asymptotically cylindrical, which will be done in §4.2.

Throughout this section M^7 will denote a compact G_2-manifold. Let $\mathcal{X}$ be the set of torsion-free G_2-structures on M and $\mathcal{D}$ the group of diffeomorphisms of M isotopic to the identity. $\mathcal{D}$ acts on $\mathcal{X}$ by pull-backs, and the moduli space of torsion-free G_2-structures on M is the quotient $\mathcal{M} = \mathcal{X}/\mathcal{D}$. The main theorem of this section is

Theorem 3.2.1 ([27, Theorem 10.4.4]). *Let M be a compact G_2-manifold. Then $\mathcal{M}$ is a smooth manifold of dimension $b^3(M)$, and the natural projection*

$$\pi_H : \mathcal{M} \to H^3(M, \mathbb{R}), \quad \varphi\mathcal{D} \mapsto [\varphi] \tag{3.3}$$

is a local diffeomorphism.

Remark 3.2.2. By the theorem the tangent space of $\mathcal{M}$ at $\varphi\mathcal{D}$ is identified with $H^3(M)$. The torsion-free G_2-structure φ determines a decomposition (cf. (2.2))

$$H^3(M) = H^3_1(M) \oplus H^3_7(M) \oplus H^3_{27}(M).$$

This decomposition is independent of the choice of representative φ for the point $\varphi\mathcal{D} \in \mathcal{M}$ and thus gives a splitting of the tangent space of $\mathcal{M}$ at $\varphi\mathcal{D}$. The $H^3_1(M)$ summand (which has dimension 1 when M is connected) corresponds to rescaling the G_2-structure. The $H^3_7(M)$ summand is the tangent space to deformations of the G_2-structure that leave the metric unchanged. The relationship between the moduli space of G_2-structures and the metrics they define is elaborated on in §5.3.

By Poincaré duality $H^3(M) \cong (H^4(M))^*$ so $H^3(M) \oplus H^4(M)$ is a symplectic vector space in a natural way.

Proposition 3.2.3 ([27, Proposition 10.4.5])**.**

$$\mathcal{M} \to H^3(M) \oplus H^4(M), \;\; \varphi \mathcal{D} \mapsto ([\varphi], [*_\varphi \varphi]) \tag{3.4}$$

is a Lagrangian immersion.

This is easy to prove once the deformation theory has been set up. In §4.2 we will give a proof for an asymptotically cylindrical analogue.

3.2.1 Plan and notation

Here we outline the plan of the proof of theorem 3.2.1 and establish some notation. We use the slice methods explained in §3.1.

Let $\mathcal{Z}^3$ be the space of smooth closed 3-forms and $\mathcal{C}^3 \subseteq \mathcal{Z}^3$ the subset of closed positive 3-forms (which is open in the uniform topology). The torsion-free G_2-structures $\mathcal{X}$ form a closed subset of $\mathcal{C}$. Let $\mathcal{Z}^3_k, \mathcal{C}^3_k, \mathcal{X}_k$ be the corresponding spaces of $C^{k,\alpha}$-sections, where $k \geq 1$.

We fix a torsion-free G_2-structure φ on M. We show that $T_\varphi \mathcal{Z}^3_k$ splits as a direct sum of the tangent space T_k of the $\mathcal{D}_{k+1}$-orbit at φ and the L^2-orthogonal complement K_k. We take a small neighbourhood $\mathcal{S}_k$ of φ in the affine space $\varphi + K_k$ as our slice for the $\mathcal{D}_{k+1}$-action at φ.

We apply the implicit function theorem to an appropriate function to deduce that $\mathcal{R}_k = \mathcal{S}_k \cap \mathcal{X}_k$ is a manifold. Regularity arguments show that the elements of $\mathcal{R}_k$, which *a priori* merely have finite $C^{k,\alpha}$-norm, are smooth. Therefore $\mathcal{R}_k$ is the *pre-moduli space* $\mathcal{R}$ *near* φ described in §3.1.

The tangent space of $\mathcal{R}$ at φ is the space $\mathcal{H}^3$ of harmonic forms. By Hodge theory for compact manifolds the natural projection map to de Rham cohomology

$$\pi_H : \mathcal{H}^m \to H^m(M), \;\; \alpha \mapsto [\alpha] \tag{3.5}$$

is an isomorphism. Therefore, if $\mathcal{R}$ is taken sufficiently small then $\mathcal{R} \to H^3(M)$ is a diffeomorphism onto its image. Theorem 3.2.1 then follows by application of theorem 3.1.9.

Abbreviate $\Lambda^m T^* M$ to Λ^m and recall that this splits into subbundles corresponding to irreducible representations of G_2.

$$\begin{aligned}
\Lambda^2 &= \Lambda_7^2 \oplus \Lambda_{14}^2, \\
\Lambda^3 &= \Lambda_1^3 \oplus \Lambda_7^3 \oplus \Lambda_{27}^3,
\end{aligned} \tag{3.6}$$

where Λ_d^m is a subbundle of the exterior cotangent bundle Λ^m of rank d. Its sections $\Omega_d^m(M)$ are the 'type d m-forms'. These type decomposition, as well as covariant derivatives and Hodge stars, are defined with respect to the G_2-structure defined by φ, unless otherwise indicated by a subscript (e.g. $\Lambda_{7,\psi}^2$ denotes the rank 7 subrepresentation of Λ^2 defined by the G_2-structure ψ).

3.2.2   The Dirac operator

We will use Fredholm properties of the Dirac operator associated to a G_2-structure φ to obtain a direct sum decomposition for the tangent space to the space of G_2-structures (proposition 3.2.6).

Recall that because $G_2 \subset Spin(7)$ a G_2-structure on a manifold M^7 induces a spin structure, a spinor bundle S, and the Dirac operator

$$\eth : \Gamma(S) \to \Gamma(S).$$

The point-wise equivalence (2.8) implies that

$$S \cong \Lambda^0 \oplus \Lambda^1. \tag{3.7}$$

Moreover, we can identify the spin representation σ_7 with the octonions and the natural representation $\mathbb{R}^7$ of G_2 with the imaginary octonions in such a way that Clifford multiplication is identified with the octonion multiplication. Since φ is the 'multiplication table' for the imaginary octonions (2.5) we find that, under the isomorphism (3.7), Clifford multiplication by some $\alpha \in \Omega^1(M)$ corresponds to

$$\Omega^0(M) \oplus \Omega^1(M) \to \Omega^0(M) \oplus \Omega^1(M), \ \ (f, \beta) \mapsto (- <\alpha, \beta>, \ f\alpha + *(\alpha \wedge \beta \wedge *\varphi)).$$

Thus the Dirac operator is identified with

$$\Omega^0(M) \oplus \Omega^1(M) \to \Omega^0(M) \oplus \Omega^1(M), \ \ (f, \beta) \mapsto (d^*\beta, \ df + *(d\beta \wedge *\varphi)). \tag{3.8}$$

We can see directly from (3.8) that the Dirac Laplacian $\eth^2$ is identified with the Hodge Laplacian on $\Omega^0(M) \oplus \Omega^1(M)$. This follows also from remark 2.1.12.

3.2.3 The slice

$\mathcal{D}_{k+1}$ acts on $\mathcal{Z}_k^3$ by pull-backs. We want to identify the tangent space to the orbits of $\mathcal{D}_{k+1}$ and find an appropriate slice for the action at φ.

Proposition 3.2.4. *Let φ be a G_2-structure on a compact manifold M^7 such that $d\varphi = 0$. Then the tangent space to the $\mathcal{D}_{k+1}$-orbit at φ is $dC^{k+1,\alpha}(\Lambda_7^2)$.*

Proof. The tangent space to the $\mathcal{D}_{k+1}$-orbit at φ consists of the Lie derivatives of φ with respect to $C^{k+1,\alpha}$ vector fields V. Since φ is closed, $\mathcal{L}_V \varphi = d(V \lrcorner \varphi)$. The forms $V \lrcorner \varphi$ are precisely the sections of Λ_7^2. $\qquad\square$

We find that we can take $K_k = \mathcal{H}^3 \oplus W$ as the complement in $\mathcal{Z}_k^3$ of the tangent space to the orbit, where W consists of the exact 3-forms of type 27.

Definition 3.2.5. Let $W = dC^{k+1,\alpha}(\Lambda^2) \cap \Omega_{27}^3(M)$.

Proposition 3.2.6. *Let M^7 be a compact G_2-manifold with G_2-structure φ, and $k \geq 0$. Then $dC^{k+1,\alpha}(\Lambda^2)$ can be written as an L^2-orthogonal direct sum*

$$dC^{k+1,\alpha}(\Lambda^2) = dC^{k+1,\alpha}(\Lambda_7^2) \oplus W \tag{3.9}$$

and the projections are bounded in Hölder norm.

Proof. We can identify the spinor bundle S both with $\Lambda^0 \oplus \Lambda_7^2$ and with $\Lambda_{1\oplus7}^3$ (shorthand for $\Lambda_1^3 \oplus \Lambda_7^3$) so that the Dirac operator $\eth : \Gamma(S) \to \Gamma(S)$ is identified with

$$\Omega^0(M) \oplus \Omega_7^2(M) \to \Omega_{1\oplus7}^3(M), \ \ (f, \eta) \mapsto \pi_{1\oplus7}d\eta + *(df \wedge \varphi).$$

The Dirac operator is elliptic and therefore Fredholm on Hölder spaces of sections, and its kernel and cokernel consists of harmonic forms. If $\beta \in dC^{k+1,\alpha}(\Lambda^2)$ then $\pi_{1\oplus7}\beta$ is L^2-orthogonal to $\mathcal{H}^3$, so lies in the image of the Dirac operator map, i.e.

$$\pi_{1\oplus7}\beta = \pi_{1\oplus7}d\eta + *(df \wedge \varphi)$$

for some $\eta \in C^{k+1,\alpha}(\Lambda^2_7)$, $f \in C^{k+1,\alpha}(\Lambda^0)$. Integrating by parts

$$49\|df\|^2_{L^2} = \|{*}d(f\varphi)\|^2_{L^2} = <{*}d(f\varphi), \beta - d\eta>_{L^2} = <{*}f\varphi, d(\beta - d\eta)>_{L^2} = 0, \qquad (3.10)$$

so $df = 0$. Hence the exact form $\beta - d\eta$ has type 27, i.e. $\beta - d\eta \in W$. $\qquad\square$

So we can indeed take $K_k = \mathcal{H}^3 \oplus W$ as a direct complement to the tangent space to the diffeomorphism orbit and use a neighbourhood $\mathcal{S}_k$ of φ in the affine space $\varphi + K_k$ as a slice.

Remark 3.2.7. For $\beta \in \Omega^3_{27}(M)$ there is a linear relation between $\pi_7 d\beta$ and $\pi_7 d^*\beta$ (see [9, Proposition 3]). Therefore $\mathcal{H}^3 \oplus W$ is precisely the $C^{k,\alpha}$ kernel of the formal adjoint $\pi_7 d^* : \Omega^3(M) \to \Omega^2_7(M)$ of $d : \Omega^2_7(M) \to \Omega^3(M)$.

Definition 3.2.8. Let P_W be the L^2-orthogonal projection $C^{k,\alpha}(\Lambda^3) \to W$.

The orthogonal projection $P_E : C^{k,\alpha}(\Lambda^3) \to dC^{k+1,\alpha}(\Lambda^2)$ is bounded by Hodge decomposition so it follows from proposition 3.2.6 that P_W is bounded in Hölder norm.

3.2.4 The pre-moduli space

We want to define a function on $\mathcal{C}^3_k$ whose zeros near φ are precisely the 3-forms giving torsion-free G_2-structures, and define it in such a way that we can apply the implicit function theorem to it.

Definition 3.2.9. For $k \geq 1$ define $F : \mathcal{C}^3_k \to W$ by

$$F(\psi) = P_W({*}\Theta(\psi)). \qquad (3.11)$$

It follows from proposition 3.1.2 and the fact that P_W is a bounded linear map that F is a smooth function between (open subsets of) Banach spaces.

We wish to show that near φ the zeros of F are precisely the torsion-free G_2-structures. Recall that, according to theorem 2.2.10(i), $\psi \in \mathcal{C}^3$ defines a torsion-free G_2-structure if and only if $d\Theta(\psi) = 0$. Closure of $\Theta(\psi)$ is in turn equivalent to $P_E({*}\Theta(\psi)) = 0$, where $P_E : C^{k,\alpha}(\Lambda^3) \to dC^{k+1,\alpha}(\Lambda^2)$ is the orthogonal projection to the exact part in the Hodge decomposition

$$C^{k,\alpha}(\Lambda^3) = \mathcal{H}^3 \oplus dC^{k+1,\alpha}(\Lambda^2) \oplus d^*C^{k+1,\alpha}(\Lambda^4).$$

$W \subseteq dC^{k+1,\alpha}(\Lambda^2)$, so certainly $P_E(*\Theta(\psi)) = 0$ implies that $P_W(*\Theta(\psi)) = 0$. It remains to show the converse, so that we do not 'lose any information' by considering zeros of F instead of $\psi \mapsto d\Theta(\psi)$.

Proposition 3.2.10. *For $k \geq 1$ and $\psi \in \mathcal{C}_k^3$ sufficiently close to φ*

$$d\Theta(\psi) = 0 \iff F(\psi) = 0.$$

Proof. If $F(\psi) = 0$ then $P_E(*\Theta(\psi))$ is L^2-orthogonal to W. At the same time, theorem 2.2.10(ii) implies that for any G_2-structure ψ with $d\psi = 0$

$$d\psi = 0 \Rightarrow \pi_{7,\psi} d^{*\psi} \psi = 0.$$

It follows that $d^*(*\Theta(\psi))$ is point-wise orthogonal to $\Lambda_{7,\psi}^2$, so that, integrating by parts, $P_E(*\Theta(\psi))$ is L^2-orthogonal to the tangent space $T_\psi = dC^{1,\alpha}(\Lambda_{7,\psi}^2)$ to the $\mathcal{D}_1$-orbit at ψ. Now the linear map

$$W \oplus C^{1,\alpha}(TM) \to dC^{1,\alpha}(\Lambda^2), \quad (w, V) \mapsto w + d(V \lrcorner \psi)$$

is surjective at $\psi = \varphi$ by proposition 3.2.6. Since it depends continuously on ψ,

$$dC^{1,\alpha}(\Lambda_7^2) = W + T_\psi$$

for any ψ sufficiently close to φ. Now the fact that $P_E(*\Theta(\psi))$ L^2-orthogonal to both W and T_ψ implies $P_E(*\Theta(\psi)) = 0$. $\qquad\square$

Remark 3.2.11. The argument above is a somewhat drier version of Hitchin [24]. Hitchin interprets $P_E(*\Theta(\psi))$ as the derivative of the volume functional (i.e. the total volume of M with respect to the volume form defined by ψ) restricted to the cohomology class of ψ. The fact that the volume functional is $\mathcal{D}_{k+1}$-invariant implies that $P_E(*\Theta(\psi))$ is L^2-orthogonal to $dC^{k,\alpha}(\Lambda_{7,\psi}^2)$

In the case when M is asymptotically cylindrical rather than compact the total volume of a G_2-structure and L^2-inner product of asymptotically translation-invariant forms do not converge, but the proof of proposition 3.2.10 works with only notational changes.

Now we compute the derivate of $F : \mathcal{C}_k^3 \to W$. $\mathcal{C}_k^3$ is an open subset of $\mathcal{Z}_k^3$, so the tangent space at φ is $\mathcal{Z}_k^3 = \mathcal{H}^3 \oplus dC^{k+1,\alpha}(\Lambda_7^2) \oplus W$.

Proposition 3.2.12. *For $k \geq 1$ the derivative $DF_\varphi : \mathcal{H}^3 \oplus dC^{k+1,\alpha}(\Lambda_7^2) \oplus W \to W$ is 0 on $\mathcal{H}^3 \oplus dC^{k+1,\alpha}(\Lambda_7^2)$ and $-id$ on W.*

Proof. Since all the forms in the $\mathcal{D}_{k+1}$-orbit of φ define torsion-free G_2-structures F is always 0 on the orbit. Therefore DF_φ must be 0 on the tangent space to the orbit, which is $dC^{k+1,\alpha}(\Lambda_7^2)$ by 3.2.4.

Using the expression (2.9) and the chain rule (3.1) to obtain the derivative of the non-linear map $\Theta : \mathcal{C}_k^3 \to C^{k,\alpha}(\Lambda^4)$ we find

$$DF_\varphi = P_W \circ (\tfrac{4}{3}\pi_1 + \pi_7 - \pi_{27}).$$

This is 0 on $\mathcal{H}^3$ since type components of harmonic forms are harmonic and therefore have no exact part. On W

$$DF_\varphi = P_W \circ (\tfrac{4}{3}\pi_1 + \pi_7 - \pi_{27}) = P_W \circ (-id) = -id. \qquad \square$$

We can now apply the implicit function theorem to F to deduce that its zero set near φ in $\mathcal{S}_k$ is a manifold with tangent space $\mathcal{H}^3$ at φ. Hence if $\mathcal{R}_k$ is a sufficiently small neighbourhood of φ in $\mathcal{S}_k \cap \mathcal{X}_k$ then $\mathcal{R}_k$ is a manifold and $\pi_H : \mathcal{R}_k \to H^3(M)$ is a diffeomorphism onto its image. The implicit function theorem shows also that φ has a neighbourhood in $\mathcal{X}_k$ that is a submanifold of $\mathcal{C}_k$. In order to deduce theorem 3.2.1 from theorem 3.1.9 it remains only to show that $\mathcal{R}_k$ consists of smooth forms.

3.2.5 Regularity

We prove that elements of $\mathcal{R}_k$ are smooth by a boot-strapping method for non-linear PDEs (for a similar solution to a similar problem see [27, p. 303]).

Proposition 3.2.13. *Let $k \geq 1$. If $\mathcal{R}_k$ is taken sufficiently small then its elements are smooth.*

Proof. If $\psi = \varphi + \beta$ with $\beta \in K_k$ then we can write $d{*}d\Theta(\psi)$ as

$$d{*}d\Theta(\psi) = -dd^*(\tfrac{4}{3}\pi_1\beta + \pi_7\beta - \pi_{27}\beta) + P(\beta, \nabla\beta, \nabla^2\beta) + R(\beta, \nabla\beta),$$

where $-dd^*(\tfrac{4}{3}\pi_1 + \pi_7 - \pi_{27})$ is the linearisation at φ, P consists of the quadratic terms that involve the second derivative $\nabla^2\beta$ and R consists of the remaining quadratic terms. P and R depend only point-wise on their arguments, and furthermore P is linear in $\nabla^2\beta$.

Now note that $-dd^* \circ (\frac{4}{3}\pi_1 + \pi_7 - \pi_{27}) = \triangle$ on $\mathcal{H}^3 \oplus W$: on $\mathcal{H}^3$ both vanish, while W consists of closed sections of Λ^3_{27}. If ψ is a zero of F near φ in $(\mathcal{H}^3 \oplus W) \cap \mathcal{C}^3_k$ then $d\Theta(\psi) = 0$, so

$$\triangle\beta + P(\beta, \nabla\beta, \nabla^2\beta) = -R(\beta, \nabla\beta). \tag{3.12}$$

Considering β and $\nabla\beta$ to be fixed we can define a second-order linear differential operator $A : \zeta \mapsto P(\beta, \nabla\beta, \nabla^2\zeta)$. Then (3.12) can be rephrased as saying that $\zeta = \beta$ is a solution of the linear second-order PDE

$$(\triangle + A)\zeta = -R. \tag{3.13}$$

If $\psi = \varphi$ then $\triangle + A = \triangle$, which is elliptic. Ellipticity is open condition so if we take $\mathcal{R}_k$ sufficiently small then $\triangle + A$ is elliptic for all $\psi \in \mathcal{R}_k$.

Now suppose that β is in the Hölder space $C^{l+1,\alpha}$ for some integer $l \geq 1$. The coefficients of $\triangle + A$ depend algebraically on β and $\nabla\beta$, so have regularity $C^{l,\alpha}$. The same goes for the RHS of (3.13). Therefore by standard regularity results about linear elliptic operators on compact manifolds (e.g. Theorem 40 in the appendix of [5]) β is $C^{l+2,\alpha}$.

By induction the elements of $\mathcal{R}_k$ are $C^{l,\alpha}$ for all l, so they are smooth. $\qquad\square$

This concludes the proof of theorem 3.2.1.

3.3 Deformations of compact $Spin(7)$-manifolds

In this section we show that the moduli space of torsion-free $Spin(7)$-structures on a compact $Spin(7)$-manifold M^8 is smooth. This was first proved by Joyce [25] and the proof here is essentially the same. We use the same slice arguments as for the deformations of compact G_2-manifolds, which will carry over without too much trouble to the asymptotically cylindrical case in §4. We will describe in detail only how to set up the slice construction.

Throughout this section M^8 will denote a compact $Spin(7)$-manifold. Let $\mathcal{X}$ be the set of torsion-free $Spin(7)$-structures on M and $\mathcal{D}$ the group of diffeomorphisms of M isotopic to the identity. $\mathcal{D}$ acts on $\mathcal{X}$ by pull-backs, and the moduli space of torsion-free $Spin(7)$-structures on M is the quotient $\mathcal{M} = \mathcal{X}/\mathcal{D}$. The main theorem of this section is

Theorem 3.3.1. *Let M^8 be a compact $Spin(7)$-manifold. Then $\mathcal{M}$ is a smooth manifold of dimension*

$$\dim \mathcal{M} = b^4_-(M) + b^1(M) + \hat{A}(M), \tag{3.14}$$

and the natural projection

$$\pi_H : \mathcal{M} \to H^4(M, \mathbb{R}), \ \ \psi\mathcal{D} \mapsto [\psi] \tag{3.15}$$

is an immersion.

Remark 3.3.2. As explained in (2.16), the term $b^1(M) + \hat{A}(M)$ is just a way of writing the dimension of the space of positive harmonic spinors on the $Spin(7)$-manifold in terms of topological invariants. In §5.3 we will see that this corresponds to deformations of the $Spin(7)$-structure that rescale the metric or leave it unchanged.

3.3.1 The Dirac operator on $Spin(7)$-manifolds

Let ψ be a fixed torsion-free $Spin(7)$-structure. Recall from §2.2.3 that we consider $Spin(7)$ to act irreducibly on $\mathbb{R}^8$ and that (considered a subgroup of $Spin(8)$) its spin representations are $\sigma_8^- \cong \mathbb{R}^8$ and $\sigma_8^+ \cong \mathbb{R} \oplus \mathbb{R}^7$ (where $\mathbb{R}^7$ is the irreducible vector representation of $Spin(7)$, factoring through $SO(7)$). Thus the negative spinor bundle $S^- \cong TM$.

We abbreviate $\Lambda^m T^*M$ to Λ^m. As explained in §2.1.2 the exterior forms decompose into subbundles corresponding to irreducible representations of $Spin(7)$.

$$
\begin{aligned}
\Lambda_+^4 &= \Lambda_1^4 \oplus \Lambda_7^4 \oplus \Lambda_{27}^4, \\
\Lambda_-^4 &= \Lambda_{35}^4, \\
\Lambda^3 &= \Lambda_8^3 \oplus \Lambda_{48}^3, \\
\Lambda^2 &= \Lambda_7^2 \oplus \Lambda_{21}^3.
\end{aligned}
\tag{3.16}
$$

We can identify subbundles isomorphic to the spinor bundles

$$S_+ \cong \Lambda_{1\oplus7}^4, \ \ S_- \cong \Lambda_8^3. \tag{3.17}$$

It is therefore natural to compare the Dirac operator $\eth_- : \Gamma(S_-) \to \Gamma(S_+)$ with

$$\pi_{1\oplus7} d : \Omega_8^3(M) \to \Omega_{1\oplus7}^4(M), \tag{3.18}$$

and if the isomorphisms (3.17) are chosen appropriately then they are in fact equal (cf. [27, p. 260]). The easiest way to check this is to note that the values of both $\eth_-$ and $\pi_{1\oplus7} d$ on $V \in \Gamma(TM) \cong \Gamma(S^-) \cong \Omega_8^3(M)$ depend algebraically on $\nabla V \in \Gamma(End(TM))$ and that as

a $Spin(7)$-representation $End(\mathbb{R}^8)$ contains a unique irreducible component of rank 1 and 7 respectively.

3.3.2 The pre-moduli space

Let M^8 be a compact $Spin(7)$-manifold and fix a torsion-free $Spin(7)$-structure ψ on M. We explain how to choose a slice for the diffeomorphism action at ψ and identify a submanifold that we use as the pre-moduli space of torsion-free $Spin(7)$-structure near ψ. Type decompositions of forms, covariant derivatives etc. will be defined with respect to the $Spin(7)$-structure ψ.

Let $\mathcal{C}^4$ be the space of smooth $Spin(7)$-structures on M. Let $\mathcal{C}_k^4$ be its $C^{k,\alpha}$ completion for some $k \geq 1$, $\alpha \in (0,1)$, and let $\mathcal{D}_{k+1}$ be the $C^{k+1,\alpha}$ completion of the group of diffeomorphisms isotopic to the identity. The tangent space to the $\mathcal{D}_{k+1}$-orbit at ψ is given by the Lie derivatives of ψ. Since ψ is closed, $\mathcal{L}_V \psi = d(V \lrcorner \psi)$ for any vector field V, and the forms $V \lrcorner \psi$ are precisely the sections of Λ_8^3. Thus

$$T_\psi(\psi \mathcal{D}_{k+1}) = dC^{k+1,\alpha}(\Lambda_8^3).$$

Recall from §2.2.3 that the tangent space to the space of $Spin(7)$-structures consists of 4-forms of type 1, 7 and 35, i.e.

$$T_\psi \mathcal{C}_k^4 = C^{k,\alpha}(\Lambda_{1\oplus7\oplus35}^4).$$

We use the interpretation of the Dirac operator above to identify the L^2-orthogonal complement of the tangent space to the orbit. Let $\mathcal{H}_{1\oplus7\oplus35}^4$ denote the harmonic forms of type 1, 7 and 35, and let W be the L^2-orthogonal complement of $\mathcal{H}_{35}^4$ in $C^{k,\alpha}(\Lambda_{35}^4)$.

Proposition 3.3.3. *Let M be a compact $Spin(7)$-manifold with $Spin(7)$-structure ψ. Then*

$$C^{k,\alpha}(\Lambda_{1\oplus7\oplus35}^4) = \mathcal{H}_{1\oplus7\oplus35}^4 \oplus dC^{k+1,\alpha}(\Lambda_8^3) \oplus W. \tag{3.19}$$

Proof. Suppose $\beta \in C^{k,\alpha}(\Lambda_{1\oplus7\oplus35}^4)$ is L^2-orthogonal to $\mathcal{H}_{1\oplus7\oplus35}^4$. Then $\pi_{1\oplus7}\beta$ is L^2-orthogonal to $\mathcal{H}_{1\oplus7}^4$, so it lies in the image of the Dirac operator map (3.18), i.e. $\beta = \pi_{1\oplus7}d\eta$ for some $\eta \in C^{k+1,\alpha}(\Lambda_8^4)$. Then $\beta - d\eta \in W$. $\qquad\square$

This means that we may use a small neighbourhood $\mathcal{S}$ of ψ in $\mathcal{H}_{1\oplus7\oplus35}^4 \oplus W$ as a slice for the diffeomorphism action at ψ. Let $\mathcal{X}_k \subset \mathcal{C}_k^4$ be the set of torsion-free $C^{k,\alpha}$ $Spin(7)$-

structures, and $\mathcal{R}_k = \mathcal{X}_k \cap \mathcal{S}$. This is the pre-moduli space of torsion-free $Spin(7)$-structures near ψ.

Proposition 3.3.4 (cf. [27, Proposition 10.7.3])**.** *ψ has a neighbourhood in $\mathcal{X}_k$ that is a submanifold of $\mathcal{C}_k^4$. If $\mathcal{S}$ is taken small enough then $\mathcal{R}_k \subseteq \mathcal{S}$ is a submanifold, with*

$$T_\psi \mathcal{R}_k = \mathcal{H}^4_{1 \oplus 7 \oplus 35}.$$

Proof. By theorem 2.2.17, $\mathcal{R}_k$ is the zero set of

$$d : \mathcal{S} \to dC^{k+1,\alpha}(\Lambda^4). \tag{3.20}$$

The RHS is the space of exact Hölder 5-forms, which is a closed subspace of $C'^{k,\alpha}(\Lambda^5)$ by Hodge decomposition. Any element can be written as $d\eta$ for some unique coexact $\eta \in C^{k+1,\alpha}(\Lambda^4)$. Then $\eta - *\eta$ is anti-self-dual and L^2-orthogonal to the harmonic forms, i.e. $\eta - *\eta \in W$. Since $W \subseteq T_\psi \mathcal{S}$ it follows that (3.20) has surjective derivative at ψ. If $\mathcal{S}$ is taken to be small then $\mathcal{R}_k$ is a submanifold by the implicit function theorem.

On the other hand any closed anti-self-dual form is harmonic, so the space of closed forms in $T_\psi \mathcal{S}$ is exactly $\mathcal{H}^4_{1 \oplus 7 \oplus 35}$.

Applying the implicit function theorem to $d : \mathcal{C}_k^4 \to dC^{k+1,\alpha}(\Lambda^4)$ shows that a small neighbourhood of ψ in $\mathcal{X}_k$ is a manifold too. $\qquad\square$

In particular the projection to de Rham cohomology $\pi_H : \mathcal{R}_k \to H^4(M)$ is an embedding. Analogously to §3.2.5 we can show that $\mathcal{R}_k$ consists of smooth elements, and then apply theorem 3.1.9 to deduce that the pre-moduli spaces can be used as coordinate charts to give the moduli space $\mathcal{M}$ a smooth structure.

It remains only to compute the dimension of the moduli space. This is just the dimension of the tangent space $\mathcal{H}^4_{1 \oplus 7 \oplus 35}$ to the pre-moduli space. $\dim \mathcal{H}^4_{35} = b^4_-(M)$, while the discussion in §3.3.1 identifies $\mathcal{H}^4_{1 \oplus 7}$ with the kernel of the Dirac operator $\eth_+ : \Gamma(S_+) \to \Gamma(S_-)$. Therefore (2.16) yields

$$\dim \mathcal{H}^4_{1 \oplus 7} = \hat{A}(M) + b^1(M).$$

This completes the proof of theorem 3.3.1.

3.4 Deformations of compact Calabi-Yau 3-folds

In this section we develop some deformation theory for compact Calabi-Yau 3-folds, which we need because the cross-sections of asymptotically cylindrical G_2-manifolds are Calabi-Yau 3-folds. It is a well-known result, independently due to Tian [52] and Todorov [53], that the moduli space of Calabi-Yau structures on a compact irreducible Calabi-Yau n-fold is smooth for $n \geq 3$. Tian and Todorov's arguments are based on Kodaira's deformation theory for complex structures, but in the case $n = 3$ it is straight-forward to give a simple proof based on the method of Hitchin [24], which we also employed in §3.2 to deal with deformations of compact G_2-manifolds. This approach is in terms of differential forms and exhibits pre-moduli spaces that provide useful coordinate charts. Such pre-moduli spaces have also been obtained by Goto [17].

Let X^6 be a compact manifold, and $\mathcal{Y}$ the set of Calabi-Yau structures (Ω, ω) on X in the sense of definition 2.2.20. Let $\mathcal{D}$ be the group of smooth diffeomorphisms isotopic to the identity. The moduli space of Calabi-Yau structures on X is $\mathcal{N} = \mathcal{Y}/\mathcal{D}$, and there is a natural projection to the de Rham cohomology

$$\pi_{\mathcal{N}} : \mathcal{N} \to H^3(X) \times H^2(X), \ \ (\Omega, \omega)\mathcal{D} \mapsto ([\Omega], [\omega]). \tag{3.21}$$

Theorem 3.4.1. *Let X^6 be a compact connected Calabi-Yau 3-fold. The moduli space $\mathcal{N}$ of Calabi-Yau structures on X is a manifold of dimension*

$$\dim \mathcal{N} = b^3(X) + b^2(X) - b^1(X) - 1, \tag{3.22}$$

and $\pi_{\mathcal{N}} : \mathcal{N} \to H^3(X) \times H^2(X)$ is an immersion.

Remark 3.4.2. The definition of a Calabi-Yau 3-fold X^6 used here allows $Hol(X)$ to be a proper subgroup of $SU(3)$. If $Hol(X)$ is exactly $SU(3)$ (so X is irreducible as a Riemannian manifold) then $b^1(X) = 0$, and the formula for the dimension simplifies to $b^3(X) + b^2(X) - 1$.

If X is an irreducible Calabi-Yau manifold then for any Calabi-Yau structure (Ω, ω) on X and $\lambda \in \mathbb{R}^+$ we can define a torsion-free product G_2-structure $\varphi = \Omega + \lambda d\theta \wedge \omega$ on $X \times S^1$. The metric defined by φ is the product of the Calabi-Yau metric on X and the metric on S^1 with radius λ (cf. proposition 2.2.24). The moduli space of such torsion-free product G_2-structures has dimension

$$\dim \mathcal{N} + 1 = b^3(X) + b^2(X) = b^3(X \times S^1),$$

which equals the dimension of the moduli space of torsion-free G_2-structures on $X \times S^1$ by theorem 3.2.1.

3.4.1 Comparison with known results

Theorem 3.4.1 agrees with the results of Tian [52] and Todorov [53]. The deformation theory for complex manifolds of Kodaira-Spencer shows that the moduli space of complex structures on X is a manifold provided that a certain 'obstruction space' vanishes. In that case the moduli space is locally diffeomorphic to $H^1(\mathcal{T})$, the first cohomology of the sheaf of holomorphic sections of the tangent bundle.

Tian and Todorov show that when X is a closed connected Calabi-Yau manifold of complex dimension $n \geq 3$ (with holonomy exactly $SU(n)$) then the deformations of the complex structures are unobstructed. For a compact Calabi-Yau manifold $H^1(\mathcal{T}) \cong (H^{n-1}(\Omega^1))^*$ by Serre duality, where Ω^1 is the sheaf of holomorphic 1-forms. $H^{n-1}(\Omega^1) \cong H^{1,n-1}(X)$ by the Dolbeault theorem, so $\dim_{\mathbb{C}} H^1(\mathcal{T}) = h^{1,n-1}(X)$.

On the other hand, Yau's solution to the Calabi conjecture shows that for a fixed complex structure there is a unique Calabi-Yau metric ω in each Kähler class, and the Kähler classes form an open subset of $H^{1,1}_{\mathbb{R}}(X)$. Our definition of a Calabi-Yau structure also involves a choice of normalised holomorphic $(n,0)$-form. This is determined up to phase by the complex structure and the metric. Hence the dimension of the moduli space of Calabi-Yau structures for a fixed complex structure is $h^{1,1}(X) + 1$.

A result of Kodaira [32, Theorem 15] shows that given a family of deformations of the complex structure on a Kähler manifold there is a corresponding family of deformations of the metric which are Kähler with respect to the deformed complex structures. This can be used together with the above to deduce that the moduli space of Calabi-Yau structures on a complex n-fold is a manifold of dimension

$$2h^{1,n-1}(X) + h^{1,1}(X) + 1 \tag{3.23}$$

(cf. [27, §6.8]). On a compact Calabi-Yau 3-fold X Serre duality gives $h^{0,2}(X) = h^{0,1}(X)$, and the Hodge decomposition of $H^*(X)$ that

$$b^3(X) = 2h^{1,2}(X) + 2,$$
$$b^2(X) = h^{1,1}(X) + 2h^{0,2}(X) = h^{1,1}(X) + b^1(X).$$

Hence for $n = 3$ the formula (3.23) for the dimension of the moduli space agrees with that stated in theorem 3.4.1.

More recently, Goto [17] gave a proof of the smoothness of the moduli space of Calabi-Yau structures that does not rely on Kodaira-Spencer theory. Theorem 3.4.1 is a special case of [17, Theorems 4.2.5 and 4.2.6] for complex dimension $n = 3$. Goto studies the problem in terms of differential forms and finds pre-moduli spaces of the form that we will need later. By specialising to the case $n = 3$ and employing the slice methods from §3.1 we obtain a simpler proof.

Remark 3.4.3. Hitchin [24] uses elementary methods to construct the moduli space of torsion-free $SL(\mathbb{C}^3)$-structures on a compact manifold X^6. In [47, §4] this is extended to another alternative proof of theorem 3.4.1. Hitchin's result implies that the Hölder completion of the space $\mathcal{Y}_1$ of torsion-free $SL(\mathbb{C}^3)$-structures is a manifold. If we let $\mathcal{Z}^2$ be the space of closed Hölder 2-forms then an implicit function theorem argument can be set up to prove essentially that the set of $(\Omega, \omega) \in \mathcal{Y}_1 \times \mathcal{Z}^2$ satisfying the point-wise algebraic conditions $\frac{1}{4}\Omega \wedge \hat{\Omega} = \frac{1}{6}\omega^3$, $\Omega \wedge \omega = 0$ is a manifold. Together with the open condition that ω defines a positive-definite metric this ensures that (Ω, ω) is an $SU(3)$-structure. Theorem 3.4.1 then follows by slice arguments like those described in section 3.1.

3.4.2 The slice

Let X^6 be a compact Calabi-Yau 3-fold, and $\mathcal{Y}$ the space of Calabi-Yau structures on X. To prove theorem 3.4.1 we first identify a slice for the action of the diffeomorphism group at each $(\Omega, \omega) \in \mathcal{Y}$.

Recall from §2.2.6 that we define an $SU(3)$-structure on X in terms of a pair of differential forms (Ω, ω), which is a section of a subbundle $\Lambda_{SU(3)}T^*X \subset \Lambda^3 T^*X \oplus \Lambda^2 T^*X$. An $SU(3)$-structure (Ω, ω) defines a metric, and it is a Calabi-Yau structure if and only if it is parallel with respect to the Levi-Civita connection.

We fix a Calabi-Yau structure (Ω, ω) on X and study a neighbourhood of (Ω, ω) in the moduli space. Abbreviate $\Lambda^m T^*X$ to Λ^m. The Calabi-Yau structure determines a type decomposition of Λ^m modelled on (2.21).

$$
\begin{aligned}
\Lambda^2 &= \Lambda_1^2 \oplus \Lambda_6^2 \oplus \Lambda_8^2, \\
\Lambda^3 &= \Lambda_{1 \oplus 1}^3 \oplus \Lambda_6^3 \oplus \Lambda_{12}^3.
\end{aligned}
\tag{3.24}
$$

If $(\sigma, \tau) \in \Lambda^3 \oplus \Lambda^2$ is tangent at (Ω, ω) to the space of $SU(3)$-structures on X then (2.18)

yields a relation between the type 1 and 6 components of σ and τ.

$$L_1(\sigma,\tau) = \sigma \wedge \hat{\Omega} - \tau \wedge \omega^2 = 0, \tag{3.25a}$$

$$L_2(\sigma,\tau) = \sigma \wedge \omega + \Omega \wedge \tau = 0. \tag{3.25b}$$

Pick $k \geq 1$ and $\alpha \in (0,1)$, and let

$$\mathcal{C}_k = \{(\Omega,\omega) \in C^{k,\alpha}(\Lambda_{SU(3)}) : d\omega = 0\}.$$

Then $\mathcal{C}_k$ is a Banach manifold, and $\mathcal{Y}$ embeds continuously in $\mathcal{C}_k$. The tangent space at (Ω,ω) is

$$T_{(\Omega,\omega)}\mathcal{C}_k = \{(\sigma,\tau) \in C^{k,\alpha}(\Lambda^{3\oplus 2}) \ : \ L_1(\sigma,\tau) = L_2(\sigma,\tau) = 0, \ d\tau = 0\}.$$

Let $\mathcal{D}_{k+1}$ be the group of $C^{k+1,\alpha}$ diffeomorphisms of X isotopic to the identity. The tangent space to the $\mathcal{D}_{k+1}$-orbit at (Ω,ω) is given by Lie derivatives

$$T_k = \{(d(V \lrcorner \Omega), d(V \lrcorner \omega)) : V \in C^{k+1,\alpha}(TM)\}.$$

To find a direct complement for T_k in $T_{(\Omega,\omega)}\mathcal{C}_k$ we make use of the Dirac operator $\eth : S^- \to S^+$. Considered as real vector bundles $S^\pm$ are both isomorphic to $\mathbb{R}^2 \oplus TX$. We can choose isomorphisms $S^+ \cong \Lambda^3_{1\oplus 1\oplus 6}$ and $S^- \cong \Lambda^2_6 \oplus \Lambda^0 \oplus \Lambda^0$ such that $\eth$ is identified with

$$\Omega^2_6(X) \oplus \Omega^0(X) \oplus \Omega^0(X) \to \Omega^3_{1\oplus 1\oplus 6}(X), \quad (\beta,f,g) \mapsto \pi_{1\oplus 1\oplus 6}d\beta + d(f\omega) + *d(g\omega). \tag{3.26}$$

To see that this is possible, observe that $\eth$ and (3.26) are both obtained algebraically from a covariant derivative taking values in $T^*X \otimes (T^*X \oplus \mathbb{R}^2)$. The vector bundle T^*X is associated to the $SU(3)$-structure by the standard representation $\mathbb{C}^3$ of $SU(3)$. Considering $\mathbb{C}^3$ as a real representation, $\mathbb{C}^3 \otimes \mathbb{C}^3$ contains a unique irreducible component isomorphic to $\mathbb{C}^3$. We can therefore make the identifications so that the $\Omega^2_6(X) \to \Omega^3_6(X)$ parts of $\eth$ and (3.26) agree. It is then easy to ensure that the remaining components agree too.

Definition 3.4.4. Let $W \subseteq C^{k,\alpha}(\Lambda^3_{12})$ be the L^2-orthogonal subspace to the harmonic forms. Let $W_2 \subset W$ be the subspace of exact forms, and $W_1 \subset W$ its L^2-orthogonal complement.

Lemma 3.4.5.

$$C^{k,\alpha}(\Lambda^3) = \mathcal{H}^3 \oplus dC^{k+1,\alpha}(\Lambda_6^2) \oplus dC^{k+1,\alpha}(\Lambda_1^2) \oplus *dC^{k+1,\alpha}(\Lambda_1^2) \oplus W, \tag{3.27a}$$

$$dC^{k+1,\alpha}(\Lambda^2) = dC^{k+1,\alpha}(\Lambda_6^2) \oplus dC^{k+1,\alpha}(\Lambda_1^2) \oplus W_2. \tag{3.27b}$$

Proof. The Dirac operator is elliptic, so the image of $\eth : C^{k+1,\alpha}(S^-) \to C^{k,\alpha}(S^+)$ is the L^2-orthogonal complement to the kernel in $C^{k,\alpha}(S^+)$. The isomorphism $S^+ \cong \Lambda^3_{1\oplus1\oplus6}$ identifies the kernel of the Dirac operator with the harmonic forms in $\Omega^3_{1\oplus1\oplus6}$ (cf. remark 2.1.12). Therefore, if $\gamma \in C^{k,\alpha}(\Lambda^3)$ is L^2-orthogonal to $\mathcal{H}^3$ then $\pi_{1\oplus1\oplus6}\gamma$ is in the image of (3.26), i.e.

$$\pi_{1\oplus1\oplus6}\gamma = \pi_{1\oplus1\oplus6}d\beta + d(f\omega) + *d(g\omega)$$

for some $\beta \in C^{k+1,\alpha}(\Lambda_6^2)$, $f, g \in C^{k+1,\alpha}(\Lambda^0)$. Then $\gamma - d\beta - d(f\omega) - *d(g\omega) \in W$. If γ is exact then integrating by parts shows that $dg = 0$. $\qquad\square$

The tangent space to the pre-moduli space of Calabi-Yau structures will turn out to be the space of harmonic tangents to the space of $SU(3)$-structures.

Definition 3.4.6. Let

$$\mathcal{H}_{SU} = \{(\sigma, \tau) \in \mathcal{H}^3 \times \mathcal{H}^2 : L_1(\sigma, \tau) = L_2(\sigma, \tau) = 0\}.$$

The map $\mathcal{H}^3 \times \mathcal{H}^2 \to \mathcal{H}^6 \times \mathcal{H}^5$, $(\sigma, \tau) \mapsto (L_1, L_2)$ is surjective, so

$$\dim \mathcal{H}_{SU} = b^3(X) + b^2(X) - b^1(X) - b^0(X). \tag{3.28}$$

Proposition 3.4.7.

$$T_{(\Omega,\omega)}\mathcal{C}_k = \mathcal{H}_{SU} \oplus T_k \oplus (W \times \{0\}).$$

Proof. Suppose $(\sigma, \tau) \in T_{(\Omega,\omega)}\mathcal{C}_k$. We wish to show that (σ, τ) lies in the RHS. By subtracting an element of $\mathcal{H}_{SU}$ we may assume that τ is exact and that σ is L^2-orthogonal to $\mathcal{H}^3$. Using (3.27a) and the fact that $dC^{k+1,\alpha}(\Lambda_6^2)$ is the space of Lie derivatives of Ω, we can also subtract an element of W from σ to ensure that

$$\sigma = d(f\omega) + *d(g\omega)$$

for some $f, g \in C^{k+1,\alpha}(\Lambda^0)$. The conditions (3.25) determine the Λ_1^2 and Λ_6^2 components of

τ from σ, giving

$$\tau = d^*(f\Omega + g\hat{\Omega}) + \beta,$$

with $\beta \in C^{k,\alpha}(\Lambda_8^2)$. Integrating by parts shows that $df = dg = 0$. Thus β is exact. But $d*\beta = -d(\omega \wedge \beta) = 0$, so $\beta = 0$. $\qquad\square$

Hence $K = \mathcal{H}_{SU} \oplus (W \times \{0\})$ is a direct complement in $T_{(\Omega,\omega)}\mathcal{C}_k$ to the tangent space to the diffeomorphism orbit at (Ω, ω). On a neighbourhood U of the zero section in $T_{(\Omega,\omega)}\mathcal{C}_k$ we may define a bundle map $\exp : U \to \mathcal{C}_k$ such that $D\exp_{(\Omega,\omega)} = id$. Then $\mathcal{S} = \exp K$ is a smooth submanifold of $\mathcal{C}_k$ with $T_{(\Omega,\omega)}\mathcal{S} = K$ and can be used as a slice for the diffeomorphism action at (Ω, ω).

3.4.3   The pre-moduli space

Let $(\Omega, \omega) \in \mathcal{Y}$, and let $\mathcal{Q} \subset \mathcal{S}$ be the subset of Calabi-Yau structures in the slice at (Ω, ω). This is the pre-moduli space of Calabi-Yau structures near (Ω, ω). We will show that $\mathcal{Q}$ is a smooth manifold with $T_{(\Omega,\omega)}\mathcal{Q} = \mathcal{H}_{SU}$ and then claim that $\mathcal{Q}$ is homeomorphic to a neighbourhood in the moduli space $\mathcal{N}$,

Let $(\Omega', \omega') \in \mathcal{C}_k$. Then $d\omega' = 0$ *a priori*, so lemma 2.2.21 implies that (Ω', ω') is a Calabi-Yau structure if and only if

$$d\Omega' = d\hat{\Omega}' = 0.$$

By lemma 3.4.5 there are unique $\beta_1 \in *dC^{k+1,\alpha}(\Lambda_1^2) \oplus W_1$ and $\beta_2 \in dC^{k+1,\alpha}(\Lambda^2) = dC^{k+1,\alpha}(\Lambda_6^2) \oplus dC^{k+1,\alpha}(\Lambda_1^2) \oplus W_2$ such that

$$d\Omega' = d\beta_1, \;\; d\hat{\Omega}' = d*\beta_2.$$

Let $F_i(\Omega')$ be the projection of β_i to W_i. We prove that $\mathcal{Q}$ is a smooth manifold by applying the implicit function theorem to

$$F : \mathcal{S} \to W_1 \oplus W_2, \;\; (\Omega', \omega') \mapsto (F_1(\Omega'), F_2(\Omega')).$$

Proposition 3.4.8. *If $(\Omega', \omega') \in \mathcal{C}_k$ is sufficiently close to (Ω, ω) then (Ω', ω') is a Calabi-Yau structure if and only if $F(\Omega') = 0$.*

Proof. Clearly $d\Omega' = 0 \Rightarrow F_1(\Omega') = 0$ and $d\hat{\Omega}' = 0 \Rightarrow F_2(\Omega') = 0$. Conversely, if $F_1(\Omega') = 0$ then $d\Omega' = d\beta_1$ with $\beta_1 = *d(f\omega')$ for some $f \in C^{k+1,\alpha}(\Lambda^0)$. But

$$<\beta_1, *d(f\omega')>_{L^2} = \int_X \beta_1 \wedge d(f\omega') = \int_X d\Omega' \wedge f\omega' = \int_X f d(\Omega' \wedge \omega') = 0. \qquad (3.29)$$

If ω' is sufficiently close to ω then it follows that $\beta_1 = 0$, so $d\Omega' = 0$.

Similarly, $d\hat{\Omega}' = d*\beta_2$ for some $\beta_2 \in dC^{k+1,\alpha}(\Lambda^2)$, and $F_2(\Omega') = 0$ implies that β_2 is L^2-orthogonal to W_2. A calculation analogous to (3.29) shows that β_2 is also L^2-orthogonal to $d(f\omega')$ for any $f \in C^{k+1,\alpha}(\Lambda^0)$.

By definition, $\Omega' + i\hat{\Omega}'$ has type $(3,0)$ with respect to the almost complex structure defined by Ω', so $d\Omega' + id\hat{\Omega}'$ has no $(1,3)$-component. As $d\Omega' = 0$ it follows that $d\hat{\Omega}'$ has type $(2,2)$. Thus for any $V \in C^{k+1,\alpha}(TX)$

$$<\beta_1, \mathcal{L}_V\Omega'>_{L^2} = \int_X \beta_1 \wedge d(V\lrcorner\Omega') = \int_X d\hat{\Omega}' \wedge (V\lrcorner\Omega') = 0.$$

If (Ω', ω') is sufficiently close to (Ω, ω) then (cf. 3.27b)

$$dC^{k+1,\alpha}(\Lambda^2) = \{\mathcal{L}_V\Omega' : V \in C^{k+1,\alpha}(TX)\} \oplus \{d(f\omega') : f \in C^{k+1,\alpha}(\Lambda^0)\} \oplus W_2.$$

Hence β_1 is L^2-orthogonal to all of $dC^{k+1,\alpha}(\Lambda^2)$ and therefore vanishes, so $d\hat{\Omega}' = 0$. $\qquad \square$

Proposition 3.4.9. *The derivative $DF_{(\Omega,\omega)} : T_{(\Omega,\omega)}\mathcal{C}_k \to W_1 \oplus W_2$ is 0 on $\mathcal{H}_{SU} \oplus T_k$ and bijective on $W \times \{0\}$.*

Proof. The derivative is obviously 0 on $\mathcal{H}_{SU}$ and on the tangent space to the diffeomorphism orbit of (Ω, ω). It is the identity on $W_1 \times \{0\}$, so the only non-trivial part is to check that the derivative is scalar on $W_2 \times \{0\}$. To see this it suffices to verify that if $\beta \in \Lambda^3_{12}$ then the derivative at Ω of $\Omega' \mapsto \hat{\Omega}'$ maps β to $-*\beta$. This is a point-wise statement, and can be deduced from proposition 2.2.4 and remark 2.2.25. $\qquad \square$

Now we can apply the implicit function theorem to F to show

Proposition 3.4.10. *If the slice $\mathcal{S}$ is taken sufficiently small then $\mathcal{Q} \subset \mathcal{S}$ is a submanifold. Its tangent space at (Ω, ω) is $\mathcal{H}_{SU}$.*

The implicit function theorem shows also that a small neighbourhood of (Ω, ω) in $\mathcal{X}_k$ is a submanifold of $\mathcal{C}_k$. Regularity arguments analogous to those in §3.2.5 show that the elements of $\mathcal{Q}$ are smooth. Theorem 3.4.1 then follows from the slice theorem 3.1.9.

Chapter 4

EAC G-manifolds

In this chapter we discuss some elementary properties of EAC manifolds with exceptional holonomy, and their deformation theory. For the G_2 case the material is largely covered in [47].

4.1 EAC G-structures

Definition 4.1.1. Let M^7 be a connected oriented manifold with cylindrical ends and cross-section X^6. A G_2-structure φ on M is said to be *exponentially asymptotically cylindrical* (EAC) if it is exponentially asymptotic (see definition 2.3.4) to a cylindrical G_2-structure on $X \times \mathbb{R}$ (see definition 2.2.23). M equipped with a torsion-free EAC G_2-structure and the associated metric is called an *EAC G_2-manifold*.

Recall that by proposition 2.2.24 a torsion-free cylindrical G_2-structure on $X^6 \times \mathbb{R}$ corresponds to a Calabi-Yau structure on X. Similarly

Definition 4.1.2. Let M^8 be a connected oriented manifold with cylindrical ends and cross-section X^7. A $Spin(7)$-structure ψ on M is said to be EAC if it is exponentially asymptotic to a cylindrical $Spin(7)$-structure on $X \times \mathbb{R}$. M equipped with a torsion-free EAC $Spin(7)$-structure and the associated metric is called an *EAC $Spin(7)$-manifold*.

The next theorem implies that an EAC G_2-manifold or $Spin(7)$-manifold is not interesting unless it has a single end. The result can be proved using the Cheeger-Gromoll line splitting theorem [12], or by more elementary methods (cf. Salur [51]).

Theorem 4.1.3. *Let M be an orientable connected asymptotically cylindrical Ricci-flat manifold. Then either M has a single end, i.e. its cross-section X is connected, or M is a cylinder $X \times \mathbb{R}$ with a product metric.*

4.1.1 Summary of EAC Hodge theory

For convenience we summarise the notation and key properties of Hodge theory on an EAC manifold M^n with cross-section X^{n-1} that we developed in §2.3.3–2.3.4.

Harmonic forms. The following denote spaces of harmonic m-forms on M.

$\mathcal{H}^m_+$	L^2-integrable
$\mathcal{H}^m_0$	bounded
$\mathcal{H}^m_-$	not exponentially growing
$\mathcal{H}^m_E$	bounded and exact
$\mathcal{H}^m_{E^*}$	bounded and coexact
$\mathcal{H}^m_{abs}$	$\mathcal{H}^m_+ \oplus \mathcal{H}^m_{E^*}$
$\mathcal{H}^m_{rel}$	$\mathcal{H}^m_+ \oplus \mathcal{H}^m_E$

In addition $\mathcal{H}^m_X$ is the space of harmonic m-forms on X and $\mathcal{H}^m_\infty$ the translation-invariant harmonic m-forms on $X \times \mathbb{R}$.

$$\mathcal{H}^m_\infty = \mathcal{H}^m_X \oplus dt \wedge \mathcal{H}^{m-1}_X.$$

By elliptic regularity any harmonic form on M that is $C^{k,\alpha}_\delta$ (for some $k \geq 1$, $\alpha \in (0,1)$ and $\delta \in \mathbb{R}$) is smooth. If $0 < \delta < \epsilon_1$ then the harmonic forms in $C^{k,\alpha}_{\pm\delta}(\Lambda^m)$ lie in $\mathcal{H}^m_\pm$. In $\mathcal{H}^m_-$

$$\mathcal{H}^m_- = \ker d + \ker d^* \qquad \mathcal{H}^m_0 = \mathcal{H}^m_{abs} + \mathcal{H}^m_{rel}$$
$$\mathcal{H}^m_0 = \ker d \cap \ker d^* \qquad \mathcal{H}^m_+ = \mathcal{H}^m_{abs} \cap \mathcal{H}^m_{rel}.$$

Boundary maps. Any bounded harmonic form on M is asymptotic to a translation-invariant form on the cylinder X. This gives a map

$$B : \mathcal{H}^m_0 \to \mathcal{H}^m_\infty.$$

We may write $B = B_a + dt \wedge B_e$, where

$$B_a : \mathcal{H}^m_0 \to \mathcal{H}^m_X, \quad B_e : \mathcal{H}^m_0 \to \mathcal{H}^{m-1}_X.$$

B_a is injective on $\mathcal{H}^m_{E*}$ and 0 on $\mathcal{H}^m_{rel}$, while B_e is injective on $\mathcal{H}^m_E$ and 0 on $\mathcal{H}^m_{abs}$. Let

$$\mathcal{A}^m = \operatorname{im} B_a, \quad \mathcal{E}^{m-1} = \operatorname{im} B_e.$$

Then

$$\mathcal{H}^m_X = \mathcal{A}^m \oplus \mathcal{E}^m \tag{4.1}$$

is an orthogonal direct sum.

Any $\psi \in \mathcal{H}^m_-$ can be written as $\chi + \alpha + t\beta + dt \wedge \gamma + tdt \wedge \delta$ with χ exponentially decaying, $\alpha, \beta \in \mathcal{H}^m_X$ and $\gamma, \delta \in \mathcal{H}^{m-1}_X$. Thus we can define a 'boundary data' map

$$BD : \mathcal{H}^m_- \to (\mathcal{H}^m_X)^2 \oplus (\mathcal{H}^{m-1}_X)^2, \quad \psi \mapsto \begin{pmatrix} \beta & \delta \\ \alpha & \gamma \end{pmatrix}.$$

The top row of the 2×2 array for $BD(\psi)$ represents the linearly growing part of ψ while the bottom row is the translation-invariant part. Let

$$BD_a : \mathcal{H}^m_- \to (\mathcal{H}^m_X)^2, \ \psi \mapsto \begin{pmatrix} \beta \\ \alpha \end{pmatrix}, \qquad BD_e : \mathcal{H}^m_- \to (\mathcal{H}^{m-1}_X)^2, \ \psi \mapsto \begin{pmatrix} \delta \\ \gamma \end{pmatrix},$$

be the projections of BD to $(\mathcal{H}^m_X)^2$ and $(\mathcal{H}^{m-1}_X)^2$ respectively. Let $\tilde{\mathcal{A}}^m = \operatorname{im} BD_a$ and $\tilde{\mathcal{E}}^{m-1} = \operatorname{im} BD_e$. Then

$$\operatorname{im} BD = \tilde{\mathcal{A}}^m \oplus \tilde{\mathcal{E}}^{m-1},$$

and $\tilde{\mathcal{A}}^m \subseteq (\mathcal{H}^m_X)^2$ and $\tilde{\mathcal{E}}^{m-1} \subseteq (\mathcal{H}^{m-1}_X)^2$ are Lagrangian subspaces.

Hodge theory. The natural map

$$\pi_H : \mathcal{H}^m_{abs} \to H^m(M)$$

is an isomorphism. The subset $\mathcal{H}^m_+ \subseteq \mathcal{H}^m_{abs}$ is mapped to the compactly supported subspace $H^m_0(M) \subseteq H^m(M)$. There is also an isomorphism

$$\mathcal{H}^m_{rel} \to H^m_{cpt}(M), \quad \alpha \mapsto \begin{cases} [\alpha + d\eta(\alpha)] & \text{for } \alpha \in \mathcal{H}^m_+ \\ \partial([B_e(\alpha)]) & \text{for } \alpha \in \mathcal{H}^m_E \end{cases}.$$

Let

$$A^m, E^m \subseteq H^m(X)$$

be the images of $\mathcal{A}^m, \mathcal{E}^m$ respectively in $H^m(X)$. A^m is the image of $j^*: H^m(M) \to H^m(X)$, and E^m is the orthogonal complement of A^m.

4.1.2 Hodge theory on EAC G_2-manifolds

Let M^7 be an EAC G_2-manifold with G_2-structure φ asymptotic to $\Omega + dt \wedge \omega$. (Ω, ω) is a Calabi-Yau structure on the cross-section X^6 and induces decompositions of the harmonic forms on X as described in §2.1.2. Maps such as $\Omega^2(X) \to \Omega^4(X)$, $\sigma \mapsto \sigma \wedge \omega$ are $SU(3)$-equivariant, so by corollary 2.1.11 they induce maps between type components of the spaces of harmonic forms. We now consider the relation between the type decompositions and the decomposition (4.1).

By remark 2.2.25, $*\varphi$ is asymptotic to $\frac{1}{2}\omega^2 - dt \wedge \hat{\Omega}$, where $\hat{\Omega}$ is the unique 3-form on X (with the boundary orientation) such that $\Omega + i\hat{\Omega}$ has type $(3,0)$ as discussed in §2.2.6.

Lemma 4.1.4. *If $\tau \in \mathcal{E}^m$ then $\tau \wedge \Omega \in \mathcal{E}^{m+3}$ and $\tau \wedge \frac{1}{2}\omega^2 \in \mathcal{E}^{m+4}$.*

If $\sigma \in \mathcal{A}^m$ then $\sigma \wedge \Omega \in \mathcal{A}^{m+3}$, $\sigma \wedge \frac{1}{2}\omega^2 \in \mathcal{A}^{m+4}$, $\sigma \wedge \omega \in \mathcal{E}^{m+2}$ and $\sigma \wedge \hat{\Omega} \in \mathcal{E}^{m+3}$.

Proof. If $\chi \in \mathcal{H}_E^{m+1}$ with $B_e\chi = \tau$ then

$$\chi \wedge \varphi \in \mathcal{H}_0^{m+4} \Rightarrow dt \wedge \tau \wedge \Omega = B(\chi \wedge \varphi) \in dt \wedge \mathcal{E}^{m+3},$$

$$\chi \wedge *\varphi \in \mathcal{H}_0^{m+5} \Rightarrow dt \wedge \tau \wedge \frac{1}{2}\omega^2 = B(\chi \wedge *\varphi) \in dt \wedge \mathcal{E}^{m+4}.$$

If $\chi \in \mathcal{H}_0^m$ with $B_a\chi = \sigma$ then

$$\chi \wedge \varphi \in \mathcal{H}_0^{m+3} \Rightarrow \sigma \wedge \Omega + dt \wedge \sigma \wedge \omega = B(\chi \wedge \varphi) \in \mathcal{A}^{m+3} \oplus dt \wedge \mathcal{E}^{m+2},$$

$$\chi \wedge *\varphi \in \mathcal{H}_0^{m+4} \Rightarrow \sigma \wedge \frac{1}{2}\omega^2 + dt \wedge \sigma \wedge \hat{\Omega} = B(\chi \wedge *\varphi) \in \mathcal{A}^{m+4} \oplus dt \wedge \mathcal{E}^{m+3}. \qquad \square$$

Hodge theory for compact manifolds allows us to identify the de Rham cohomology of X with the harmonic m-forms on X. The L^2-inner product on $\mathcal{H}_X^m$ therefore gives an inner product on $H^m(X)$, and the Hodge star $*: \mathcal{H}_X^m \to \mathcal{H}_X^{6-m}$ gives isomorphisms

$$*: H^m(X) \to H^{6-m}(X).$$

This map is the composition of the metric isomorphism $H^m(X) \cong (H^m(X))^*$ with Poincaré duality $(H^m(X))^* \cong H^{6-m}(X)$. Proposition 2.3.39 implies that there is an orthogonal direct sum

$$H^m(X) = A^m \oplus E^m,$$

where $A^m = j^*(H^m(M))$ and $E^m = *A^{6-m}$. Set $A_d^k = A^k \cap H_d^k(X)$ and $E_d^k = E^k \cap H_d^k(X)$.

Proposition 4.1.5. *Let M^7 an EAC G_2-manifold with cross-section X. Then*

$$H_6^2(X) = A_6^2 \oplus E_6^2, \quad H_6^4(X) = A_6^4 \oplus E_6^4,$$

and the sums are orthogonal. Furthermore

  (i) $H_6^2(X) \to H_6^4(X), [\alpha] \mapsto *[\alpha]$ *maps A_6^2 to E_6^4 and E_6^2 to A_6^4,*

  (ii) $H^1(X) \to H_6^4(X), [\alpha] \mapsto [\alpha] \cup [\Omega]$ *maps A^1 to A_6^4 and E^1 to E_6^4,*

  (iii) $H^1(X) \to H^5(X), [\alpha] \mapsto [\alpha] \cup [\frac{1}{2}\omega^2]$ *maps A^1 to A^5 and E^1 to E^5.*

Proof. (i) is obvious, since $*$ maps $A^m \leftrightarrow E^{6-m}$.

$[\alpha] \mapsto [\alpha] \cup [\Omega]$ is a bijection $H^1(X) \to H_6^4(X)$. It maps A^1 into A^4 and E^1 into E^4 by lemma 4.1.4, which implies (ii). It follows that $A^1 \to A_6^4$ and $E^1 \to E_6^4$ are both surjective and that $H_6^4(X)$ splits as $A_6^4 \oplus E_6^4$. $H_6^2(X)$ splits too by (i).

(iii) follows from lemma 4.1.4 in the same way. $\qquad\square$

When X is Kähler, the complex structure J is parallel and the Hodge Laplacian on forms commutes with the action of J. Thus $\mathcal{H}_X^1$ inherits a complex structure from X in the Kähler case and the inner product on $\mathcal{H}_X^1$ is Hermitian. We identify $H^1(X) \leftrightarrow \mathcal{H}_X^1$ as usual to make $H^1(X)$ a Hermitian vector space.

Proposition 4.1.6. *Let M^7 be an EAC G_2-manifold with cross-section X. Then*

$$j^* : H^1(M) \hookrightarrow H^1(X)$$

embeds $H^1(M)$ as a Lagrangian subspace of $H^1(X)$ with the symplectic form $< \cdot, J\cdot >$. In particular $b^1(M) = \frac{1}{2}b^1(X)$.

Proof. By proposition 2.3.41, j^* maps $H^1(M)$ isomorphically to its image A^1. The complex structure on $H^1(X)$ can be expressed as

$$J[\alpha] = *([\alpha] \cup [\tfrac{1}{2}\omega^2]).$$

It follows from proposition 4.1.5 that J restricts to an isomorphism $A^1 \to E^1$. Thus

$$H^1(X) = A^1 \oplus JA^1. \qquad\square$$

4.1.3 Spinors on EAC $Spin(7)$-manifolds

For compact $Spin(7)$-manifolds the dimension of the space of positive harmonic spinors appears as a term in the dimension of the moduli space of torsion-free $Spin(7)$-structures and also determines whether a simply-connected compact $Spin(7)$-manifold has holonomy exactly $Spin(7)$. For EAC $Spin(7)$-manifolds the dimension of the space of bounded harmonic positive spinors plays a similar role.

Let M^8 be an EAC $Spin(7)$-manifold with cross-section X^7, and $S^{\pm}$ the real spinor bundles on M. As in §2.3.5 we let $\mathcal{H}_0^{S^{\pm}}$ denote the spaces of bounded and decaying harmonic spinors on M with respect to the Dirac Laplacian $\eth^2$, and let $\mathcal{H}_{\infty}^{S}$ be the translation-invariant spinors on the cylinder $X \times \mathbb{R}$. Let S_X be the bundle of spinors on the cross-section, and $\mathcal{H}_X^{S}$ the harmonic spinors on X. Recall that the negative spinor bundle S^- is isomorphic to T^*M and that, because M is scalar-flat, the elements of $\mathcal{H}_0^{S^{\pm}}$ and $\mathcal{H}_X^{S}$ are parallel by the Lichnerowicz formula.

Proposition 4.1.7. *On an EAC $Spin(7)$-manifold M^8*

$$\dim \mathcal{H}_0^{S^+} = 1 + b^1(X) - b^1(M).$$

Proof. Because harmonic spinors are parallel, the boundary maps $B : \mathcal{H}_0^{S^{\pm}} \to \mathcal{H}_{\infty}^{S}$ are injective and it follows from proposition 2.3.44 that

$$\dim \mathcal{H}_0^{S^+} = \dim \mathcal{H}_{\infty}^{S} - \dim \mathcal{H}_0^{S^-}.$$

Now $\mathcal{H}_0^{S^-} \cong \mathcal{H}_0^1$, while $\mathcal{H}_{\infty}^{S} \cong \mathcal{H}_X^0 \oplus \mathcal{H}_X^1$. If X is connected (i.e. M has a single end) then $\dim \mathcal{H}_X^0 = 1$. Corollary 2.3.40 implies that $\dim \mathcal{H}_0^1 = b^1(M)$, so the formula holds. If X is not connected then M is a product cylinder, and the result is easy. $\square$

We can also deduce a splitting result similar to proposition 4.1.5. Fix a torsion-free EAC $Spin(7)$-structure ψ on M. Denote bounded harmonic m-forms of type d on M by $\mathcal{H}_{0,d}^m$. The asymptotic limit of ψ is $*_{\varphi}\varphi + dt \wedge \varphi$, where φ is a torsion-free G_2-structure on the cross-section X^7 and $*_{\varphi}$ is the induced Hodge star (see proposition 2.2.26). φ induces type decompositions of $\Omega^*(M)$. Let $\mathcal{A}_d^4 = \mathcal{A}^4 \cap \Omega_d^4(X)$ and $\mathcal{E}_d^4 = \mathcal{E}^4 \cap \Omega_d^4(X)$. Clearly $\mathcal{A}_1^4 \cong \mathbb{R}(*_{\varphi}\varphi)$, and $\mathcal{E}_1^4 = 0$ unless M is a product cylinder. In any case $\mathcal{H}_{X,1}^4 = \mathcal{A}_1^4 \oplus \mathcal{E}_1^4$, but it is not clear *a priori* that $\mathcal{H}_{X,7}^4$ and $\mathcal{H}_{X,27}^4$ split in a similar way.

Proposition 4.1.8. *Let M be an EAC Spin(7)-manifold. The boundary maps give iso-morphisms*

$$B_a : \mathcal{H}^4_{0,7} \overset{\sim}{\longrightarrow} \mathcal{A}^4_7,$$
$$B_e : \mathcal{H}^5_{0,8} \overset{\sim}{\longrightarrow} \mathcal{E}^4_1 \oplus \mathcal{E}^4_7,$$

and $\mathcal{H}^4_X$ splits as

$$\mathcal{H}^4_X = \mathcal{A}^4_1 \oplus \mathcal{A}^4_7 \oplus \mathcal{A}^4_{27} \oplus \mathcal{E}^4_1 \oplus \mathcal{E}^4_7 \oplus \mathcal{E}^4_{27}.$$

Proof. The boundary maps are injective since both $\mathcal{H}^4_{0,7}$ and $\mathcal{H}^5_{0,8}$ consist of parallel forms. The dimensions of the images $B_e(\mathcal{H}^5_{0,8}) \subseteq \mathcal{E}^4_{1\oplus7}$ and $B_a(\mathcal{H}^4_{0,1\oplus7}) \subseteq \mathcal{A}^4_{1\oplus7}$ are therefore the same as $B(\mathcal{H}^{S^+})$ and $B(\mathcal{H}^{S^-})$, respectively. Dimension-counting using proposition 2.3.44 tells us that equality holds and that $\mathcal{H}^4_{X,1\oplus7} = \mathcal{A}^4_{1\oplus7} \oplus \mathcal{E}^4_{1\oplus7}$.

Since $\mathcal{H}^4_{X,1}$ splits so does $\mathcal{H}^4_{X,7}$ and hence also $\mathcal{H}^4_{X,27}$. $\square$

4.1.4 A topological criterion for $Hol = G_2$

In this section we obtain a topological criterion for when the holonomy group of an EAC G_2-manifold M^7 is precisely G_2 and not a proper subgroup. As stated in corollary 2.2.8 the holonomy group of a metric defined by a torsion-free G_2-structure is always a subgroup of G_2. For compact G_2-manifolds theorem 2.2.12 states that a necessary and sufficient condition for full holonomy G_2 is that the manifold has finite fundamental group. We review the proof from Joyce [27, p. 245] and then generalise the result to the EAC case.

We first state two lemmas. The first one is due to Cheeger and Gromoll, and is a corollary of their line splitting theorem [12]. Note that all covering spaces will be presumed to be equipped with the Riemannian metric pulled back by the covering map. In particular all covering maps will be local isometries, and all covering transformations are isometries.

Lemma 4.1.9 ([5, Corollary 6.67]). *Let M be a compact Ricci-flat Riemannian manifold. Then M has a finite cover isometric to a Riemannian product $T^k \times N$, where T^k is a flat torus (of dimension k possibly 0) and N is compact and simply-connected.*

Lemma 4.1.10. *Let G be a closed connected subgroup of $SO(n)$ and let M^n be a connected Riemannian manifold with $Hol(M) \subseteq G$. Then $Hol(M) = G$ if and only if $Hol(\tilde{M}) = G$ for any cover $\tilde{M}$ of M.*

Proof. Let $\overline{M}$ be the universal cover of M. For any cover $\tilde{M}$ of M

$$Hol(\overline{M}) \subseteq Hol(\tilde{M}) \subseteq Hol(M).$$

To prove the lemma it therefore suffices to show that if $Hol(M) = G$ then $Hol(\overline{M}) = G$. It is easy to see that $Hol(\overline{M})$ is in fact the restricted holonomy group $Hol^0(M)$ of M, and by [5, Corollary 10.48] this is the identity component of $Hol(M)$. Since G connected it follows that if $Hol(M) = G$ then $Hol^0(M) = G$. $\qquad\square$

Now we prove the previously stated result that a compact G_2-manifold has holonomy exactly G_2 if and only if it has finite fundamental group.

Proof of theorem 2.2.12. If $\pi_1(M)$ is infinite then let $\tilde{M} = T^k \times N$ be a finite cover of M as given by lemma 4.1.9. Clearly $\pi_1(M)$ infinite implies $k > 0$, so $Hol(\tilde{M}) = Hol(N) \neq G_2$. Hence $Hol(M)$ is a proper subgroup of G_2.

If $\pi_1(M)$ is finite consider the universal cover $\overline{M}$ of M. From theorem 2.1.3 we find that up to conjugacy the only proper subgroups of G_2 that can be the holonomy group of a simply-connected Riemannian manifold are $1, SU(2)$ and $SU(3)$. Thus if $Hol(\overline{M})$ is not G_2 then it fixes at least one vector in its action on $\mathbb{R}^7$. By proposition 2.1.2, there must exist a non-zero parallel 1-form on $\overline{M}$ which, by corollary 2.1.9, is harmonic. But since $\overline{M}$ is compact there is an isomorphism $\mathcal{H}^1 \to H^1(\overline{M})$ between harmonic forms and de Rham cohomology. $\overline{M}$ is simply-connected, so $b^1(\overline{M}) = 0$ and there can be no non-trivial harmonic 1-forms on $\overline{M}$. Hence $Hol(M) = Hol(\overline{M}) = G_2$. $\qquad\square$

When stating an EAC analogue of theorem 2.2.12 we need to take into account that M^7 could be a product cylinder $X^6 \times \mathbb{R}$. If X is an irreducible Calabi-Yau manifold then the fundamental group is finite, but $Hol(M)$ is of course contained in $SU(3)$. The correct statement in the EAC setting is:

Theorem 4.1.11. *Let M^7 be an EAC G_2-manifold. Then $Hol(M) = G_2$ if and only if the fundamental group $\pi_1(M)$ is finite and neither M nor any double cover of M is homeomorphic to a cylinder.*

To prove the theorem we use that by proposition 2.3.41 the space of parallel 1-forms on M is exactly $\mathcal{H}_0^1$ and that, by corollary 2.3.40, the natural map $\mathcal{H}_0^1 \to H^1(M)$ from bounded harmonic forms to de Rham cohomology is an isomorphism when M has a single end. In order to obtain the information we need about the structure of the fundamental group of

M in the asymptotically cylindrical case, we quote a result about groups of polynomial growth.

Definition 4.1.12. Let G be a finitely generated group and S a finite set of generators of G. Let $N(k)$ be the number of elements of G that can be written as a product of less than k elements of S. If $N(k)$ is bounded by a polynomial in k then G is said to have *polynomial growth*.

This definition is independent of the choice of finite generating set S. The next result is the main theorem in Gromov [20] (see also Kleiner [31] for a shortened proof).

Theorem 4.1.13. *If G is finitely generated and has polynomial growth then G has a nilpotent subgroup of finite index.*

Corollary 4.1.14. *Let M be an asymptotically cylindrical manifold with non-negative Ricci curvature. Then the fundamental group $\pi_1(M)$ has a nilpotent subgroup of finite index.*

Sketch proof. M is homotopy equivalent to a compact manifold with boundary so $\pi_1(M)$ is finitely generated. Volume comparison arguments show that the volume of balls in the universal cover of M grows polynomially and this can be used to deduce that $\pi_1(M)$ has polynomial growth (see Milnor [43] for details). Hence theorem 4.1.13 applies. $\qquad\square$

Lemma 4.1.15. *Let M be a Ricci-flat EAC manifold.*

(i) *If M has a finite normal cover homeomorphic to a cylinder then M or a double cover of M is homeomorphic to a cylinder.*

(ii) *If $\pi_1(M)$ is infinite then M has a finite cover $\tilde{M}$ with $b^1(\tilde{M}) > 0$.*

Proof. (i) If $\tilde{M}$ is a finite normal cover of M homeomorphic to a cylinder then it is isometric to a product cylinder $Y \times \mathbb{R}$. M is a quotient of $Y \times \mathbb{R}$ by a finite group A of isometries. The isometries are products of isometries of Y and of $\mathbb{R}$ (since they preserve the set of globally distance minimising geodesics $\{\{y\} \times \mathbb{R} : y \in Y\}$). The elements of A have finite order, so they must act on the $\mathbb{R}$ factor as either the identity or as reflections. Therefore the subgroup $B \subseteq A$ which acts as the identity on $\mathbb{R}$ is either all of A, in which case M is the cylinder $(Y/B) \times \mathbb{R}$, or a normal subgroup of index 2, in which case $(Y/B) \times \mathbb{R}$ is a cylindrical double cover of M.

(ii) Let $G_0 \subseteq \pi_1(M)$ be a nilpotent subgroup of finite index. G_0 is soluble, so the derived series $G_{i+1} = [G_i, G_i]$ reaches 1. Therefore there is a largest i such that $G_i \subseteq \pi_1(M)$ has finite index. Let $\tilde{M}$ be the cover of M corresponding to $G_i \subseteq \pi_1(M)$. G_i/G_{i+1} is an infinite Abelian group, so has non-zero rank. Hence

$$b^1(\tilde{M}) = rk\left(\pi_1(\tilde{M})/[\pi_1(\tilde{M}), \pi_1(\tilde{M})]\right) = rk(G_i/G_{i+1}) > 0. \qquad \square$$

The lemma implies that if M is an EAC G_2-manifold then one of 4 possible cases holds:

(i) $\pi_1(M)$ is finite and M is homeomorphic to a cylinder. Then M is isometric to $Y \times \mathbb{R}$ for some compact Calabi-Yau manifold Y^6. The same arguments as in the proof of theorem 2.2.12 show that the holonomy of Y cannot be a proper subgroup of $SU(3)$. Thus $Hol(M) = SU(3)$.

(ii) $\pi_1(M)$ is finite, M has a single end and has a double cover homeomorphic to a cylinder. This double cover has holonomy $SU(3)$ so $Hol(M) \neq G_2$.

(iii) $\pi_1(M)$ is infinite. Then M has a finite cover $\tilde{M}$ with $b^1(M) > 0$. By theorem 2.3.33 together with proposition 2.3.41 there is a parallel 1-form on $\tilde{M}$, so $Hol(\tilde{M}) \subseteq SU(3)$ and $Hol(M) \neq G_2$.

(iv) $\pi_1(M)$ is finite and neither M nor any double cover of M is homeomorphic to a cylinder. Then the universal cover $\tilde{M}$ of M is an EAC G_2-manifold with a single end. The only proper subgroups of G_2 that can be the holonomy group of a complete simply-connected manifold are 1, $SU(2)$ and $SU(3)$, so if $Hol(\tilde{M})$ is not G_2 then there is a parallel vector field on $\tilde{M}$. But $b^1(\tilde{M}) = 0$, so by corollary 2.3.40 and proposition 2.3.41 there are no parallel 1-forms on $\tilde{M}$. Hence $Hol(M) = G_2$.

$Hol(M)$ is exactly G_2 only in case (iv), so we have proved theorem 4.1.11. We will find examples of EAC manifolds with holonomy exactly G_2 in chapter 7 and examples of the cases (ii) and (iii) are provided below. We can also show that an EAC G_2-manifold has holonomy exactly G_2 if its cross-section (which is a Calabi-Yau 3-fold) has holonomy exactly $SU(3)$.

Corollary 4.1.16. *Let M^7 be an EAC G_2-manifold with cross-section X and suppose that M is not finitely covered by a cylinder. If $Hol(X) = SU(3)$ then $Hol(M) = G_2$.*

Proof. Suppose that $Hol(M)$ is a proper subgroup of G_2. Then $\pi_1(M)$ is infinite, so M has a finite cover $\tilde{M}$ with $b^1(\tilde{M}) > 0$. Let $\tilde{X}$ be the cross-section of $\tilde{M}$. By proposition 4.1.6 $b^1(\tilde{X}) = 2b^1(\tilde{M}) > 0$, so $Hol(\tilde{X})$ is a proper subgroup of $SU(3)$. $\tilde{X}$ is a finite cover of X, so it follows that $Hol(X)$ is a proper subgroup of $SU(3)$. $\qquad\square$

Proposition 4.1.6 also implies that if $b^1(X) > 0$ then $b^1(M) > 0$, so M is reducible. It is, however, not true in general that M is reducible if X is reducible. In §7.3.3(ii) we will see an example of an EAC G_2-manifold with holonomy exactly G_2, whose cross-section is a quotient of $T^2 \times K3$ with $b^1 = 0$.

Example 4.1.17. There exist EAC manifolds W^6 with holonomy precisely $SU(3)$ (see Kovalev [34, Theorem 2.7]). Then we can define a torsion-free G_2-structure on the product $W \times S^1$ as in proposition 2.2.24. Of course $Hol(W \times S^1)$ is not all of G_2, but just $SU(3)$. Furthermore $b^1(W \times S^1) > 0$, so by theorem 4.1.11 *no* EAC G_2-structure on $W \times S^1$ can have holonomy exactly G_2.

Example 4.1.18. Let $Y \subset \mathbb{C}P^5$ be the complex projective variety defined by the equations $\sum X_i^2 = 0$, $\sum X_i^4 = 0$. Y is a complete intersection of hypersurfaces, so is a smooth complex 3-fold. As described in [27, p. 40] the adjunction formula can be used to show that the first Chern class $c_1(Y)$ vanishes, and the Lefschetz hyperplane theorem, stated in the form [7, Theorem I], can be applied to show that $\pi_1(Y) = 1$.

Since the polynomials defining Y are real, the complex conjugation map on $\mathbb{C}P^5$ restricts to an involution $a : Y \to Y$. a is anti-holomorphic, and since the defining polynomials have no roots over $\mathbb{R}$ the involution has no fixed points.

Let ω_{FS} be the restriction of the Fubini-Study Kähler form to Y. The involution is anti-symplectic, i.e. $a^*\omega_{FS} = -\omega_{FS}$. Since $c_1(Y) = 0$, Yau's solution to the Calabi conjecture [56] implies that there is a unique Kähler form ω in the cohomology class of ω_{FS} such that the corresponding metric is Ricci-flat, making Y into a Calabi-Yau manifold. The cohomology class of ω_{FS} is preserved by $-a^*$ and $-a^*\omega$ is a Kähler form defining a Ricci-flat metric, so the uniqueness part of Yau's theorem implies that $-a^*\omega = \omega$ (cf. [27, Proposition 15.2.2]).

Pick a global holomorphic non-vanishing 3-form ϕ on Y. $\overline{a^*\phi}$ is also holomorphic and therefore equal to $\lambda^2\phi$ for some $\lambda \in \mathbb{C}$. Replacing ϕ with $\lambda\phi$ we can assume without loss of generality that $\lambda = 1$. Then $\Omega = \operatorname{re}\phi$ is preserved by a^*. We can rescale Ω to ensure that (Ω, ω) is a Calabi-Yau structure in the sense of definition 2.2.20.

Now define a G_2-structure on $Y \times \mathbb{R}$ by $\varphi = \Omega + dt \wedge \omega$. By proposition 2.2.24, φ is torsion-free. Let M be the quotient of $Y \times \mathbb{R}$ by $a \times (-1)$. M has a single end and $\pi_1(M)$

has order 2. $(a \times (-1))^* \varphi = a^* \Omega + (-dt) \wedge a^* \omega = \varphi$, so φ induces a well-defined torsion-free G_2-structure on M.

4.1.5 A topological criterion for $Hol = Spin(7)$

As stated in corollary 2.2.15 the holonomy group of a metric defined by a torsion-free $Spin(7)$-structure is always a subgroup of $Spin(7)$. In this section we obtain a topological criterion for when the holonomy group of an asymptotically cylindrical $Spin(7)$-manifold M^8 is precisely $Spin(7)$ and not a proper subgroup. For a compact $Spin(7)$-manifold theorem 2.2.18 states that $Hol(M) = Spin(7)$ if and only if $\pi_1(M) = 1$ and $\hat{A}(M) = 1$. We sketch the proof from Joyce [27, §10] since the argument for the EAC case is similar, even though the criterion obtained looks different.

Proof of theorem 2.2.18. Recall that any harmonic spinor on a compact $Spin(7)$-manifold is parallel and that the negative spinor bundle is isomorphic to T^*M. As in the G_2 case, any compact manifold M^8 with holonomy exactly $Spin(7)$ must have finite fundamental group. Then $b^1(M) = 0$, so there are no non-zero harmonic negative spinors. Because $Spin(7)$ fixes a unique line in the positive spin representation σ_8^+ the dimension of the space of harmonic positive spinors is 1. Hence the index $\hat{A}(M)$ of the Dirac operator is 1. The same is true for any finite cover of M. Since $\hat{A}(M)$ is a characteristic class it follows that M can have no non-trivial finite cover, so M is simply-connected.

Conversely, if M^8 is a simply-connected $Spin(7)$-manifold then Berger's list (theorem 2.1.3) implies that the only possible holonomy groups of M other than $Spin(7)$ are $SU(4)$, $Sp(2)$ and $SU(2) \times SU(2)$. These fix a 2-, 3- and 4-dimensional subspace of σ_8^+, respectively. For a compact simply-connected $Spin(7)$-manifold we can therefore use the dimension $\hat{A}(M)$ of the space of parallel positive spinors to distinguish between the four possible holonomy groups. $\qquad\square$

In the EAC case it is still true that bounded harmonic spinors are parallel, and the dimension of the space of parallel spinors is given in terms of Betti numbers by proposition 4.1.7. Unlike the compact case this is not a characteristic class, so it is not immediately clear that EAC manifolds with holonomy $Spin(7)$ need to be simply-connected. It would be interesting to decide if that is the case, but there are currently no known examples at all of EAC manifolds with holonomy $Spin(7)$.

Theorem 4.1.19. *Let M^8 be an EAC $Spin(7)$-manifold. Then $Hol(M) = Spin(7)$ if and only if $\pi_1(M)$ is finite, neither M nor any double cover of M is homeomorphic to a cylinder, and the cross-section $\tilde{X}$ of the universal cover has $b^1(\tilde{X}) = 0$.*

Proof. If these conditions are satisfied then the universal cover $\tilde{M}$ is a simply-connected manifold with cylindrical ends – a single end, in fact, by lemma 4.1.15(i). As in the G_2 case we can deduce from corollary 2.3.40 and proposition 2.3.41 that there are no parallel 1-forms on $\tilde{M}$. As in the compact case the only possibile holonomy groups of $\tilde{M}$ other than $Spin(7)$ are $SU(4)$, $Sp(2)$ and $SU(2) \times SU(2)$, and we can use the dimension of the space of parallel positive spinors to distinguish between them. The condition $b^1(\tilde{X}) = 0$ implies that this is 1 by proposition 4.1.7. Therefore $\tilde{M}$, and hence also M, has holonomy exactly $Spin(7)$.

Conversely, if $\pi_1(M)$ is infinite then M has a finite cover $\tilde{M}$ with $b^1(\tilde{M}) > 0$ by lemma 4.1.15(ii). If M has a cylindrical double cover then that must have a product metric. If $\pi_1(M)$ is finite and the universal cover $\tilde{M}$ has cross-section $\tilde{X}$ with $b^1(\tilde{X}) > 0$ then the dimension of the space of parallel spinors on $\tilde{M}$ is at least 2 by proposition 4.1.7. In each of the three cases M has a finite cover with holonomy a proper subgroup of $Spin(7)$, so M cannot have holonomy exactly $Spin(7)$. $\qquad\square$

4.2 Deformations of EAC G_2-manifolds

In this section we construct the moduli space of exponentially asymptotically cylindrical (EAC) torsion-free G_2-structures on a manifold with a cylindrical end. We prove that the moduli space is smooth and find its dimension. We also study the boundary map to the moduli space of Calabi-Yau structures on the cross-section.

4.2.1 Results

Let M^7 be a connected oriented manifold with cylindrical ends and cross-section X^6. For $\delta > 0$ let $\mathcal{X}_\delta$ be the space of torsion-free EAC G_2-structures with rate δ on M (see definition 4.1.1). $\mathcal{X}_\delta$ is topologised as a subspace of the space of exponentially asymptotically translation-invariant 3-forms.

Let $\mathcal{X}_+ = \bigcup_{\delta > 0} \mathcal{X}_\delta$. If $\delta_1 > \delta_2 > 0$ then the inclusion $\mathcal{X}_{\delta_1} \hookrightarrow \mathcal{X}_{\delta_2}$ is continuous, so we can give $\mathcal{X}_+$ the direct limit topology, i.e. $U \subseteq \mathcal{X}_+$ is open if and only if $U \cap \mathcal{X}_\delta$ is open in $\mathcal{X}_\delta$ for all $\delta > 0$. Similarly let $\mathcal{D}_+$ be the group of EAC diffeomorphisms of M with any positive

rate (in the sense of definition 2.3.6) that are isotopic to the identity. $\mathcal{D}_+$ acts on $\mathcal{X}_+$ by pull-backs, and the *moduli space of torsion-free EAC G_2-structures* on M is the quotient $\mathcal{M}_+ = \mathcal{X}_+/\mathcal{D}_+$.

Remark 4.2.1. The definition of an EAC G_2-structure φ that is used involves a normalisation: if t is the cylindrical coordinate on M then $\|\frac{\partial}{\partial t}\| \to 1$ uniformly on X as $t \to \infty$ (in the metric defined by φ) so a scalar multiple $\lambda\varphi$ is not an EAC G_2-structure. This normalisation is the most convenient to work with, but a different choice of normalisation (e.g. that $Vol(X) = 1$ in the induced metric on the boundary) would of course give the same results. Another interpretation is that $\mathbb{R}^+$ acts on the moduli space of unnormalised EAC G_2-structures by rescaling and that $\mathcal{M}_+$ is the resulting quotient.

In the compact case theorem 3.2.1 gives a description of the moduli space of torsion-free G_2-structures using the natural projection map to the de Rham cohomology. In the EAC case, however, it is not enough to consider

$$\mathcal{M}_+ \to H^3(M), \;\; \varphi\mathcal{D}_+ \mapsto [\varphi].$$

We also need to consider the boundary values of φ to get an adequate description. Any $\varphi \in \mathcal{X}_+$ is asymptotic to some $\Omega + dt \wedge \omega$ with $(\Omega, \omega) \in \Omega^3(X) \times \Omega^2(X)$. Let

$$\pi_{\mathcal{M}} : \mathcal{M}_+ \to H^3(M) \times H^2(X), \;\; \varphi\mathcal{D}_+ \mapsto ([\varphi], [\omega]). \tag{4.2}$$

The main result of this section is

Theorem 4.2.2. $\mathcal{M}_+$ *is a smooth manifold, and* $\pi_{\mathcal{M}} : \mathcal{M}_+ \to H^3(M) \times H^2(X)$ *is an immersion.*

In order to prove theorem 4.2.2 we make use of our understanding of the deformations of the 'boundary' of an EAC G_2-manifold. By proposition 2.2.24, the cross-section X is a compact Calabi-Yau 3-fold, and the moduli space $\mathcal{N}$ of Calabi-Yau structures on X is a smooth manifold by theorem 3.4.1.

In subsection 4.2.8 we look at some local properties of $\mathcal{M}_+$. Its dimension is given by

Proposition 4.2.3. $\dim \mathcal{M}_+ = b^4(M) + \frac{1}{2}b^3(X) - b^1(M) - 1$.

We also study the properties of the *boundary map* on $\mathcal{M}_+$. This is the natural map $B : \mathcal{M}_+ \to \mathcal{N}$ which sends a G_2-structure on M to the Calabi-Yau structure on X defined by its asymptotic limit. As before we denote by $A^m \subseteq H^m(X)$ the image of the pull-back

map $j^* : H^m(M) \to H^m(X)$ in the long exact sequence for relative cohomology (2.32). If φ is asymptotic to $\Omega + dt \wedge \omega$ then

$$[\Omega] = j^*([\varphi]) \in A^3,$$
$$\tfrac{1}{2}[\omega^2] = j^*([*_\varphi \varphi]) \in A^4,$$

so the image of $B : \mathcal{M}_+ \to \mathcal{N}$ is contained in

$$\mathcal{N}_A = \{(\Omega, \omega)\mathcal{D}_X \in \mathcal{N} : [\Omega] \in A^3, [\omega^2] \in A^4\}. \tag{4.3}$$

It turns out that, locally at least, these necessary conditions for a point to be in the image are also sufficient.

Theorem 4.2.4. *The image of*

$$B : \mathcal{M}_+ \to \mathcal{N}_A \tag{4.4}$$

is open in $\mathcal{N}_A$ and a submanifold of $\mathcal{N}$. The map is a submersion onto its image.

Since the methods used are entirely local they do not tell us anything about the global properties of $\mathcal{M}_+$ or the image of (4.4).

We will show that the fibres of the submersion (4.4) are locally diffeomorphic to the compactly supported subspace $H_0^3(M) \subseteq H^3(M)$. The fibre over (Ω, ω) corresponds to the moduli space of torsion-free G_2-structures asymptotic to $\Omega + dt \wedge \omega$. Thus

Corollary 4.2.5. *The moduli space of torsion-free G_2-structures on M^7 exponentially asymptotic to a fixed cylindrical G_2-structure on $X^6 \times \mathbb{R}$ is a manifold. It is mapped locally diffeomorphically to an affine translate of $H_0^3(M)$ by $\pi_H : \mathcal{M}_+ \to H^3(M)$.*

In the proof of proposition 4.2.3 we find that $\dim H_0^3(M) = b^3(M) - \tfrac{1}{2}b^3(X)$, so

$$\dim \mathcal{N}_A = b^4(M) - b^3(M) + b^3(X) - b^1(M) - 1.$$

It follows from theorem 4.2.2 that

$$\mathcal{M}_+ \to H^3(M) \oplus H^4(M), \quad \varphi \mathcal{D}_+ \to ([\varphi], [*_\varphi \varphi]) \tag{4.5}$$

is an immersion. The image in $H^3(M) \oplus H^4(M)$ of a fibre of $B : \mathcal{M}_+ \to \mathcal{N}$ lies in an affine translate of $H_0^3(M) \oplus H_0^4(M)$, which has a natural symplectic structure since $H_0^3(M) \cong (H_0^4(M))^*$.

Proposition 4.2.6. *The restriction of* (4.5) *to the moduli space of torsion-free G_2-structures on M^7 exponentially asymptotic to a fixed cylindrical G_2-structure on $X^6 \times \mathbb{R}$ is a Lagrangian immersion to an affine translate of $H_0^3(M) \oplus H_0^4(M)$.*

Theorem 4.1.3 implies that if M is a G_2-manifold then either M is a cylinder $X \times \mathbb{R}$ (with a product metric) or M has a single end. If M is a cylinder $X \times \mathbb{R}$ then the only possible torsion-free G_2-structure asymptotic to a given cylindrical G_2-structure φ_∞ is φ_∞ itself, so the moduli space of asymptotically cylindrical torsion-free G_2-structures on M is equivalent to the moduli space of Calabi-Yau structures on X (we can compute that $H_0^m(X \times \mathbb{R}) = 0$ for all m, so this agrees with corollary 4.2.5). The moduli space will therefore only be interesting when M has a single end, though we will not need to assume this in the proof of theorem 4.2.2.

4.2.2   Proof outline

We wish to prove that $\mathcal{M}_+$ has the structure of a smooth manifold. In order to do this we construct, as an intermediate step, moduli spaces of torsion-free EAC G_2-structures with some fixed rate $\delta > 0$. The arguments are similar to the proof of the compact version (theorem 3.2.1), but each step needs to be adapted to the EAC setting.

Each $\varphi \in \mathcal{X}_\delta$ defines an EAC metric and hence a parameter $\epsilon_1(\varphi)$ such that the associated Hodge Laplacian is Fredholm on Hölder spaces with weights smaller than $\epsilon_1(\varphi)$ (see proposition 2.3.21). When we study a neighbourhood of φ we need to assume that $\delta < \epsilon_1(\varphi)$, so we let

$$\mathcal{X}_\delta' = \{\varphi \in \mathcal{X}_\delta : \epsilon_1(\varphi) > \delta\}.$$

By lemma 2.3.19, ϵ_1 depends lower semi-continuously on the asymptotic model so $\mathcal{X}_\delta'$ is an open subset of $\mathcal{X}_\delta$. Let $\mathcal{D}_\delta$ be the group of exponentially cylindrical diffeomorphisms of M with rate δ and $\mathcal{M}_\delta = \mathcal{X}_\delta'/\mathcal{D}_\delta$.

In generalising the proof from the compact case we make use of the Hodge theory for EAC manifolds developed in §2.3.4. In order to apply the implicit function theorem to show that the pre-moduli space is a smooth manifold we need an EAC version of the chain rule proposition 3.1.2.

Proposition 4.2.7. *Let M be an EAC manifold with rate δ, E and F vector bundles associated to TM and $\Psi : E \to F$ a smooth fibre-preserving map that is exponentially asymptotically translation-invariant with rate δ. Then Ψ induces a smooth map of sections $C_\delta^{k,\alpha}(E) \to C_\delta^{k,\alpha}(F)$.*

We now consider how to adapt the steps of §3.2 in order to construct pre-moduli spaces of torsion-free EAC G_2-structures near $\varphi \in \mathcal{X}'_\delta$.

Given $\varphi \in \mathcal{X}'_\delta$, let $\Omega + dt \wedge \omega = B(\varphi)$ denote the asymptotic limit of φ. If we identify $\Omega + dt \wedge \omega$ with the pair (Ω, ω) then, by proposition 2.2.24, $\Omega + dt \wedge \omega$ defines a Calabi-Yau structure on X. In order to simplify the problem of finding a slice for the $\mathcal{D}$-action at φ we use the deformation theory for compact Calabi-Yau 3-folds developed in §3.4. In particular, proposition 3.4.10 ensures that there is a pre-moduli space $\mathcal{Q}$ of Calabi-Yau structures near $\Omega + dt \wedge \omega$. Let

$$\mathcal{X}_\mathcal{Q} = \{\psi \in \mathcal{X}'_\delta : B(\psi) \in \mathcal{Q}\}$$

and let $\mathcal{D}_\mathcal{Q} \subseteq \mathcal{D}_\delta$ be the subgroup of diffeomorphisms asymptotic to automorphisms of the cylindrical Calabi-Yau structure $\Omega + dt \wedge \omega$. By proposition 3.1.10, $\mathcal{D}_\mathcal{Q}$ acts on $\mathcal{X}_\mathcal{Q}$, and we will see that $\mathcal{X}_\mathcal{Q}/\mathcal{D}_\mathcal{Q}$ maps homeomorphically to an open subset of $\mathcal{M}_\delta$.

We use slice arguments to study a neighbourhood of $\varphi \mathcal{D}_\mathcal{Q}$ in $\mathcal{X}_\mathcal{Q}/\mathcal{D}_\mathcal{Q}$. In order to be able to apply analysis results we need to use Banach spaces of forms, so we work with weighted Hölder $C^{k,\alpha}_\delta$ spaces for some fixed $k \geq 1$ and $\alpha \in (0, 1)$. Note that the boundary values of elements of $\mathcal{X}_\mathcal{Q}$ must lie in

$$\mathcal{Q}_A = \{\Omega' + dt \wedge \omega' \in \mathcal{Q} : [\Omega'] \in A^3, [\omega'^2] \in A^4\}, \tag{4.6}$$

where A^m is the image of $j^* : H^m(M) \to H^m(X)$ (cf. discussion before theorem 4.2.4). We use a cut-off function ρ for the cylinder on M to consider $\rho \mathcal{Q}_A$ as a subspace of smooth asymptotically translation-invariant 3-forms supported on the cylinder of M and set

$$\mathcal{Z}^3_\mathcal{Q} \subseteq C^{k,\alpha}_\delta(\Lambda^3) + \rho \mathcal{Q}_A$$

to be the subspace of closed forms. Then $\mathcal{X}_\mathcal{Q}$ embeds continuously into $\mathcal{Z}^3_\mathcal{Q}$.

The main technical steps in the construction of the pre-moduli space near φ are

  (i) to show that $\mathcal{Q}_A$ is a submanifold of $\mathcal{Q}$ (proposition 4.2.9), so that $\mathcal{Z}^3_\mathcal{Q}$ is a manifold,

  (ii) to find a complement K in $T_\varphi \mathcal{Z}^3_\mathcal{Q}$ for the tangent space to the $\mathcal{D}_\mathcal{Q}$-orbit at φ (proposition 4.2.15) and to pick a submanifold $\mathcal{S} \subseteq \mathcal{Z}^3_\mathcal{Q}$ with $T_\varphi \mathcal{S} = K$,

  (iii) to show that the space of torsion-free G_2-structures $\mathcal{R}_\delta \subseteq \mathcal{S}$ is a submanifold (proposition 4.2.19),

  (iv) to show that the elements of $\mathcal{R}_\delta$ are smooth and EAC with rate δ (proposition 4.2.24).

To complete the proof of theorem 4.2.2 it then only remains to explain how to adapt the slice arguments from §3.1 to the EAC case. This is done in §4.2.7. Repeated use is made of the regularity result for isometries of EAC manifolds proposition 2.3.7. We show that $\mathcal{R}_\delta$ is homeomorphic to a neighbourhood in $\mathcal{X}_\mathcal{Q}/\mathcal{D}_\mathcal{Q}$ (and therefore in $\mathcal{M}_\delta$). Hence $\mathcal{M}_\delta$ is a manifold for any $\delta > 0$. We then show that $\mathcal{M}_\delta$ is homeomorphic to an open subset of $\mathcal{M}_+$ for any $\delta > 0$ and deduce that $\mathcal{M}_+$ is a manifold.

Remark 4.2.8. The last step means that if $\varphi \in \mathcal{X}_+$ is EAC with rate $\delta_0(\varphi)$ then for any $0 < \delta < \min\{\delta_0(\varphi), \epsilon_1(\varphi)\}$ the pre-moduli space $\mathcal{R}_\delta$ gives a chart near φ not only in $\mathcal{M}_\delta$, but also in $\mathcal{M}_{\delta'}$ for any $\delta' > \delta$ and hence in $\mathcal{M}_+$. In other words, $\mathcal{R}_\delta$ is essentially independent of δ if δ is chosen sufficiently small.

4.2.3 The boundary values

As explained above we are restricting our attention to determining the space of torsion-free G_2-structures in $\mathcal{Z}^3_\mathcal{Q}$ whose boundary values lie in a certain space $\mathcal{Q}_A$. To make this approach work we first of all need to show that $\mathcal{Q}_A$ is a manifold.

Let X^6 be the cross-section of an EAC G_2-manifold M^7, and (Ω, ω) a Calabi-Yau structure on X defined by the limit of a torsion-free EAC G_2-structure on M. Let $\mathcal{Q}$ be the pre-moduli space of Calabi-Yau structures near (Ω, ω) and equivalently to (4.6) define

$$\mathcal{Q}_A = \{(\Omega', \omega') \in \mathcal{Q} : [\Omega'] \in A^3, [\omega'^2] \in A^4\}.$$

Since $\mathcal{Q}$ is diffeomorphic to a neighbourhood in the moduli space $\mathcal{N}$ of Calabi-Yau structures on X, $\mathcal{Q}_A$ is homeomorphic to a neighbourhood in the subspace $\mathcal{N}_A \subseteq \mathcal{N}$ defined by (4.3).

Recall that by proposition 3.4.10 the tangent space at (Ω, ω) to the pre-moduli space $\mathcal{Q}$ is the space $\mathcal{H}_{SU}$ of harmonic tangents to the space of $SU(3)$-structures. As before let $E^m \subseteq H^m(X)$ be the orthogonal complement of A^m and let $\mathcal{A}^m, \mathcal{E}^m \subseteq \mathcal{H}^m_X$ denote the respective spaces of harmonic representatives. By lemma 4.1.4, $\tau \mapsto \omega \wedge \tau$ maps $\mathcal{A}^2 \to \mathcal{E}^4$ and $\mathcal{E}^2 \to \mathcal{A}^4$. Hence the linearisation of the condition $[\omega'^2] \in A^4$ is $[\tau] \in E^2$, and we would expect the tangent space to $\mathcal{Q}_A$ at (Ω, ω) to be

$$\mathcal{H}_{SU,A} = \{(\sigma, \tau) \in \mathcal{H}_{SU} : \sigma \in \mathcal{A}^3, \tau \in \mathcal{E}^2\}.$$

This is indeed the case.

Proposition 4.2.9. *Let (Ω, ω) be the Calabi-Yau structure induced on the cross-section X^6 of an EAC G_2-manifold M^7 and $\mathcal{Q}$ the pre-moduli space of Calabi-Yau structures near (Ω, ω). Then $\mathcal{Q}_A \subseteq \mathcal{Q}$ is a submanifold and*

$$T_{(\Omega,\omega)}\mathcal{Q}_A = \mathcal{H}_{SU,A}.$$

Proof. The map $\mathcal{Q} \to H^3(X)$ is a submersion, so

$$\mathcal{Q}' = \{(\Omega', \omega') \in \mathcal{Q} : [\Omega'] \in A^3\}$$

is a submanifold of $\mathcal{Q}$. By proposition 4.1.5

$$H^4(X) = A_1^4 \oplus A_6^4 \oplus A_8^4 \oplus E_6^4 \oplus E_8^4,$$

where as before $A_d^4 = H_d^4(X) \cap A^4$ and $E_d^4 = H_d^4(X) \cap E^4$. Let

$$P_{E,8} : H^4(X) \to E_8^4$$

be the orthogonal projection. For $(\Omega', \omega') \in \mathcal{Q}'$, let $E^{m\prime}$ be the orthogonal complement of A^m with respect to the metric defined by (Ω', ω'), and let $P_{A'} : H^m(X) \to A^m$ and $P_{E'} : H^m(X) \to E^{m\prime}$ be the respective projections. Let

$$F : \mathcal{Q}' \to E_8^4, \quad (\Omega', \omega') \mapsto P_{E,8}P_{E'}[\omega' \wedge \omega'].$$

We prove that $\mathcal{Q}_A$ is a submanifold of $\mathcal{Q}'$ by showing that it is the zero set of F and that F has surjective derivative at (Ω, ω).

Suppose $F(\Omega', \omega') = 0$ and let $a = P_{E'}[\omega' \wedge \omega']$. Write $a = b + c$, with $b \in A^4$, $c \in E^4$. $P_{E,8}a = 0 \Rightarrow \pi_8 c = 0$, so $c \in E_6^4$. Since $E^1 \to E_6^4$, $v \mapsto [\Omega] \cup v$ is an isomorphism $c = [\Omega] \cup v$ for some $v \in E^1$. In the inner product $<,>'$ on $H^*(X)$ defined by (Ω', ω'),

$$<a, a>' = <a, [\Omega] \cup v>' = <a, [\Omega] \cup v - [\Omega'] \cup P_{E'}v>' \leq \|a\|'(\|[\Omega - \Omega']\|'\|v\|' + \|P_{A'}v\|').$$

The RHS can be estimated by $\|[\Omega - \Omega']\|(\|a\|')^2$ for (Ω', ω') close to (Ω, ω). Hence for (Ω', ω') sufficiently close to (Ω, ω)

$$F(\Omega', \omega') = 0 \Rightarrow P_{E'}[\omega'^2] = 0 \Rightarrow [\omega'^2] \in A^4.$$

So $\mathcal{Q}_A \subseteq \mathcal{Q}'$ is the zero set of F. It remains to show that F has surjective derivative. If $(\sigma, \tau) \in (\mathcal{A}^3 \times \mathcal{H}_X^2) \cap \mathcal{H}_{SU} = T_{(\Omega,\omega)}\mathcal{Q}'$ then, since $[\omega^2] \in A^4$,

$$DF_{(\Omega,\omega)}(\sigma, \tau) = P_{E,8}P_E(2[\omega \wedge \tau]) = 2P_{E,8}[\omega \wedge \tau].$$

Since $\mathcal{A}_8^2 \to \mathcal{E}_8^4$, $\tau \mapsto \omega \wedge \tau$ is a bijection, the derivative maps the space $0 \times \mathcal{A}_8^2$ onto E_8^4.

By the implicit function theorem $\mathcal{Q}_A$ is a manifold and the tangent space at (Ω, ω) is

$$\ker DF_{(\Omega,\omega)} = \mathcal{H}_{SU,A}. \qquad \square$$

Corollary 4.2.10. *The map $\mathcal{H}_{SU,A} \to \mathcal{A}^3$, $(\sigma, \tau) \mapsto \sigma$ is surjective with kernel $0 \times \mathcal{E}_8^2$.*

Proof. The last part of the proof of the proposition actually shows that $\mathcal{Q}_A \to A^3$ is a submersion, so $\mathcal{H}_{SU,A} \to \mathcal{A}^3$ is surjective. This could also be deduced from lemma 4.1.4. By definition of $\mathcal{H}_{SU}$, the kernel consists of those $(0, \tau) \in 0 \times \mathcal{E}^2$ satisfying the conditions (2.18), which reduce to $\pi_1 \tau = \pi_6 \tau = 0$. $\qquad \square$

Proposition 4.2.9 implies directly that a neighbourhood of the image of $B : \mathcal{M}_+ \to \mathcal{N}_A$ is a manifold. The rest of theorem 4.2.4 follows too, once we have proved the main result that $\mathcal{M}_+$ is a manifold. We will return to this in §4.2.8.

Remark 4.2.11. If $b^1(X) = 0$ then the proof of proposition 4.2.9 simplifies and it is possible to show that $\mathcal{N}_A$ is itself a submanifold of $\mathcal{N}$. However, we do not want to exclude the case $b^1(X) \neq 0$, since many of the interesting examples for the gluing construction discussed in §6 have reducible cross-section.

4.2.4 The slice

Fix $k \geq 1$, $\delta > 0$, $\alpha \in (0,1)$ and $\varphi \in \mathcal{X}_\delta'$. We will find a direct complement K in $T_\varphi \mathcal{Z}_\mathcal{Q}^3$ to the $\mathcal{D}_\mathcal{Q}$-orbit. Then we will define a submanifold $\mathcal{S} \subseteq \mathcal{Z}_\mathcal{Q}^3$ whose tangent space at φ is K. $\mathcal{S}$ will be used as a slice in $\mathcal{Z}_\mathcal{Q}^3$ for the $\mathcal{D}_\mathcal{Q}$-action at φ.

The fixed G_2-structure φ is used to define an EAC metric and a Hodge star. It also defines type decompositions of the exterior bundles (3.6) and spaces of harmonic forms, as described in §2.1.2. Recall that the map $TM \to \Lambda_7^2$, $v \mapsto v \lrcorner \varphi$ is a bundle isomorphism and that Lie derivatives $\mathcal{L}_V \varphi = d(V \lrcorner \varphi)$ are precisely exterior derivatives of 2-forms of type 7.

Restricting our attention to G_2-structures in $\mathcal{Z}_\mathcal{Q}^3$ is convenient because the asymptotic values of elements of $T_\varphi \mathcal{Z}_\mathcal{Q}^3$ are harmonic. Recall the notation for harmonic forms from the

summary in §2.3.4, in particular that $\mathcal{H}_0^m$ and $\mathcal{H}_\infty^m$ denote the spaces of bounded harmonic forms on M and harmonic translation-invariant forms on $X \times \mathbb{R}$ respectively.

For convenience we identify translation-invariant 3-forms on $X \times \mathbb{R}$ with pairs of 3- and 2-forms on X by $\sigma + dt \wedge \tau \leftrightarrow (\sigma, \tau)$. This identifies the tangent spaces $\mathcal{H}_{SU}$ and $\mathcal{H}_{SU,A}$ of $\mathcal{Q}$ and $\mathcal{Q}_A$ with subspaces $\mathcal{H}_{SU}^3$ and $\mathcal{H}_{SU,A}^3 \subseteq \mathcal{H}_\infty^3$. Let

$$\mathcal{Z}_{cyl}^3 \subseteq C_\delta^{k,\alpha}(\Lambda^3) \oplus \rho \mathcal{H}_{SU,A}^3$$

be the subspace of closed forms. Clearly $T_\varphi \mathcal{Z}_{\mathcal{Q}}^3 \subseteq \mathcal{Z}_{cyl}^3$; we show below that equality holds. The tangent space to the pre-moduli space of torsion-free G_2-structures at φ will turn out to be the subspace

$$\mathcal{H}_{cyl}^3 \subseteq \mathcal{Z}_{cyl}^3$$

of harmonic forms. This is exactly the subspace of elements of $\mathcal{H}_0^3$ which are tangent to cylindrical deformations of the G_2-structure, i.e. whose boundary values lie in $\mathcal{H}_{SU}^3$. The boundary map $B : \mathcal{H}_0^3 \to \mathcal{H}_\infty^3$ maps $\mathcal{H}_{cyl}^3$ precisely onto $\mathcal{H}_{SU,A}^3$. Together with the Hodge decomposition theorem 2.3.27 it follows that

$$\mathcal{Z}_{cyl}^3 = \mathcal{H}_{cyl}^3 \oplus C_\delta^{k,\alpha}[d\Lambda^2]. \tag{4.7}$$

Remark 4.2.12. $dC_\delta^{k+1,\alpha}(\Lambda^{m-1})$ is the space of exterior derivatives of decaying forms, while we use $C_\delta^{k,\alpha}[d\Lambda^m]$ to denote the space of exact decaying forms. $dC_\delta^{k+1,\alpha}(\Lambda^{m-1}) \subseteq C_\delta^{k,\alpha}[d\Lambda^m]$ is a closed subspace of finite codimension.

Lemma 4.2.13. *$\mathcal{Z}_{\mathcal{Q}}^3$ is a manifold and $T_\varphi \mathcal{Z}_{\mathcal{Q}}^3 = \mathcal{Z}_{cyl}^3$.*

Proof. If ψ is a 3-form asymptotic to an element $(\Omega', \omega') \in \mathcal{Q}_A$ then the condition $[\Omega'] \in A^3$ implies that $d\psi \in dC_\delta^{k,\alpha}(\Lambda^3)$. Therefore

$$d : C_\delta^{k,\alpha}(\Lambda^3) + \rho \mathcal{Q}_A \to dC_\delta^{k,\alpha}(\Lambda^3)$$

is a submersion and the result follows from the implicit function theorem. $\square$

Let $\mathcal{D}_{\mathcal{Q}}^{k+1}$ be the group of diffeomorphisms of M which are isotopic to the identity and $C_\delta^{k+1,\alpha}$-asymptotic to a cylindrical automorphism of the cylindrical G_2-structure $\Omega + dt \wedge \omega$. The elements of a neighbourhood of the identity in $\mathcal{D}_{\mathcal{Q}}^{k+1}$ can be written as $\exp(V + \rho V_\infty)$, where V is a $C_\delta^{k+1,\alpha}$ vector field on M and V_∞ is a translation-invariant vector field on

$X \times \mathbb{R}$ with $\mathcal{L}_{V_\infty}(\Omega + dt \wedge \omega) = 0$, i.e. $V_\infty \in (\mathcal{H}_\infty^1)^\sharp$. Therefore if we let

$$D = \rho(\mathcal{H}_\infty^1)^\sharp \lrcorner \varphi \subseteq \Omega_7^2(M)$$

then the tangent space to the $\mathcal{D}_\mathcal{Q}^{k+1}$-orbit at φ is

$$\{\mathcal{L}_{V+\rho V_\infty}\varphi : V \in C_\delta^{k+1,\alpha}(TM), V_\infty \in (\mathcal{H}_\infty^1)^\sharp\} = d(C_\delta^{k+1,\alpha}(\Lambda_7^2) \oplus D).$$

As a direct complement in $\mathcal{Z}_{cyl}^3$ we can take $K = \mathcal{H}_{cyl}^3 \oplus W$, where W is the space of decaying exact forms of type 27. Like in the compact case this is the kernel of the formal adjoint of $d : \Omega_7^2(M) \to \Omega^3(M)$ (see remark 3.2.7).

Definition 4.2.14. Let $W = C_\delta^{k,\alpha}[d\Lambda^2] \cap \Omega^{27}(M)$.

Proposition 4.2.15.

$$\mathcal{Z}_{cyl}^3 = d(C_\delta^{k+1,\alpha}(\Lambda_7^2) \oplus D) \oplus \mathcal{H}_{cyl}^3 \oplus W.$$

Proof. By (4.7) it suffices to show

$$C_\delta^{k,\alpha}[d\Lambda^2] = d(C_\delta^{k+1,\alpha}(\Lambda_7^2) \oplus D) \oplus W. \tag{4.8}$$

As in the proof of proposition 3.2.6 we can identify the spinor bundle on the G_2-manifold M both with $\Lambda^0 \oplus \Lambda_7^2$ and with $\Lambda_{1\oplus7}^3$ so that the Dirac operator on exponentially asymptotically translation-invariant spinors (2.39) is identified with

$$C_\delta^{k+1,\alpha}(\Lambda^0 \oplus \Lambda_7^2) \to C_\delta^{k,\alpha}(\Lambda_{1\oplus7}^3), \;\; (f, \eta) \mapsto \pi_{1\oplus7}d\eta + *(df \wedge \varphi).$$

As explained in §2.3.5, the image of this map is the L^2-orthogonal complement of the decaying harmonic spinors. On a scalar-flat manifold there are no harmonic decaying spinors, so the map is surjective. Thus, if $\beta \in C_\delta^{k,\alpha}[d\Lambda^2]$ then

$$\pi_{1\oplus7}\beta = \pi_{1\oplus7}d\eta + *(df \wedge \varphi)$$

for some $\eta \in C_\delta^{k+1,\alpha}(\Lambda_7^2) \oplus D$, $f \in C_\delta^{k+1,\alpha}(\Lambda^0) \oplus \rho\mathcal{H}_\infty^0$. Integrating by parts like in (3.10) shows $df = 0$. Hence the decaying exact form $\beta - d\eta$ has type 27, i.e. $\beta - d\eta \in W$. $\qquad\square$

We want to take as our slice for the $\mathcal{D}_\delta$-action at φ a submanifold $\mathcal{S} \subseteq \mathcal{Z}_\mathcal{Q}^3$ with

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$T_\varphi \mathcal{S} = K$. To this end we pick a map

$$\exp : U \to \mathcal{Z}_\mathcal{Q}^3 \tag{4.9}$$

on a neighbourhood U of 0 in $\mathcal{Z}_{cyl}^3 = T_\varphi \mathcal{Z}_\mathcal{Q}^3$, such that $D\exp_0 = id$. We also insist that exp is a translation on the decaying forms and that it maps smooth forms to smooth forms. We can do this since by (4.7) the decaying forms have a finite-dimensional complement of smooth forms in $\mathcal{Z}_{cyl}^3$. We then choose

$$\mathcal{S} = \exp(K \cap U). \tag{4.10}$$

4.2.5 Smoothness of the pre-moduli space

Let $\mathcal{R}_\delta \subseteq \mathcal{S}$ be the subset of $C_\delta^{k,\alpha}$ torsion-free G_2-structures. $\mathcal{R}_\delta$ is the pre-moduli space of torsion-free G_2-structures near φ. In order to show that $\mathcal{R}_\delta$ is a submanifold we will exhibit it as the zero set of a function F with surjective derivative and apply the implicit function theorem.

Recall that by theorem 2.2.10(i), a closed G_2-structure ψ is torsion-free if and only if $d\Theta(\psi) = 0$, where Θ is the non-linear map $\psi \mapsto *_\psi \psi$. Thus

$$\mathcal{R}_\delta = \{\psi \in \mathcal{S} : d\Theta(\psi) = 0\}.$$

If $\psi \in \mathcal{Z}_\mathcal{Q}^3$ then ψ is asymptotic to a torsion-free cylindrical G_2-structure $\Omega' + dt \wedge \omega'$, so $d\Theta(\psi)$ decays. Moreover, elements of $\mathcal{Z}_\mathcal{Q}^3$ are asymptotic to elements of $\mathcal{Q}_A \subseteq \mathcal{Q}$ by definition. Therefore $\Theta(\psi)$ is asymptotic to $\frac{1}{2}\omega'^2 - dt \wedge \hat{\Omega}'$, with $[\omega'^2] \in A^4$ (cf. (4.6) and remark 2.2.25). This implies that $d\Theta(\psi) \in dC_\delta^{k+1,\alpha}(\Lambda^4)$.

$\Theta : \Lambda_+^3 T^* M \to \Lambda^4 T^* M$ is point-wise smooth, so by the chain rule

$$\mathcal{Z}_\mathcal{Q}^3 \to dC_\delta^{k+1,\alpha}(\Lambda^4), \ \ \psi \to d\Theta(\psi)$$

is a smooth function. We need to adjust this map to obtain a function with surjective derivative. If β is a 3-form such that $d^*\beta \in d^* C_\delta^{k+1,\alpha}(\Lambda^3)$ then by the Hodge decomposition theorem 2.3.27 there is a unique $P_E(\beta) \in C_\delta^{k+1,\alpha}[d\Lambda^2]$ such that $d^*\beta = d^* P_E(\beta)$. We can think of $P_E(\beta)$ as the exact part of β. We then let $P_W(\beta)$ be the image of $P_E(\beta)$ under the projection $C_\delta^{k+1,\alpha}[d\Lambda^2] \to W$ in the direct sum decomposition (4.8).

Definition 4.2.16. For ψ close to φ in $\mathcal{Z}_{\mathcal{Q}}^3$ let

$$F(\psi) = P_W(*\Theta(\psi)). \tag{4.11}$$

Clearly $d\Theta(\psi) = 0 \Rightarrow F(\psi) = 0$. We need to show that that the converse also holds, so that we do not 'lose any information' by considering zeros of F instead of $\psi \mapsto d\Theta(\psi)$.

Proposition 4.2.17. *For $\psi \in \mathcal{Z}_{\mathcal{Q}}^3$ sufficiently close to φ, ψ is torsion-free if and only if $F(\psi) = 0$.*

Proof. We adapt the proof of proposition 3.2.10. For $\psi \in \mathcal{Z}_{\mathcal{Q}}^3$ the tangent space to the $\mathcal{D}_{\mathcal{Q}}^1$-orbit at ψ can be written as

$$T_\psi = d(C_\delta^{1,\alpha}(\Lambda_{7,\psi}^2) \oplus \rho(\mathcal{H}_\infty^1)^\sharp \lrcorner \psi),$$

where $\Lambda_{7,\psi}^2$ is the bundle of type 7 2-forms defined by the G_2-structure ψ. The linear map

$$W \oplus C_\delta^{1,\alpha}(TM) \oplus \mathcal{H}_\infty^1 \to C_\delta^{0,\alpha}[d\Lambda^2], \quad (w, V, \beta) \mapsto w + d((V + \rho\beta^\sharp) \lrcorner \psi)$$

is surjective at $\psi = \varphi$ by (4.8). Since it depends continuously on ψ,

$$C_\delta^{0,\alpha}[d\Lambda^2] = W + T_\psi$$

for any ψ sufficiently close to φ.

We deduce from theorem 2.2.10(ii) that $*d\Theta(\psi) = d^* P_E(*\Theta(\psi))$ is point-wise orthogonal to $\Lambda_{7,\psi}^2$ when $d\psi = 0$. Therefore $P_E(*\Theta(\psi))$ is L^2-orthogonal to T_ψ (integrating by parts). $F(\psi) = 0$ means that $P_E(*\Theta(\psi))$ is L^2-orthogonal to W too, so $P_E(*\Theta(\psi)) = 0$. $\square$

By proposition 4.2.7, $F : \mathcal{Z}_{\mathcal{Q}}^3 \to W$ is a smooth function. Next we show that it satisfies the hypotheses of the implicit function theorem.

Proposition 4.2.18. $DF_\varphi : \mathcal{Z}_{cyl}^3 \to W$ *is 0 on* $d(C_\delta^{k+1,\alpha}(\Lambda_7^2) \oplus D) \oplus \mathcal{H}_{cyl}^3$ *and* $-id$ *on* W.

Proof. Same as the compact version 3.2.12. $\square$

We have taken the pre-moduli space $\mathcal{R}_\delta$ near φ to consist of the torsion-free EAC G_2-structures in the slice $\mathcal{S}$. However, we can only prove that it has the properties we want close to φ. We will therefore repeatedly replace $\mathcal{S}$ by a neighbourhood of φ in $\mathcal{S}$ in order to ensure that $\mathcal{R}_\delta \subseteq S$ has the desired properties. The first step is to ensure that $\mathcal{R}_\delta$ is a

manifold. Proposition 4.2.17 shows that if $\mathcal{S}$ is sufficiently small then $\mathcal{R}_\delta$ is the zero set of F in $\mathcal{S}$. Applying the implicit function theorem to $F : \mathcal{S} \to W$ we deduce

Proposition 4.2.19. *If the slice $\mathcal{S}$ near φ is shrunk sufficiently then the space $\mathcal{R}_\delta$ of torsion-free EAC G_2-structures in $\mathcal{S}$ is a manifold. Its tangent space at φ is $\mathcal{H}^3_{cyl}$.*

The implicit function theorem implies also that a small neighbourhood of φ in $\mathcal{X}_{k,\mathcal{Q}}$, the space of torsion-free G_2-structures in $\mathcal{Z}^3_{\mathcal{Q}}$, is a manifold.

4.2.6 Regularity

To finish the proof of theorem 4.2.2 we need to show that if the slice $\mathcal{S}$ is taken sufficiently small then the torsion-free G_2-structures in $\mathcal{S}$ are smooth and asymptotically cylindrical. We first prove a regularity result about solutions of elliptic operators which are '$C^{l,\alpha}_\delta$-asymptotically cylindrical', and then generalise the regularity argument for the compact case from §3.2.5. We start by quoting an interior estimate.

Theorem 4.2.20. *Let $B_R, B_{2R} \subset \mathbb{R}^q$ be the balls centred on the origin of radius R and $2R$ respectively. Let A be an elliptic order r operator with $C^{l,\alpha}$ coefficients acting on the $\mathbb{R}^p$-valued functions on B_{2R}.*

If $u \in C^{r,\alpha}(B_{2R}, \mathbb{R}^p)$ and $Au \in C^{l,\alpha}(B_{2R}, \mathbb{R}^p)$ then $u|_{B_R} \in C^{l+r,\alpha}(B_R, \mathbb{R}^p)$. Furthermore

$$\|u|_{B_R}\|_{C^{l+r,\alpha}} \leq K \left(\|Au\|_{C^{l,\alpha}} + \|u\|_{C^0} \right), \tag{4.12}$$

where the constant K depends on q, k, α, the $C^{l,\alpha}$-norm of the coefficients of A and a lower bound for the operator norm of the symbol of A evaluated at unit vectors.

Proof. This is essentially a paraphrase of Theorems 6.2.5 and 6.2.6 in [45]. $\qquad\square$

Definition 4.2.21. Say that the differential operator A on M is $C^{l,\alpha}_\delta$-*asymptotic* to the translation-invariant operator A_∞ on the cylinder $X \times \mathbb{R}$ if the difference between the coefficients (as in definition 2.3.11) of A and A_∞ is $C^{l,\alpha}_\delta$.

It is well-known how to go from an interior estimate like theorem 4.2.20 to global estimates for appropriate norms on an asymptotically cylindrical manifold. We wish to use $C^{l,\alpha}_\delta$-norms. The theorem below is similar to for example [40, Theorem 6.3].

Theorem 4.2.22. *Let M^n be an asymptotically cylindrical manifold, E a vector bundle on M associated to TM and A a linear elliptic order r differential operator on the sections of E that is $C^{l,\alpha}_\delta$-asymptotic to a translation-invariant operator with $C^{l,\alpha}$ coefficients.*

If $s \in C^{r,\alpha}_\delta(E)$ and $As \in C^{l,\alpha}_\delta(E)$ then $s \in C^{l+r,\alpha}_\delta(E)$. Furthermore

$$\|s\|_{C^{l+r,\alpha}_\delta} < K \left(\|As\|_{C^{l,\alpha}_\delta} + \|s\|_{C^0_\delta} \right) \tag{4.13}$$

for some $K > 0$ independent of s.

Proof. Working just on the compact part M_0 of M this is easy. We can take coordinate charts that identify neighbourhoods $U \subset M_0$ with $B_{2R} \subset \mathbb{R}^n$ and apply theorem 4.2.20 to deduce that on the inverse image U' of B_R

$$\|s|_{U'}\|_{C^{l+r,\alpha}} < K \left(\|As|_U\|_{C^{l,\alpha}} + \|s|_U\|_{C^0} \right) \tag{4.14}$$

for some $K > 0$ independent of s. K is allowed to depend on U, but since we can cover M_0 by finitely many U we can take K large enough to ensure that

$$\|s|_{M_0}\|_{C^{l+r,\alpha}} < K \left(\|As|_{M_0}\|_{C^{l,\alpha}} + \|s|_{M_0}\|_{C^0} \right). \tag{4.15}$$

Now we consider the non-compact part $M_\infty = X \times \mathbb{R}^+$. Let V be a neighbourhood in X, $f : V \to B_{2R} \subset \mathbb{R}^{n-1}$ a diffeomorphism and V' the inverse image of B_R under f. For $t \in \mathbb{R}^+$ let I_t and I'_t be the intervals $(t - 2R, t + 2R), (t - R, t + R)$ respectively. By theorem 4.2.20

$$\|s|_{V' \times I'_t}\|_{C^{l+r,\alpha}} < K \left(\|As|_{V \times I_t}\|_{C^{l,\alpha}} + \|s|_{V \times I_t}\|_{C^0} \right). \tag{4.16}$$

Here the constant K depends on q, k, α, a bound on $V \times I_t$ for the $C^{l,\alpha}$-norm of the coefficients of A, a lower bound on $V \times I_t$ for the operator norm of the symbol of A at unit vectors and finally also on the diffeomorphism f (e.g. on how f distorts the metric). Since A is $C^{l,\alpha}_\delta$-asymptotically translation-invariant we can find global bounds for the $C^{l,\alpha}$-norms of the coefficients of A and for the operator norm of the symbol. Thus we can take K in (4.16) independent of t. Furthermore we can scale (4.16) by $e^{\delta t}$ in order to replace the norms by weighted norms. Hence we deduce

$$\|s|_{V' \times \mathbb{R}^+}\|_{C^{l+r,\alpha}_\delta} < K \left(\|As|_{V \times \mathbb{R}^+}\|_{C^{l,\alpha}_\delta} + \|s|_{V \times \mathbb{R}^+}\|_{C^0_\delta} \right).$$

Since we can cover M_∞ by finitely many strips $V' \times \mathbb{R}^+$ it follows that we can take K large enough that

$$\|s|_{M_\infty}\|_{C^{l+r,\alpha}_\delta} < K \left(\|As|_{M_\infty}\|_{C^{l,\alpha}_\delta} + \|s|_{M_\infty}\|_{C^0_\delta} \right).$$

Together with (4.15) this proves the theorem. □

Remark 4.2.23. With some additional algebra theorem 4.2.22 can be a generalised to show regularity of '$C_\delta^{r,\alpha}$-asymptotically translation-invariant' solutions.

Now consider again a G_2-manifold M with a torsion-free G_2-structure φ, and the pre-moduli space $\mathcal{R}_\delta$ of torsion-free G_2-structures in the slice $\mathcal{S} = \exp(K \cap U) \subseteq \mathcal{Z}_\mathcal{Q}^3$ for the $\mathcal{D}_\mathcal{Q}$-action at φ. We use theorem 4.2.22 in a boot-strapping argument to show that the elements of $\mathcal{R}_\delta$ are EAC. A priori they are $C_\delta^{k,\alpha}$-asymptotic to elements of $\mathcal{Q}_A$. As in proposition 4.2.19 we can only work close to φ, and must replace $\mathcal{S}$ by a neighbourhood of φ in $\mathcal{S}$.

Proposition 4.2.24. *If the slice $\mathcal{S}$ near φ is shrunk sufficiently then the pre-moduli space $\mathcal{R}_\delta \subseteq \mathcal{S}$ consists of smooth exponentially asymptotically translation-invariant forms.*

Proof. We want to show that if $\psi \in \mathcal{S}$ is sufficiently close to φ and $d\Theta(\psi) = 0$ then ψ is smooth and exponentially asymptotically translation-invariant. Set $\psi = \varphi + \beta$. Then

$$D(d*d\Theta)_\varphi = -dd^* \circ (\tfrac{4}{3}\pi_1 + \pi_7 - \pi_{27}),$$

so we can write

$$d*d\Theta(\varphi + \beta) = -dd^*(\tfrac{4}{3}\pi_1\beta + \pi_7\beta - \pi_{27}\beta) + P(\beta, \nabla\beta, \nabla^2\beta) + R(\beta, \nabla\beta),$$

where P consists of the quadratic terms of $d*d\Theta(\varphi + \beta)$ that involve $\nabla^2\beta$, and R consists of the remaining quadratic terms. P and R depend only point-wise on their arguments and P is linear in $\nabla^2\beta$.

By the definition of the map $\exp$ (4.9) we can write $\beta = \kappa + \gamma$, with $\kappa \in W$ and γ smooth and exponentially asymptotic to some element of $\mathcal{Q}_A$. As κ is closed of type 27

$$-dd^*(\tfrac{4}{3}\pi_1\kappa + \pi_7\kappa - \pi_{27}\kappa) = \triangle\kappa.$$

Considering β and $\nabla\beta$ as fixed we can define a second-order linear differential operator $A : \zeta \mapsto P(\beta, \nabla\beta, \nabla^2\zeta)$. Then the condition $d*d\Theta(\psi) = 0$ is equivalent to

$$(\triangle + A)\kappa = -R + dd^*(\tfrac{4}{3}\pi_1\gamma + \pi_7\gamma - \pi_{27}\gamma) - A\gamma. \qquad (4.17)$$

If $\beta = 0$ then $A = 0$, so $\triangle + A$ is elliptic. Ellipticity is an open condition, so $\triangle + A$ is in fact elliptic for any β sufficiently small in the uniform norm.

Now suppose κ is $C_\delta^{l+1,\alpha}$ and is a solution of (4.17). R and the coefficients of A depend smoothly on κ and $\nabla\kappa$. Therefore $\triangle + A$ and the RHS of (4.17) are $C_\delta^{l,\alpha}$-asymptotically translation-invariant. Since the RHS of (4.17) is decaying *a priori* it is $C_\delta^{l,\alpha}$. If β is sufficiently small that $\triangle + A$ is elliptic then by theorem 4.2.22 κ is $C_\delta^{l+2,\alpha}$. Since κ is $C_\delta^{1,\alpha}$ it is $C_\delta^{l,\alpha}$ for all l by induction.

Hence $\psi = \varphi + \kappa + \gamma$ is smooth and exponentially asymptotically translation-invariant.

$\square$

4.2.7 Constructing the moduli space

For each $\varphi \in \mathcal{X}_\delta'$ we have constructed a pre-moduli space $\mathcal{R}_\delta$. $\mathcal{R}_\delta$ is a manifold, its elements are smooth and EAC and its tangent space at φ is $\mathcal{H}_{cyl}^3$. To complete the proof of the main theorem 4.2.2 we now provide slice arguments to show that we can take the pre-moduli spaces $\mathcal{R}_\delta$ as coordinate charts to define a smooth structure on $\mathcal{M}_\delta$. We use the same ideas as in the compact case and study the charts for $\mathcal{M}_+$ in terms of the projection to the de Rham cohomology which appears in the statement of the main theorem 4.2.2.

$$\pi_\mathcal{M} : \mathcal{X}_+ \to H^3(M) \times H^2(X), \quad \varphi \mapsto ([\varphi], [B_e(\varphi)]).$$

We first check that $\pi_\mathcal{M}$ is an embedding on $\mathcal{R}_\delta$. If we allow ourselves to shrink $\mathcal{R}_\delta$ this amounts to showing that the derivative of $\pi_\mathcal{M}$ at φ is injective. The derivative is

$$(\pi_H, \pi_{H,e}) : \mathcal{H}_{cyl}^3 \to H^3(M) \times H^2(X), \quad \beta \mapsto ([\beta], [B_e(\beta)])$$

and the kernel consists of harmonic, exact, decaying forms, so it is trivial.

Recall from subsection 4.2.2 that we chose a pre-moduli space $\mathcal{Q}$ near the Calabi-Yau structure (Ω, ω) on X defined by the asymptotic limit of φ, and that $\mathcal{X}_\mathcal{Q} \subseteq \mathcal{X}_\delta'$ is the subset of torsion-free G_2-structures whose asymptotic limits lie in $\mathcal{Q}$. $\mathcal{D}_\mathcal{Q} \subseteq \mathcal{D}_\delta$ is the subgroup of smooth EAC diffeomorphisms of M whose asymptotic limits lie in the automorphism group of (Ω, ω). $\mathcal{D}_\mathcal{Q}$ acts on $\mathcal{X}_\mathcal{Q}$ by proposition 3.1.10 and as an intermediate step for our slice result we prove that $\mathcal{R}_\delta$ is a coordinate chart for $\mathcal{X}_\mathcal{Q}/\mathcal{D}_\mathcal{Q}$.

Proposition 4.2.25. *The natural map $\mathcal{R}_\delta \to \mathcal{X}_\mathcal{Q}/\mathcal{D}_\mathcal{Q}$ is a homeomorphism onto a neighbourhood of $\varphi\mathcal{D}_\mathcal{Q}$.*

Proof. Recall that a small neighbourhood of φ in $\mathcal{X}_{k,\mathcal{Q}}$, the space of torsion-free G_2-structures in $\mathcal{Z}_\mathcal{Q}^3$, is a manifold. Its tangent space at φ is $d(C_\delta^{k+1,\alpha}(\Lambda_7^2) \oplus D) \oplus \mathcal{H}_{cyl}^3$ by proposition

4.2.18. The first term is the tangent space to the $\mathcal{D}_{\mathcal{Q}}^{k+1}$-orbit of φ, so the derivative at (φ, id) of

$$\mathcal{R}_\delta \times \mathcal{D}_{\mathcal{Q}}^{k+1} \to \mathcal{Z}_{\mathcal{Q}}^3, \quad (\beta, \phi) \mapsto \phi^* \beta$$

is surjective. By the submersion theorem it is an open map on a neighbourhood of (φ, id). Using the regularity of isometries of EAC manifolds from proposition 2.3.7 it follows (as in corollary 3.1.6) that $\mathcal{R}_\delta \to \mathcal{X}_{\mathcal{Q}}/\mathcal{D}_{\mathcal{Q}}$ is an open map. It is also injective, since $\pi_{\mathcal{M}}$ is $\mathcal{D}_\delta$-invariant and injective on $\mathcal{R}_\delta$. $\qquad\square$

For our argument to work we may need to shrink $\mathcal{Q}$ by replacing it with a neighbourhood of (Ω, ω) in $\mathcal{Q}$.

Lemma 4.2.26. *If the pre-moduli $\mathcal{Q}$ of Calabi-Yau structures is shrunk sufficiently then $\mathcal{X}_{\mathcal{Q}}/\mathcal{D}_{\mathcal{Q}}$ is homeomorphic to a neighbourhood of $\varphi \mathcal{D}_\delta$ in $\mathcal{X}'_\delta/\mathcal{D}_\delta$.*

Proof. The natural map $f : \mathcal{X}_{\mathcal{Q}}/\mathcal{D}_{\mathcal{Q}} \to \mathcal{X}'_\delta/\mathcal{D}_\delta$ is injective by proposition 3.1.10.

Let $\mathcal{Y}$ be the space of Calabi-Yau structures on X. One of the properties of the pre-moduli space $\mathcal{Q}$ is that there is a neighbourhood U of (Ω, ω) in $\mathcal{Y}$ and a continuous map

$$P : U \to C^\infty(TX) \times \mathcal{Q}, \quad (\beta, \gamma) \mapsto (V, \Omega', \omega')$$

such that $(\beta, \gamma) = (\exp V)^*(\Omega', \omega')$ for any $(\beta, \gamma) \in U$ (the map is given by restricting a local right inverse of the submersion (3.2) to a small U).

Let $\mathcal{X}_U = \{\psi \in \mathcal{X}'_\delta : B(\psi) \in U\}$. If $\psi \in \mathcal{X}_U$ let $V = P(B(\psi))$, $\phi = \exp \rho V \in \mathcal{D}_\delta$ and $g(\psi) = \phi^*\psi$. Then $B(g(\psi)) \in \mathcal{Q}$, so $\psi \in \mathcal{X}_{\mathcal{Q}}$. Obviously $f(g(\psi)\mathcal{D}_{\mathcal{Q}})) = \psi \mathcal{D}_\delta$. Since f is injective g induces a well-defined map $\mathcal{X}_U \mathcal{D}_\delta \to \mathcal{X}_{\mathcal{Q}}/\mathcal{D}_{\mathcal{Q}}$. g is an inverse for f on a neighbourhood of $\varphi \mathcal{D}_\delta$ in $\mathcal{X}'_\delta/\mathcal{D}_\delta$, so the result follows. $\qquad\square$

Theorem 4.2.27. $\mathcal{M}_\delta$ *has a unique smooth structure such that*

$$\pi_{\mathcal{M}} : \mathcal{M}_\delta \to H^3(M) \times H^2(X)$$

is an immersion.

Proof. Same as theorem 3.1.9. $\qquad\square$

If $\delta_1 > \delta_2 > 0$ then $\mathcal{M}_{\delta_1} \to \mathcal{M}_{\delta_2}$ is injective by proposition 2.3.7, and $\mathcal{M}_{\delta_1}$ must be an open submanifold of $\mathcal{M}_{\delta_2}$ since $\pi_{\mathcal{M}}$ is an immersion on both spaces. Similarly $\mathcal{M}_\delta \to \mathcal{M}_+$

is injective for any $\delta > 0$, so

$$\mathcal{M}_+ = \bigcup_{\delta > 0} \mathcal{M}_\delta.$$

To finish the proof of the main theorem 4.2.2 it remains only to observe

Lemma 4.2.28. *For any $\delta > 0$ the natural map $\mathcal{M}_\delta \to \mathcal{M}_+$ is a homeomorphism to an open subset.*

Proof. We need to show that $\mathcal{M}_\delta \to \mathcal{M}_+$ is open, i.e. that if $U \subseteq \mathcal{X}'_\delta$ with $U\mathcal{D}_\delta$ open in $\mathcal{X}_\delta$ then $U\mathcal{D}_+$ is open in $\mathcal{X}_+$. By the definition of the topology on $\mathcal{X}_+$ this means that $U\mathcal{D}_+ \cap \mathcal{X}_{\delta'}$ is open in $\mathcal{X}_{\delta'}$ for any $\delta' > \delta$. But proposition 2.3.7 implies that $U\mathcal{D}_+ \cap \mathcal{X}_{\delta'} = U\mathcal{D}_{\delta'}$, which is open in $\mathcal{X}_{\delta'}$ since $\mathcal{M}_\delta \to \mathcal{M}_{\delta'}$ is a local diffeomorphism. $\qquad\square$

This concludes the proof of theorem 4.2.2.

4.2.8 Properties of the moduli space

We look at some local properties of the moduli space $\mathcal{M}_+$ on an EAC G_2-manifold M, which follow from the existence of a pre-moduli space $\mathcal{R}$ with tangent space $\mathcal{H}^3_{cyl}$.

First, the boundary map B maps $\mathcal{H}^3_{cyl}$ onto $\mathcal{H}^3_{SU,A}$, so proposition 4.2.9 implies that $B : \mathcal{R} \to \mathcal{Q}_A$ is a submersion. As $\mathcal{Q}_A$ is homeomorphic to an open set in $\mathcal{N}_A$ it follows that $B : \mathcal{M}_+ \to \mathcal{N}_A$ is a submersion onto its image and we have proved theorem 4.2.4.

We can now deduce corollary 4.2.5. It suffices to show that the fibres of $B : \mathcal{M}_+ \to \mathcal{N}_A$ are locally diffeomorphic to the compactly supported subspace $H^3_0(M) \subseteq H^3(M)$.

Lemma 4.2.29. *Let φ be an EAC torsion-free G_2-structure on M^7, $\mathcal{R}$ the pre-moduli space of EAC torsion-free G_2-structures near φ and $\mathcal{Q}$ the pre-moduli space of Calabi-Yau structures near $B(\varphi)$.*

$$\pi_H : \mathcal{R} \to H^3(M)$$

maps a neighbourhood of the fibre of $B : \mathcal{R} \to \mathcal{Q}_A$ containing φ diffeomorphically to an open subset of the affine space $[\varphi] + H^3_0(M)$.

Proof. If ψ is in the same fibre as φ then $\psi - \varphi$ is exponentially decaying, so lemma 2.3.36 implies that $[\psi - \varphi] \in H^3_0(M)$. The tangent space to the fibre at φ is the kernel of the derivative of the submersion B, i.e. the subspace $\mathcal{H}^3_+$ of decaying forms in $\mathcal{H}^3_{cyl} = T_\varphi \mathcal{R}$. By theorem 2.3.35 $\mathcal{H}^3_+ \cong H^3_0(M)$, and the result follows. $\qquad\square$

We can also use the pre-moduli space charts to check that $\mathcal{M}_+ \to H^3(M) \oplus H^4(M)$ restricts to Lagrangian immersions of the fibres of the boundary map. To prove proposition 4.2.6 it suffices to prove that for each φ the derivative is a Lagrangian embedding on the tangent space to the pre-moduli space.

Lemma 4.2.30. *Let φ be an EAC torsion-free G_2-structure on M^7 and $\mathcal{R}$ the pre-moduli space near φ. Then the derivative of $\mathcal{R} \to H^3(M) \oplus H^4(M)$ embeds the tangent space at φ to the fibre of $B : \mathcal{R} \to \mathcal{Q}_A$ as a Lagrangian subspace of $H_0^3(M) \oplus H_0^4(M)$.*

Proof. As above the tangent space to the fibre of φ is $T_\varphi \mathcal{R} = \mathcal{H}_+^3$. Since harmonic 3-forms of type 1 or 7 are parallel the elements of $\mathcal{H}_+^3$ have type 27. By proposition 2.2.4, the derivative on the fibre is

$$\mathcal{H}_+^3 \to H_0^3(M) \oplus H_0^4(M), \quad \chi \mapsto ([\chi], -[*\chi]).$$

If $\chi_1, \chi_2 \in \mathcal{H}_+^3$ then the symplectic pairing of their images in $H_0^3(M) \oplus H_0^4(M)$ is just

$$<\chi_1, -\chi_2>_{L^2} - <-\chi_1, \chi_2>_{L^2} = 0.$$

Hence the derivative on the fibre is a Lagrangian inclusion. $\qquad\square$

Finally, to confirm the formula for the dimension in proposition 4.2.3 we just have to compute the dimension of $\mathcal{H}_{cyl}^3$. Recall from subsection 2.3.4 that A^m is the image of $j^* : H^m(M) \to H^m(X)$, that $H^m(X) = A^m \oplus E^m$ is an orthogonal direct sum and that the harmonic representatives of the summands are denoted by $\mathcal{A}^m$ and $\mathcal{E}^m$ respectively.

Lemma 4.2.31. *Let M^{4k+3} be an oriented EAC manifold with cross-section X. Then $A^{2k+1} \subseteq H^{2k+1}(X)$ has dimension $\frac{1}{2}b^{2k+1}(X)$.*

Proof. $H^{2k+1}(X)$ is a symplectic vector space under the Poincaré pairing. In particular $b^{2k+1}(X)$ is even. $* : H^{2k+1}(X) \to H^{2k+1}(X)$ maps A^{2k+1} isomorphically to its orthogonal complement E^{2k+1}. The Poincaré pairing on $H^{2k+1}(X)$ can be expressed as $< \cdot, *\cdot >$, so $A^{2k+1} \subseteq H^{2k+1}(X)$ is a Lagrangian subspace. $\qquad\square$

In particular for any EAC G-manifold M with cross-section X the long exact sequence (2.32) for relative cohomology gives

$$\dim H_0^3(M) = b^3(M) - \tfrac{1}{2}b^3(X). \tag{4.18}$$

Lemma 4.2.32. $\dim \mathcal{H}^3_{cyl} = b^4(M) + \frac{1}{2}b^3(X) - b^1(M) - 1$.

Proof. As before we let $E^2_d = E^2 \cap H^2_d(X)$. As a consequence of corollary 4.2.10 and theorem 2.3.33 we find that $\pi_H : \mathcal{H}^3_{cyl} \to H^3(M)$ is surjective and the kernel is mapped bijectively to E^2_8 by $\pi_{H,e} : \mathcal{H}^3_{cyl} \to H^2(X)$. Hence $\dim \mathcal{H}^3_{cyl} = b^3(M) + \dim E^2_8$.

The dimension of E^2 can be computed from the long exact sequence (2.32) for relative cohomology together with (4.18).

$$\dim E^2 = \dim \ker(e : H^3_{cpt}(M) \to H^3(M))$$
$$= b^4(M) - \dim H^3_0(M) = b^4(M) - b^3(M) + \tfrac{1}{2}b^3(X).$$

By propositions 4.1.5 and 2.3.41

$$\dim E^2_6 = \dim A^1 = b^1(M), \quad \dim E^2_1 = \dim A^0 = 1.$$

Hence
$$\dim E^2_8 = b^4(M) - b^3(M) + \tfrac{1}{2}b^3(X) - b^1(M) - 1. \qquad \square$$

4.3 Deformations of EAC $Spin(7)$-manifolds

In this section we prove that the moduli space of torsion-free exponentially asymptotically cylindrical $Spin(7)$-structures on a manifold with cylindrical ends is smooth and study some of its local properties. The method is very similar to that for the G_2 case.

4.3.1 Results

Let M^8 be a connected oriented manifold with cylindrical ends and cross-section X^7. For $\delta > 0$ let $\mathcal{X}_\delta$ be the space of torsion-free EAC $Spin(7)$-structures with rate δ on M (see definition 4.1.2). $\mathcal{X}_\delta$ has the topology of a subspace of the space of exponentially asymptotically translation-invariant 4-forms.

Let $\mathcal{X}_+ = \bigcup_{\delta>0} \mathcal{X}_\delta$. If $\delta_1 > \delta_2 > 0$ then the inclusion $\mathcal{X}_{\delta_1} \hookrightarrow \mathcal{X}_{\delta_2}$ is continuous, so we can give $\mathcal{X}_+$ the direct limit topology. Let $\mathcal{D}_+$ be the group of EAC diffeomorphisms of M with any positive rate (in the sense of definition 2.3.6) that are isotopic to the identity. $\mathcal{D}_+$ acts on $\mathcal{X}_+$ by pull-backs, and the *moduli space of torsion-free EAC $Spin(7)$-structures* on M is the quotient $\mathcal{M}_+ = \mathcal{X}_+/\mathcal{D}_+$.

Remark 4.3.1. The definition of an EAC $Spin(7)$-structure involves a normalisation of the asymptotic limit (cf. remark 4.2.1).

The main result of this section is

Theorem 4.3.2. $\mathcal{M}_+$ *is a smooth manifold and* $\pi_H : \mathcal{M}_+ \to H^4(M)$ *is an immersion.*

Since M is non-compact, the notion of self-duality does not make sense for arbitrary classes in $H^4(M)$, but only for compactly supported ones. Let $H^4_+(M) \subseteq H^4_0(M)$ be the self-dual subspace and denote its dimension by $b^4_+(M)$. The dimension of the moduli space is given by

Proposition 4.3.3. $\dim \mathcal{M}_+ = b^4(M) - b^4_+(M) - b^1(M) + 1 + b^1(X)$.

Remark 4.3.4. The term $b^4(M) - b^4_+(M)$ is the dimension of the space of bounded anti-self-dual forms, while $-b^1(M) + 1 + b^1(X)$ is the dimension of the space of bounded parallel positive spinors on the $Spin(7)$-manifold M. This is analogous to the terms in the dimension formula for the moduli space on a compact $Spin(7)$-manifold (see remark 3.3.2).

The asymptotic limit of a torsion-free EAC $Spin(7)$-structure is of the form $*_\varphi \varphi + dt \wedge \varphi$, where φ is a torsion-free G_2-structure on the cross-section X^7 and $*_\varphi$ the induced Hodge star on X (see proposition 2.2.26). This gives a natural boundary map $B : \mathcal{M}_+ \to \mathcal{N}$ where $\mathcal{N}$ is the moduli space of torsion-free G_2-structures on X. As before, let $A^m \subseteq H^m(X)$ denote the image of the pull-back map $j^* : H^m(M) \to H^m(X)$. If a torsion-free $Spin(7)$-structure ψ is asymptotic to $*_\varphi \varphi + dt \wedge \varphi$ then

$$[*_\varphi \varphi] = j^*([\psi]) \in A^4,$$

so the image of $B : \mathcal{M}_+ \to \mathcal{N}$ is contained in a subset determined by the topology of the pair (M, X)

$$\mathcal{N}_A = \{\varphi \mathcal{D}_X \in \mathcal{N} : [*\varphi] \in A^4\}. \tag{4.19}$$

$\mathcal{N} \to H^4(X)$ is a local diffeomorphism by theorem 3.2.1, so $\mathcal{N}_A$ is a submanifold of $\mathcal{N}$.

Theorem 4.3.5.

$$B : \mathcal{M}_+ \to \mathcal{N}_A \tag{4.20}$$

is a submersion.

Since the methods used are entirely local they do not tell us anything about the global properties of $\mathcal{M}_+$ or the image of (4.20).

We will show that the fibres of the submersion (4.20) are locally diffeomorphic to the compactly supported anti-self-dual cohomology $H^4_-(M) \subseteq H^4_0(M)$. The fibre over φ corresponds to the moduli space of torsion-free $Spin(7)$-structures asymptotic to $*\varphi + dt \wedge \varphi$. Thus

Corollary 4.3.6. *The moduli space of torsion-free $Spin(7)$-structures on M exponentially asymptotic to a fixed cylindrical $Spin(7)$-structure on $X \times \mathbb{R}$ is a manifold locally diffeomorphic to $H^4_-(M)$.*

4.3.2 Proof outline

We use a set-up entirely analogous to the G_2 case. Each $\psi \in \mathcal{X}_\delta$ defines an EAC metric, and hence a parameter $\epsilon_1(\psi)$ (cf. proposition 2.3.21). Let

$$\mathcal{X}'_\delta = \{\psi \in \mathcal{X}_\delta : \epsilon_1(\psi) > \delta\}.$$

ϵ_1 depends lower semi-continuously on the asymptotic model by lemma 2.3.19, so $\mathcal{X}'_\delta$ is an open subset of $\mathcal{X}_\delta$. Let $\mathcal{D}_\delta$ be the group of exponentially cylindrical diffeomorphisms of M with rate δ, and $\mathcal{M}_\delta = \mathcal{X}'_\delta / \mathcal{D}_\delta$.

We now consider how to adapt the steps of section 3.2 to construct pre-moduli spaces of torsion-free EAC $Spin(7)$-structures near $\psi \in \mathcal{X}'_\delta$.

Given $\psi \in \mathcal{X}'_\delta$, the asymptotic limit has the form $B(\psi) = *\varphi + dt \wedge \varphi$ where φ is a torsion-free G_2-structure on X. In §3.2 we saw that there is a pre-moduli space of torsion-free G_2-structures near φ which we now denote by $\mathcal{Q}$. We can identify $\mathcal{Q}$ with a space of torsion-free cylindrical $Spin(7)$-structures on $X \times \mathbb{R}$, and let

$$\mathcal{X}_\mathcal{Q} = \{\psi \in \mathcal{X}'_\delta : B(\psi) \in \mathcal{Q}\}.$$

Note that the boundary values of elements of $\mathcal{X}_\mathcal{Q}$ must lie in

$$\mathcal{Q}_A = \{*\varphi' + dt \wedge \varphi' \in \mathcal{Q} : [*\varphi'] \in A^4\} \tag{4.21}$$

where A^m is the image of $j^* : H^m(M) \to H^m(X)$ (cf. discussion before theorem 4.3.5). Let ρ be a cut-off function for the cylinder on M. $\rho \mathcal{Q}_A$ can be identified with a subspace of smooth asymptotically translation-invariant 4-forms supported on the cylinder of M. Let

$$\mathcal{C}^4_\mathcal{Q} \subseteq C^{k,\alpha}_\delta(\Lambda^4) + \rho \mathcal{Q}_A$$

be the subspace of forms which take values in $\Lambda_{Spin(7)}T^*M$, in other words the space of $Spin(7)$-structures which are $C_\delta^{k,\alpha}$-asymptotic to elements of $\mathcal{Q}_A$. Then $\mathcal{X}_\mathcal{Q} \hookrightarrow \mathcal{C}_\mathcal{Q}^4$ continuously.

Since $\mathcal{Q} \to H^4(X)$ is a local diffeomorphism it is, unlike in the G_2 case, immediately clear that $\mathcal{Q}_A \subseteq \mathcal{Q}$ is a submanifold. Its tangent space can be considered as a subspace of the translation-invariant harmonic 4-forms on the cylinder,

$$\mathcal{H}_{G_2,A}^4 = T_\varphi \mathcal{Q}_A \subset \mathcal{H}_\infty^4.$$

We will see that it is precisely the image under the boundary map B of the space $\mathcal{H}_{cyl}^4$ of harmonic tangents at ψ to the space of $C_\delta^{k,\alpha}$-asymptotically translation-invariant $Spin(7)$-structures.

In the next subsections we describe how to set up a slice and find a smooth pre-moduli space $\mathcal{R}_\delta$ near ψ with $T_\psi \mathcal{R}_\delta = \mathcal{H}_{cyl}^4$, generalising from the compact case. The regularity and slice arguments that complete the proof of theorem 4.3.2 are very similar to the G_2 case and therefore omitted. Just as in the G_2 case, one finds that the elements of $\mathcal{R}_\delta$ are smooth, that $\mathcal{R}_\delta \to \mathcal{M}_\delta$ can be used as a coordinate chart, and that $\mathcal{M}_+ = \bigcup_{\delta>0} \mathcal{M}_\delta$ is therefore a smooth manifold.

Once the coordinate charts are set up it is easy to deduce proposition 4.3.3, theorem 4.3.5 and corollary 4.3.6. This is done in §4.3.5.

4.3.3 The slice

Fix $k \geq 1$, $\delta > 0$, $\alpha \in (0,1)$ and $\psi \in \mathcal{X}_\delta'$. We find a submanifold $\mathcal{S} \subseteq \mathcal{C}_\mathcal{Q}^4$ whose tangent space at ψ is a direct complement to the diffeomorphism orbit.

The fixed $Spin(7)$-structure ψ is used to define an EAC metric and a Hodge star. It also defines type decompositions of the exterior bundles (3.16) and of the spaces of harmonic forms, as described in subsection 2.1.2. Note that the map $TM \to \Lambda_8^3$, $v \mapsto v \lrcorner \psi$ is a bundle isomorphism. Therefore Lie derivatives $\mathcal{L}_V \psi = d(V \lrcorner \psi)$ are precisely exterior derivatives of 3-forms of type 8.

Recall the notation for harmonic forms from the summary in §4.1.1, in particular that $\mathcal{H}_0^m$ and $\mathcal{H}_\infty^m$ denote the spaces of bounded harmonic forms on M and harmonic translation-invariant forms on $X \times \mathbb{R}$ respectively. G_2-structures on X are identified with cylindrical $Spin(7)$-structures on $X \times \mathbb{R}$ by $\varphi' \leftrightarrow *_{\varphi'}\varphi' + dt \wedge \varphi'$.

The torsion-free G_2-structure φ on the cross-section X^7 induced by ψ defines type decompositions of $\Omega^*(X)$. In view of proposition 2.2.4, the tangent space to $\mathcal{Q}_A$ corresponds to

$$\mathcal{H}^4_{G_2,A} = \{\chi + dt \wedge *(\tfrac{3}{4}\pi_1 + \pi_7 - \pi_{27})\chi \; : \; \chi \in \mathcal{A}^4\} \subseteq \mathcal{H}^4_\infty.$$

Recall that the tangent space to the space of $Spin(7)$-structures consists of sections of $\Lambda^4_{1\oplus7\oplus35}$. $T_\psi \mathcal{C}^4_\mathcal{Q}$ is therefore the space of 4-forms of type 1, 7 and 35 which are $C^{k,\alpha}_\delta$-asymptotic to elements of $\mathcal{H}^4_{G_2,A}$. Let $\mathcal{H}^4_{cyl} \subset T_\psi \mathcal{C}^4_\mathcal{Q}$ be the subspace of harmonic forms.

Lemma 4.3.7. $B : \mathcal{H}^4_{cyl} \to \mathcal{H}^4_{G_2,A}$ *is surjective.*

Proof. Proposition 4.1.8 implies that $\mathcal{A}^4$ splits into type components, so $\mathcal{H}^4_{G_2,A}$ splits as

$$\mathcal{H}^4_{G_2,A} = \{\chi + dt \wedge *\tfrac{3}{4}\chi : \chi \in \mathcal{A}^4_1\} \oplus \{\chi + dt \wedge *\chi : \chi \in \mathcal{A}^4_7\} \oplus \{\chi - dt \wedge *\chi : \chi \in \mathcal{A}^4_{27}\}.$$

It implies also that $\mathcal{H}^4_{0,7} \subset \mathcal{H}^4_{cyl}$ is mapped onto the middle term by the boundary map B.

For $\chi \in \mathcal{A}^4_{27}$, the EAC Hodge decomposition theorem 2.3.33 implies that there is some $\phi \in \mathcal{H}^4_{abs}$ such that $B(\phi) = \chi$. Then $\phi - *\phi$ lies in the space $\mathcal{H}^4_{0,35}$ of bounded harmonic anti-self-dual forms and $B(\phi - *\phi) = \chi - dt \wedge *\chi$, so $\phi - *\phi \in \mathcal{H}^4_{cyl}$. Finally, $*\varphi + dt \wedge \tfrac{3}{4}\varphi$ can be written as $\tfrac{7}{8}B(\psi) + \tfrac{1}{8}(*\varphi - dt \wedge \varphi)$, and the second term lies in the image of $\mathcal{H}^4_{0,35}$. $\square$

Let $\mathcal{D}^{k+1}_\mathcal{Q}$ be the group of diffeomorphisms of M which are isotopic to the identity and $C^{k+1,\alpha}_\delta$-asymptotic to a cylindrical automorphism of the cylindrical $Spin(7)$-structure $*_\varphi\varphi + dt \wedge \varphi$. $\mathcal{D}^{k+1}_\mathcal{Q}$ acts on $\mathcal{X}_\mathcal{Q}$ by proposition 3.1.10. The elements of a neighbourhood of the identity in $\mathcal{D}^{k+1}_\mathcal{Q}$ can be written as $\exp(V + \rho V_\infty)$, where V is a $C^{k+1,\alpha}_\delta$ vector field on M and V_∞ is a translation-invariant vector field on $X \times \mathbb{R}$ with $\mathcal{L}_{V_\infty}(*_\varphi\varphi + dt \wedge \varphi) = 0$, i.e. $V_\infty \in (\mathcal{H}^1_\infty)^\sharp$. Therefore if we set

$$D = \rho(\mathcal{H}^1_\infty)^\sharp \lrcorner \psi \subseteq \Omega^3_8(M)$$

then the tangent space to the $\mathcal{D}^{k+1}_\mathcal{Q}$-orbit at ψ is

$$\{\mathcal{L}_{V+\rho V_\infty}\psi : V \in C^{k+1,\alpha}_\delta(TM), V_\infty \in (\mathcal{H}^1_\infty)^\sharp\} = d(C^{k+1,\alpha}_\delta(\Lambda^3_8) \oplus D).$$

Definition 4.3.8. Let $W \subseteq C^{k,\alpha}_\delta(\Lambda^4_{35})$ be the L^2-orthogonal subspace to the decaying harmonic forms $\mathcal{H}^4_+$.

Proposition 4.3.9.
$$T_\psi \mathcal{C}^4_\mathcal{Q} = d(C^{k+1,\alpha}_\delta(\Lambda^3_8) \oplus D) \oplus \mathcal{H}^4_{cyl} \oplus W.$$

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Proof. The Hodge decomposition theorem 2.3.27 implies that a non-zero decaying exact form cannot be anti-self-dual, so $W \cap d(C_\delta^{k+1,\alpha}(\Lambda_8^3) \oplus D) = 0$ and the sum is direct. It follows from lemma 4.3.7 that any element of $T_\psi \mathcal{C}_\mathcal{Q}^4$ can be written as $\phi + \beta$, with $\phi \in \mathcal{H}_{cyl}^4$ and $\beta \in C_\delta^{k,\alpha}(\Lambda_{1\oplus7\oplus35}^4)$ L^2-orthogonal to the decaying harmonic forms. It therefore suffices to show that $\pi_{1\oplus7}\beta$ lies in the image of

$$\pi_{1\oplus7}d : C_\delta^{k+1,\alpha}(\Lambda_8^3) \oplus D \to C_\delta^{k,\alpha}(\Lambda_{1\oplus7}^4).$$

Just as in the compact case proposition 3.3.3 this is easily proved by interpreting the map as the Dirac operator (cf. (2.39) is Fredholm). $\qquad\square$

We want to take as our slice for the $\mathcal{D}_\delta$-action at ψ a submanifold $\mathcal{S} \subseteq \mathcal{C}_\mathcal{Q}^4$ with $T_\psi \mathcal{S} = \mathcal{H}_{cyl}^4 \oplus W$. To this end we pick a map

$$\exp : U \to \mathcal{C}_\mathcal{Q}^4 \tag{4.22}$$

on a neighbourhood U of 0 in $T_\psi \mathcal{C}_\mathcal{Q}^4$, such that $D\exp_0 = id$. We also insist that exp maps decaying forms to decaying forms and smooth forms to smooth forms. We can do this since the decaying forms have a finite-dimensional complement of smooth forms in $T_\psi \mathcal{C}_\mathcal{Q}^4$. We then choose

$$\mathcal{S} = \exp((\mathcal{H}_{cyl}^4 \oplus W) \cap U). \tag{4.23}$$

4.3.4 Smoothness of the pre-moduli space

Let $\mathcal{R}_\delta \subseteq \mathcal{S}$ be the subset of $C_\delta^{k,\alpha}$ torsion-free $Spin(7)$-structures. $\mathcal{R}_\delta$ is the pre-moduli space of torsion-free $Spin(7)$-structures near ψ.

If $\psi' \in \mathcal{C}_\mathcal{Q}^4$ then ψ' is asymptotic to a torsion-free cylindrical $Spin(7)$-structure, so $d\psi'$ decays. Moreover, elements of $\mathcal{C}_\mathcal{Q}^4$ are asymptotic to elements of $\mathcal{Q}_A \subseteq \mathcal{Q}$ by definition. Therefore ψ' is asymptotic to $*_{\varphi'}\varphi' + dt \wedge \varphi'$ with $[*_{\varphi'}\varphi'] \in A^4$, so that $d\psi' \in dC_\delta^{k+1,\alpha}(\Lambda^4)$.

Recall that by theorem 2.2.17, a closed $Spin(7)$-structure ψ' is torsion-free if and only if $d\psi' = 0$. Thus $\mathcal{R}_\delta$ is the zero set of

$$d : \mathcal{S} \to dC_\delta^{k+1,\alpha}(\Lambda^4).$$

To show that $\mathcal{R}_\delta$ is a manifold we check that the derivative at ψ is surjective.

Proposition 4.3.10. *The map $d : T_\psi \mathcal{C}_\mathcal{Q}^4 \to dC_\delta^{k+1,\alpha}(\Lambda^4)$ is 0 on $d(C_\delta^{k+1,\alpha}(\Lambda_7^2) \oplus D) \oplus \mathcal{H}_{cyl}^4$ and bijective on W.*

Proof. Any closed anti-self-dual form is harmonic, so W contains no closed forms. Conversely, any element of $dC_\delta^{k+1,\alpha}(\Lambda^4)$ can be written as $d\beta$ with β coexact by the Hodge decomposition theorem 2.3.27. Then $\beta - *\beta \in W$ and $d(\beta - *\beta) = d\beta$. $\qquad \square$

Thus we can apply the implicit function theorem to deduce that a small neighbourhood of φ in $\mathcal{X}_{k,\mathcal{Q}}$, the space of torsion-free $Spin(7)$-structures in $\mathcal{C}_\mathcal{Q}^4$, is a manifold. Also, $\mathcal{R}_\delta$ is a submanifold of $\mathcal{S}$.

Proposition 4.3.11. *If the slice $\mathcal{S}$ near ψ is shrunk sufficiently then the space $\mathcal{R}_\delta$ of torsion-free EAC $Spin(7)$-structures in $\mathcal{S}$ is a manifold. Its tangent space at ψ is $\mathcal{H}_{cyl}^4$.*

The projection to de Rham cohomology $\pi_H : \mathcal{H}_{cyl}^4 \to H^4(M)$ is injective. This is because the boundary conditions on $\phi \in \mathcal{H}_{cyl}^4$ imply that if $j^*[\phi] = 0$ then ϕ is decaying, and therefore cannot be exact by integration by parts. Hence

$$\pi_H : \mathcal{R}_\delta \to H^4(M)$$

is an embedding if $\mathcal{R}_\delta$ is taken small enough. We can then use regularity and slice arguments just like for the G_2 case (§4.2.6–4.2.7) to show

Proposition 4.3.12. *If the slice $\mathcal{S}$ is taken small enough then the elements of $\mathcal{R}_\delta \subset \mathcal{S}$ are smooth and EAC.*

Proposition 4.3.13. $\mathcal{R}_\delta \to \mathcal{M}_\delta$ *is a homeomorphism onto an open subset.*

Using pre-moduli spaces as charts we deduce

Theorem 4.3.14. *Let M^8 be an EAC $Spin(7)$-manifold. Then $\mathcal{M}_\delta$ has a unique smooth structure such that*
$$\pi_H : \mathcal{M}_\delta \to H^4(M)$$
is an immersion.

Finally $\mathcal{M}_\delta \to \mathcal{M}_+$ is a homeomorphism onto an open subset for any $\delta > 0$, so the union $\mathcal{M}_+ = \bigcup_{\delta > 0} \mathcal{M}_\delta$ is a smooth manifold. This completes the proof of the main theorem 4.3.2.

4.3.5 Properties of the moduli space

Now that we have found pre-moduli spaces that can be used as charts for $\mathcal{M}_+$ we can deduce some simple results about the local properties of the moduli space. For a start, lemma 4.3.7 means that $B : T_\psi \mathcal{R} \to T_\varphi \mathcal{Q}_A$ is surjective, so $B : \mathcal{R} \to \mathcal{Q}_A$ is a submersion and theorem 4.3.5 follows.

To find the dimension of the moduli space we just need to evaluate the dimension of $T_\psi \mathcal{R} = \mathcal{H}^4_{cyl}$. From the proof of lemma 4.3.7 we see that

$$\dim \mathcal{H}^4_{cyl} = \dim \mathcal{H}^4_{0,1 \oplus 7} + \dim \mathcal{H}^4_{0,35}.$$

The first term corresponds to the bounded harmonic positive spinors, with dimension given by proposition 4.1.7. The second term corresponds to the bounded anti-self-dual harmonic forms. This splits into two parts: the decaying anti-self-dual forms $\mathcal{H}^4_{+,35}$ and $\{\phi - *\phi : \phi \in \mathcal{H}^4_E\}$. Its image in $H^4(M)$ is therefore a direct complement of the self-dual compactly supported cohomology $H^4_+(M)$, so

$$\dim \mathcal{H}^4_{0,35} = b^4(M) - b^4_+(M).$$

This proves proposition 4.3.3.

Finally, $\mathcal{H}^4_{+,35}$ is the tangent space to the fibre of ψ in the submersion $\mathcal{R} \to \mathcal{Q}_A$. The decaying anti-self-dual forms represent precisely the compactly supported anti-self-dual cohomology $H^4_-(M)$, so the projection from $\mathcal{R}$ to the affine subspace $[\psi] + H^4_-(M) \subset H^4(M)$ is a local diffeomorphism. This proves corollary 4.3.6.

Chapter 5

G-metrics and Ricci-flat deformations

Let G be one of the exceptional holonomy groups $Spin(7)$ or G_2. We have previously discussed how a torsion-free G-structure on a manifold M (of dimension 8 or 7 respectively) determines a G-*metric*, i.e. a Riemannian metric of holonomy $\subseteq G$, and studied the deformation theory of EAC G-structures. We now consider instead the deformations of G-metrics on compact and EAC manifolds.

G-metrics on a manifold M are Ricci-flat for $G = SU(n)$, $Sp(n)$, $Spin(7)$ or G_2. Wang [55] proves a local converse for the case when M is compact: any small Ricci-flat deformation of a G-metric still has holonomy contained in G. In other words the moduli space $\mathcal{W}_G$ of G-metrics is an open subset of the moduli space $\mathcal{W}$ of Ricci-flat metrics. Wang proves the result case by case, but asks if there is a general proof. We observe that the problem can be reduced in a uniform way to showing unobstructedness for deformations of torsion-free G-structures. This has in turn been given a uniform treatment by Goto [17].

We prove furthermore that $\mathcal{W}_G$ is a smooth manifold. For $G = Spin(7)$ or G_2 we show also that the moduli space $\mathcal{M}$ of torsion-free G-structures is a locally trivial fibre bundle over $\mathcal{W}_G$ and generalise the results to the case when M is an EAC G-manifold.

5.1   Results

Let G be one of the Ricci-flat holonomy groups $SU(n)$, $Sp(n)$, $Spin(7)$ or G_2. In the cases $Spin(7)$, G_2 or $SU(3)$ we explained in §2.2 how a G-metric on a manifold M of the appropriate dimension can be defined in terms of a G-structure, i.e. a section of a subbundle $\Lambda_G T^*M \subset \Lambda^* T^*M$, which is torsion-free and in particular closed. The tangent space to $\Gamma(\Lambda_G T^*M)$ at a G-structure χ consists of the sections of the bundle of point-wise tangents

to $\Lambda_G T^*M$ at χ, which is a vector subbundle $E_\chi \subseteq \Lambda^* T^*M$ associated to the G-structure. Since E_χ is a bundle of forms the Hodge Laplacian acts on $\Gamma(E_\chi)$. When χ is torsion-free this is the same as the Lichnerowicz Laplacian from §2.1.2. The same approach can be taken for $SU(n)$ and $Sp(n)$. We do not need the details, but they can be found in [17].

The group $\mathcal{D}$ of diffeomorphisms of M isotopic to the identity acts on the space of torsion-free G-structures by pull-backs and the quotient is the moduli space $\mathcal{M}$ of torsion-free G-structures. Goto [17] proves that the deformations of torsion-free G-structures are unobstructed in the following sense:

Proposition 5.1.1. *Let $G = SU(n)$, $Sp(n)$, $Spin(7)$ or G_2, M a compact G-manifold, and χ a torsion-free G-structure on M. Then there is a submanifold $\mathcal{R}$ of the space of C^1 G-structures such that*

(i) *the elements of $\mathcal{R}$ are smooth torsion-free G-structures,*

(ii) *the tangent space to $\mathcal{R}$ at χ is the space of harmonic sections of E_χ,*

(iii) *the natural map $\mathcal{R} \to \mathcal{M}$ is a homeomorphism onto a neighbourhood of $\chi\mathcal{D}$ in $\mathcal{M}$.*

We proved this proposition for the cases $Spin(7)$, G_2 and $SU(3)$ in §3. The spaces $\mathcal{R}$ are pre-moduli spaces of torsion-free G-structures and can be used as coordinate charts for $\mathcal{M}$.

$\mathcal{D}$ also acts on the space of Riemannian metrics, and we let $\mathcal{W}_G$ and $\mathcal{W}$ denote the moduli spaces of G-metrics and Ricci-flat metrics respectively. In §5.2 we review the deformation theory of Ricci-flat metrics and prove

Theorem 5.1.2. *Let $G = SU(n)$, $Sp(n)$, $Spin(7)$ or G_2, and let M be a compact G-manifold. Then $\mathcal{W}_G$ is open in $\mathcal{W}$. Moreover, $\mathcal{W}_G$ is a smooth manifold and the natural map*

$$m : \mathcal{M} \to \mathcal{W}_G$$

that sends a torsion-free G-structure to the metric it defines is a submersion.

Remark 5.1.3. It is easy to see that $\mathcal{W}_G$ is also closed in $\mathcal{W}$, so it is a union of connected components. It is an open problem whether there exist *any* compact Ricci-flat manifolds without a holonomy reduction.

Remark 5.1.4. The quotient of the space of G-metrics by the group of *all* diffeomorphisms of M (not just the ones isotopic to the identity) is a quotient of $\mathcal{W}_G$ with discrete fibres and in general an orbifold (cf. remark 5.2.8).

The case $G = G_2$ of theorem 5.1.2 was proved by M.Y. Wang [55, Theorem 3.1B]. For $G = Sp(n)$ or $Spin(7)$, Wang showed that $\mathcal{W}_G \subseteq \mathcal{W}$ is open. The case $G = SU(n)$ is a special case of a more general result by Koiso on Einstein deformations of Kähler-Einstein metrics.

Let X^{2n} be a compact Kähler-Einstein manifold. Koiso [33, Theorem 0.7] shows that if the Einstein constant e (equivalently the first Chern class $c_1(X)$) is non-positive and the complex deformations of X are unobstructed, then any small Einstein deformation of the metric is Kähler with respect to some perturbed complex structure. In other words the map from the moduli space of Kähler-Einstein structures to the moduli space of Einstein metrics is open (see e.g. [5, §12J] for a discussion). The proof shows that near any Kähler-Einstein metric there is a smooth pre-moduli space of Einstein metrics, so that the moduli space of Kähler-Einstein metrics is an orbifold. As explained in §3.4.1, Tian [52] and Todorov [53] show that on a compact Calabi-Yau manifold the obstructions to the complex deformations vanish. Hence most of theorem 5.1.2 for $G = SU(n)$ follows from Koiso's theorem.

Remark 5.1.5. Dai, X. Wang and Wei [14] use the fact that $\mathcal{W}_G$ is open in $\mathcal{W}$ to deduce that any scalar-flat deformation of a Ricci-flat G-metric on a compact manifold remains a G-metric.

The proof of theorem 5.1.2 given in §5.2 is a simplification of Wang's argument for the case $G = G_2$. We observe that the point-wise surjectivity of the derivative of m follows from the equivariance properties of the Laplacians on manifolds with reduced holonomy discussed in subsection 2.1.2. This makes it easy to see that the proof applies also for the other Ricci-flat holonomy groups, provided that the deformations of torsion-free G-structures are unobstructed.

The proof works also on non-compact manifolds if the deformation theory for the G-structures is in place. For EAC G_2-manifolds and $Spin(7)$-manifolds we proved in §4 that there is a smooth moduli space $\mathcal{M}_+$ of torsion-free EAC G-structures. Let $\mathcal{W}_+$ denote the moduli space of EAC Ricci-flat metrics.

Theorem 5.1.6. *Let $G = Spin(7)$ or G_2, and let M be an EAC G-manifold. Then $\mathcal{W}_G$ is open in $\mathcal{W}_+$. Moreover, $\mathcal{W}_G$ is a smooth manifold and the natural map*

$$m : \mathcal{M}_+ \to \mathcal{W}_G$$

is open.

In [35] Kovalev proves the analogous result for EAC Calabi-Yau manifolds, by an extension of Koiso's arguments for the compact Kähler-Einstein case. The discussion in §5.2.4 of deformations of EAC Ricci-flat metrics is similar to that in [35].

For $G = Spin(7)$ and G_2 we also study the structure of the map m in greater detail. As described in chapter 4 the topology of a compact G-manifold M determines whether each G-metric on M has holonomy *exactly* G. In §5.3 we explain that the fibres of m are determined up to diffeomorphism by the topology in a similar way, and show

Theorem 5.1.7. *Let $G = Spin(7)$, G_2 or $SU(3)$, and let M be a compact G-manifold. Then*

$$m : \mathcal{M} \to \mathcal{W}_G$$

is a locally trivial fibre bundle. The typical fibre is a disjoint union of real projective spaces.

When M has holonomy exactly G the typical fibre of m is a point if $G = Spin(7)$ or G_2, and S^1 if $G = SU(3)$.

The proof uses the fact that point-wise the space of G-structures defining the same inner product is $SO(8)/Spin(7) \cong SO(7)/G_2 \cong SO(6)/SU(3) \cong \mathbb{R}P^7$. To be precise, the point-wise fibre can in each case be identified with the space of lines in a real rank 8 spin representation. In principle, the result is an application of the observation of Wang [54] that a torsion-free G-structure can be recovered explicitly from a parallel spinor (or an appropriate set of spinors). In the case $G = SU(n)$ for $n \geq 4$ the fibres of m would be identified in terms of the space of parallel *pure* spinors, which is less straight-forward to determine topologically. Nevertheless theorem 5.1.7 is, like theorem 5.1.2, the analogue of a result for Calabi-Yau manifolds: [5, Theorem 12.103] states that the moduli space of Calabi-Yau structures on a compact manifold is a locally trivial fibration with compact fibres over the moduli space of Calabi-Yau metrics (but does not describe the fibres further).

The fact that $\mathcal{W}_G$ is smooth suggests that it may be as natural to consider in general as the moduli space of G-structures $\mathcal{M}$ (of course it makes no difference for irreducible G_2-manifolds and $Spin(7)$-manifolds). For example Karigiannis and Leung [30] and Grigorian and Yau [19] both study the geometry of the moduli space $\mathcal{M}$ of torsion-free G_2-structures on a compact G_2-manifold. They find that the curvature computations are more involved for G_2-manifolds M^7 with $b^1(M) > 0$. This is because curvature has different expressions on the three components $H_1^3(M)$, $H_7^3(M)$ and $H_{27}^3(M)$ of the tangent space to $\mathcal{M}$ (which is isomorphic to $H^3(M)$). If $b^1(M) = 0$ then the $H_7^3(M)$ component vanishes,

and the computations simplify. On the other hand the tangent space to $\mathcal{W}_{G_2}$ is isomorphic to $H^3_{1 \oplus 27}(M)$, whether $b^1(M) = 0$ or not.

The main requirement for proving theorem 5.1.7 is theorem 5.1.2, so it can be generalised to non-compact manifolds under similar assumptions. In §5.3.3 we prove

Theorem 5.1.8. *Let $G = Spin(7)$ or G_2, and let M be an EAC G-manifold. Then*

$$m : \mathcal{M}_+ \to \mathcal{W}_G$$

is a locally trivial fibre bundle. The typical fibre is a disjoint union of real projective spaces. If M has holonomy exactly G then m is a diffeomorphism.

5.2 Ricci-flat deformations of G-metrics

5.2.1 Killing vector fields

Before we discuss the deformations of Ricci-flat metrics we make some remarks about *Killing vector fields*. These are the infinitesimal isometries of a Riemannian manifold (M, g), i.e. vector fields V such that the Lie derivative $\mathcal{L}_V g$ vanishes.

Definition 5.2.1. Given a metric g on M let $\delta^* : \Omega^1(M) \to \Gamma(S^2(T^*M))$ be the symmetric part of the Levi-Civita connection $\nabla : \Omega^1(M) \to \Gamma(T^*M \otimes T^*M)$.

The formal adjoint δ of δ^* is the restriction of $\nabla^* : \Gamma(T^*M \otimes T^*M) \to \Omega^1(M)$ to the symmetric part $\Gamma(S^2(T^*M))$.

Proposition 5.2.2 ([5, Lemma 1.60]). *Let g be a Riemannian metric on a manifold M and V a vector field. Then $\mathcal{L}_V g = 2\delta^* V^\flat$, where $V^\flat$ denotes the 1-form $g(V, \cdot)$.*

The second Bianchi identity implies that

$$(2\delta + d \operatorname{tr})Ric = 0 \tag{5.1}$$

for any Riemannian metric. The operator $2\delta + d\operatorname{tr}$ is sometimes called the Bianchi operator, and it satisfies also the following useful identity.

Lemma 5.2.3 ([35, (14)]). *If (M, g) is a Ricci-flat manifold then*

$$(2\delta + d \operatorname{tr})\delta^* = \triangle.$$

Proof. The anti-symmetric part of ∇ on $\Omega^1(M)$ is $\frac{1}{2}d$, so $\delta^* = \nabla - \frac{1}{2}d$. Also $\operatorname{tr} \delta^* = d^*$ on $\Omega^1(M)$. Using proposition 2.1.8 we obtain

$$(2\delta + d\operatorname{tr})\delta^* = 2\nabla^*\nabla - \nabla^*d + d\operatorname{tr}\delta^* = 2\nabla^*\nabla - d^*d - dd^* = \triangle. \qquad \square$$

Proposition 5.2.4. *Let (M, g) be a Ricci-flat manifold. If V is a Killing field then the 1-form $V^\flat$ is harmonic. If M is compact then the converse also holds.*

Proof. $\delta^* V^\flat = 0 \Rightarrow \triangle V^\flat = 0$ by lemma 5.2.3. Trivially $\nabla^* V^\flat = 0 \Rightarrow \delta^* V^\flat = 0$, and if M is compact then $\triangle V^\flat = 0 \Rightarrow \nabla^* V^\flat = 0$ by corollary 2.1.9. $\qquad \square$

This implies that, for any of the Ricci-flat holonomy groups G, the space of infinitesimal automorphisms of a compact G-manifold is $(\mathcal{H}^1)^\sharp$.

5.2.2 Deformations of Ricci-flat metrics

We summarise some deformation theory for Ricci-flat metrics. This is mostly taken from the deformation theory for Einstein metrics as explained in [5, §12C], specialised to the Ricci-flat case. It turns out that in general we cannot prove that the moduli space of Ricci-flat metrics is a manifold.

Let M^n be a compact manifold. The diffeomorphism group $\mathcal{D}$ acts on the space of Ricci-flat metrics on M by pull-backs. We define the moduli space $\mathcal{W}$ of Ricci-flat metrics to be the quotient of the space of Ricci-flat metrics by $\mathcal{D}$. (We do not divide by the rescaling action of $\mathbb{R}^+$ too, as is done in [5].)

Take $k \geq 2$, and let g be a Ricci-flat Riemannian metric on M. In order to study a neighbourhood of $g\mathcal{D}$ in $\mathcal{W}$ we use slice arguments explained in §3.1. We include the space of smooth Riemannian metrics in the Hölder space $C^{k,\alpha}(S^2 T^* M)$, and let $\mathcal{D}_{k+1}$ be the $C^{k+1,\alpha}$ completion of $\mathcal{D}$ ($\mathcal{D}_{k+1}$ is generated by exp of $C^{k+1,\alpha}$ vector fields).

By proposition 5.2.2 the tangent space to the $\mathcal{D}_{k+1}$-orbit at g is $\delta_g^* C^{k+1,\alpha}(\Lambda^1)$. Let K be the kernel of $2\delta_g + d\operatorname{tr}_g$ in $C^{k,\alpha}(S^2 T^* M)$. Because g is Ricci-flat harmonic 1-forms are parallel, and therefore L^2-orthogonal to the image of $2\delta_g + d\operatorname{tr}_g$. It follows from lemma 5.2.3 and the Fredholm alternative for $\triangle_g$ on $\Omega^1(M)$ that there is a direct sum decomposition

$$C^{k,\alpha}(S^2 T^* M) = \delta_g^* C^{k+1,\alpha}(\Lambda^1) \oplus K.$$

We use a neighbourhood $\mathcal{S}$ of g in K as a slice for the $\mathcal{D}$-action.

Remark 5.2.5. This is not exactly the same choice of slice as in [5]. It has been used before by Biquard [6] and Kovalev [35].

Let $\mathcal{Q}$ be the space of Ricci-flat (not *a priori* smooth) metrics in $\mathcal{S}$ – this is the *pre-moduli space of Ricci-flat metrics* near g. The linearisation of the Ricci curvature functional at a Ricci-flat metric is given by (cf. [5, (12.28')])

$$(DRic)_g h = \triangle_L h + \delta_g^*(2\delta_g + d\,\mathrm{tr}_g)h, \tag{5.2}$$

where $\triangle_L$ denotes the Lichnerowicz Laplacian on $S^2 T^* M$ in the sense of definition 2.1.6. In particular, on the tangent space K to the slice the linearisation reduces to $\triangle_L$. It is elliptic so its kernel has finite dimension. Moreover, the kernel is contained in K: differentiating the Bianchi identity (5.1) at the Ricci-flat metric g gives

$$(2\delta_g + d\,\mathrm{tr}_g)(DRic)_g = 0,$$

and hence

$$\triangle_L h = 0 \;\Rightarrow\; \triangle(2\delta_g + d\,\mathrm{tr}_g)h = 0 \;\Rightarrow\; (2\delta_g + d\,\mathrm{tr}_g)h = 0.$$

Definition 5.2.6. The space of *infinitesimal Ricci-flat deformations* of g is the kernel $\varepsilon(g)$ of $\triangle_L$ in $\Gamma(S^2(T^* M))$.

If $h \in \Gamma(S^2 T^* M)$ is tangent to a curve of Ricci-flat metrics in the slice $\mathcal{S}$ then of course $h \in \varepsilon(g)$. The converse is not true; in general there may be elements in $\varepsilon(g)$ which are not tangent to any curve of Ricci-flat metrics. Thus $\mathcal{Q}$ need not be a manifold with tangent space $\varepsilon(g)$.

The image of $DRic_g$ is the L^2-orthogonal complement K' to $\varepsilon(g)$ in K. Let P_g be the L^2-orthogonal projection to K'. The Ricci curvature functional is real analytic. We can apply the implicit function theorem to the composition

$$F : \mathcal{S} \to K' : \; h \mapsto P_g Ric(h) \tag{5.3}$$

to deduce that there is a real analytic submanifold $Z \subseteq \mathcal{S}$ whose tangent space at g is precisely $\varepsilon(g)$ and which contains $\mathcal{Q}$ as a real analytic subset. The analyticity implies that *if* every element of $\varepsilon(g)$ is tangent to a curve of Ricci-flat metrics then in fact $\mathcal{Q}$ contains a neighbourhood of g in Z. Thus the pre-moduli space $\mathcal{Q}$ is a manifold in this case (cf. [33, Corollary 3.5]).

The elements of Z are smooth by elliptic regularity, and when $\mathcal{Q} = Z$ one can deduce from corollary 3.1.6 that $\mathcal{Q} \to \mathcal{W}$ is open. In general the argument needs to be modified a bit. One needs to extend (5.3) to a function on a neighbourhood U of g in $C^{k,\alpha}(S^2 T^* M)$ such that $F^{-1}(0)$ is a manifold containing the Ricci-flat metrics *and* $Z\mathcal{D}_{k+1} \cap U \subseteq F^{-1}(0)$. One way to do this is to first define a smooth map $f : U \to g\mathcal{D}_{k+1}$ with the property that $f(\phi^* g') = \phi^* g$ when $\phi \in \mathcal{D}_{k+1}$ and $g' \in Z$. If $f(h) = \phi^* g$ then $P_{f(h)}$ is a projection to $\phi^* K'$, and we can take

$$ F : U \to K', \ \ h \mapsto P_g P_{f(h)} Ric(h). \tag{5.4} $$

Then

$$ Z \times \mathcal{D}_{k+1} \to F^{-1}(0) $$

is an open map near (g, id) by the submersion theorem, and we can proceed as in the proof of theorem 3.1.4 and corollary 3.1.6 to deduce that $\mathcal{Q} \to \mathcal{W}$ is open. Theorem 3.1.7 implies that in fact this is injective up to the action of the stabiliser $\mathcal{I}_g$. Moreover, proposition 5.2.4 implies that all stabilisers of elements in $\mathcal{Q}$ have the same dimension, so at least their identity components must be equal. As $\mathcal{I}_g$ is compact its action on $\mathcal{Q}$ factors through a finite group (cf. [5, 12.25]). Hence

Theorem 5.2.7. *Let M be a compact manifold and g a Ricci-flat metric on M. Let $\mathcal{Q}$ be the pre-moduli space of Ricci-flat metrics near g and $\mathcal{I}_g$ the stabiliser of g in $\mathcal{D}$. Then $\mathcal{Q}/\mathcal{I}_g$ is homeomorphic to a neighbourhood of $g\mathcal{D}$ in $\mathcal{W}$. In particular, if every element of $\varepsilon(g)$ is integrable then $\mathcal{W}$ is an orbifold near $g\mathcal{D}$.*

Remark 5.2.8. Clearly the argument would give the same result even if we were to consider the moduli space of Ricci-flat metrics given by dividing by the action of the full diffeomorphism group of M.

5.2.3 Proof of theorem 5.1.2

Let M be a compact G-manifold, $\Gamma(\Lambda_G T^* M)$ the space of G-structures on M and

$$ m : \Gamma(\Lambda_G T^* M) \to \Gamma(S^2 T^* M), \ \ \chi \mapsto g_\chi \tag{5.5} $$

the natural map that sends a G-structure to the metric it defines. In order to prove theorem 5.1.2 we show first that for any torsion-free G-structure χ the derivative of m maps the tangent space to the pre-moduli space $\mathcal{R}$ at χ onto the space $\varepsilon(g_\chi)$ of infinitesimal Ricci-flat deformations.

The tangent space to $\Gamma(\Lambda_G T^*M)$ at χ is the space of differential forms $\Gamma(E_\chi)$, where $E_\chi \subseteq \Lambda^* T^*M$ is a vector subbundle associated to the G-structure defined by χ. Fibre-wise $\Lambda_G T^*M$ is a $GL(\mathbb{R}^n)$-orbit and E_χ is the tangent space $\mathfrak{gl}_n \chi$ to the orbit. Because m is $GL(\mathbb{R}^n)$-equivariant its derivative takes $a\chi \mapsto ag_\chi$ for any $a \in \mathfrak{gl}_n$, which maps onto the fibre of $S^2 T^*M$. Hence the derivative

$$Dm_\chi : \Gamma(E_\chi) \to \Gamma(S^2 T^*M) \tag{5.6}$$

is surjective. Furthermore, the derivative is G-equivariant with respect to the G-structure defined by χ. Since $\triangle_L$ is the Lichnerowicz Laplacian on $S^2 T^*M$, lemma 2.1.10 implies that the diagram below commutes.

$$
\begin{array}{ccc}
\Gamma(E_\chi) & \xrightarrow{Dm_\chi} & \Gamma(S^2 T^*M) \\
\downarrow{\scriptstyle\triangle} & & \downarrow{\scriptstyle\triangle_L} \\
\Gamma(E_\chi) & \xrightarrow{Dm_\chi} & \Gamma(S^2 T^*M)
\end{array}
$$

Hence

Lemma 5.2.9. *If χ is a torsion-free G-structure then Dm_χ maps the harmonic sections of E_χ onto the space $\varepsilon(g_\chi)$ of infinitesimal Ricci-flat deformations.*

So let χ be any torsion-free G-structure on M and $\mathcal{R}$ the pre-moduli space of torsion-free G-structures near χ. As described in subsection 5.2.2, there is a slice at g_χ for the $\mathcal{D}$-action on the metrics and a local projection map P to the slice. The Ricci-flat metrics in the slice are contained in a submanifold Z, and the tangent space to Z at χ is $\varepsilon(g_\chi)$.

$$P \circ m : \mathcal{R} \to Z \tag{5.7}$$

is a well-defined smooth map and lemma 5.2.9 means that its derivative at χ is surjective. Therefore every element of $\varepsilon(g_\chi)$ is tangent to a path of Ricci-flat metrics, so $\mathcal{Q}$ is a manifold. By the submersion theorem, the image $\mathcal{W}_G$ in $\mathcal{W}$ contains a neighbourhood of $g\mathcal{D}$.

The pre-images of g_χ under m are defined by differential forms which are harmonic with respect to g_χ. By Hodge theory they represent distinct cohomology classes. Because

elements of $\mathcal{I}_{g_\chi}$ are isotopic to the identity they must therefore fix the fibre, so $\mathcal{I}_{g_\chi} = \mathcal{I}_\chi$. Now, if $g' \in \mathcal{Q}$ then $g' = \phi^* m(\chi')$ for some $\chi' \in \mathcal{R}$ and $\phi \in \mathcal{D}_{k+1}$ because (5.7) is a submersion. As $\mathcal{I}_\chi$ acts trivially on $\mathcal{R}$ by proposition 3.1.10 it follows that the conjugate $\mathcal{I}_{g_\chi}^\phi$ fixes g'. But then $\mathcal{I}_{g_\chi}^\phi \subseteq \mathcal{I}_{g'} \subseteq \mathcal{I}_{g_\chi}$ by theorem 3.1.7, so in fact $\mathcal{I}_{g_\chi}^\phi = \mathcal{I}_{g_\chi}$. Thus $\mathcal{I}_{g_\chi}$ fixes g'.

Now theorem 5.2.7 implies that $\mathcal{Q}$ is homeomorphic to a neighbourhood of $\mathcal{W}$. Thus $\mathcal{W}_G$ is a manifold near $g\mathcal{D}$ and the proof of theorem 5.1.2 is complete.

5.2.4   The asymptotically cylindrical case

The proof of theorem 5.1.2 only used the compactness assumption to access certain deformation results for G-structures and Ricci-flat metrics. For the cases $G = G_2$ and $Spin(7)$ we found pre-moduli spaces of EAC G-structures in §4, with properties analogous to proposition 5.1.1.

Proposition 5.2.10. *Let $G = Spin(7)$ or G_2, M an EAC G-manifold and χ a torsion-free EAC G-structure on M. Then there is a submanifold $\mathcal{R}$ of the space of C^1 G-structures such that*

(i) *the elements of $\mathcal{R}$ are smooth EAC torsion-free G-structures,*

(ii) *the tangent space to $\mathcal{R}$ at χ is the space of bounded harmonic sections of E_χ,*

(iii) *the natural map $\mathcal{R} \to \mathcal{M}_+$ is a homeomorphism onto a neighbourhood of $\chi\mathcal{D}_+$ in $\mathcal{M}_+$.*

In order to prove the EAC version theorem 5.1.6 we only need to explain how to set up the deformation theory for EAC Ricci-flat metrics. We define the slices with same equations as in the compact case in §5.2.2 and use the same reasoning as for deformations of EAC G_2-manifolds in §4.2.7 to make the slice arguments work on EAC manifolds. The resulting approach is similar to that of Kovalev [35], who considers Ricci-flat deformations of EAC Calabi-Yau manifolds.

Let M^n be a manifold with cylindrical ends and cross-section X^{n-1}. Let $\mathcal{W}_+$ be the quotient of the space of EAC Ricci-flat metrics (with any exponential rate) by the group $\mathcal{D}_+$ of EAC diffeomorphisms of M isotopic to the identity. We pick an EAC Ricci-flat metric g on M and study a neighbourhood of $g\mathcal{D}_+$ in $\mathcal{W}_+$. By definition, the asymptotic limit of g is a cylindrical metric $dt^2 + g_X$ on $X \times \mathbb{R}$, where g_X is a Ricci-flat metric on X.

We work with weighted Hölder spaces of sections. Let $k \geq 2$, $\alpha \in (0, 1)$, and $\delta > 0$ be less than the exponential rate of g. The metric g defines a Hodge Laplacian on 1-forms

and a Lichnerowicz Laplacian on symmetric bilinear forms, which are both asymptotically translation-invariant operators. We require that δ is small enough that the Laplacians are Fredholm on $C_\delta^{k,\alpha}$ spaces, as we may according to theorem 2.3.17.

We proved in §5.2.2 that there is a real analytic submanifold $Z \subset C^{k,\alpha}(S^2 T^* X)$ which contains representatives of all diffeomorphism classes of Ricci-flat metrics on X close to g_X. Its tangent space $T_{g_X} Z = \varepsilon(g_X)$ is the space of Lichnerowicz harmonic sections of $S^2 T^* X$.

Let $\mathscr{M}_Z^k$ denote the space of $C^{k,\alpha}$ metrics on M which are $C_\delta^{k,\alpha}$-asymptotic to cylindrical metrics $dt + g_X^2$ such that $g_X \in Z$. If ρ is a cut-off function for the cylinder then ρZ can be identified with a space of bilinear forms on M, and $\mathscr{M}_Z^k$ is an open subset

$$\mathscr{M}_Z^k \subset C_\delta^{k,\alpha}(S^* T^* M) + \rho Z.$$

Similarly let $\mathcal{D}_Z^{k+1}$ be the set of EAC diffeomorphisms with rate δ which are asymptotic to elements of the isometry group $\mathcal{I}_{g_X}$ of g_X. Then $\mathscr{M}_Z^k$ contains representatives of all diffeomorphism classes of Ricci-flat metrics near g and, because Z is $\mathcal{I}_{g_X}$-invariant, theorem 3.1.7 implies that any isometry between elements of $\mathscr{M}_Z^k$ must lie in $\mathcal{D}_Z^{k+1}$ (a similar argument for simplifying the problem by a slice at the boundary was used for the G_2 case in lemma 4.2.26).

We therefore identify a slice in $\mathscr{M}_Z^k$ for the action of $\mathcal{D}_Z^{k+1}$ at g. The tangent space to $\mathscr{M}_Z^k$ at g is

$$T_g \mathscr{M}_Z^k = C_\delta^{k,\alpha}(S^* T^* M) \oplus \rho\varepsilon(g_X).$$

The tangent space at the identity of $\mathcal{D}_Z^{k+1}$ corresponds to vector fields which are $C_\delta^{k,\alpha}$-asymptotic to translation-invariant Killing vector fields on the cylinder, i.e. to elements of $(\mathcal{H}_\infty^1)^\sharp$, where $\mathcal{H}_\infty^1$ denotes the translation-invariant harmonic 1-forms on the cylinder $X \times \mathbb{R}$. By proposition 5.2.2 the tangent space to the $\mathcal{D}_Z^{k+1}$-orbit at g is

$$\delta_g^*(C_\delta^{k,\alpha}(\Lambda^1) \oplus \rho\mathcal{H}_\infty^1).$$

Let K be the kernel of $2\delta_g + d\operatorname{tr}_g$ in $T_g\mathscr{M}_Z^k$.

Lemma 5.2.11. *Let M be a Ricci-flat EAC manifold with a single end. Then*

$$T_g\mathscr{M}_Z^k = K \oplus \delta_g^*(C_\delta^{k,\alpha}(\Lambda^1) \oplus \rho\mathcal{H}_\infty^1). \tag{5.8}$$

Proof. $(2\delta_g + d\operatorname{tr}_g)\delta^* = \triangle_g$ according to lemma 5.2.3, so it suffices to show that the image

of $2\delta_g + d\operatorname{tr}_g : T_g \mathscr{M}_Z^k \to C_\delta^{k-1,\alpha}(\Lambda^1)$ is contained in the image of

$$\triangle : C_\delta^{k+1,\alpha}(\Lambda^1) \oplus \rho\mathcal{H}_\infty^1 \to C_\delta^{k-1,\alpha}(\Lambda^1).$$

It follows from proposition 2.3.21 that this has index 0, so its image is the L^2-orthogonal complement to its kernel $\mathcal{H}_0^1$, the space of bounded harmonic 1-forms.

Now, if $h \in T_g\mathscr{M}_Z^k$ and $\beta \in \mathcal{H}_0^1$ then the difference between $<\delta_g h, \beta>$ and $<h, \delta_g^* \beta> = 0$ is the divergence of the contraction of h with β. The boundary condition on h ensures that the asymptotic limit of the contraction has no dt-component, so the integral of the divergence is 0. Hence

$$<\delta_g h, \beta>_{L^2} = 0.$$

The hypothesis that M has a single end ensures that the asymptotic limit of β has no dt-component (corollary 2.3.40), so integration by parts also applies to show that $<d\operatorname{tr}_g h, \beta>_{L^2} = 0$. Thus the image of $2\delta_g + d\operatorname{tr}_g$ is L^2-orthogonal to $\mathcal{H}_0^1$. $\qquad\square$

Now we can use a real analytic $\mathcal{I}_g$-invariant submanifold $\mathcal{S} \subset \mathscr{M}_Z^k$ with $T_g\mathcal{S} = K$ as a slice for the $\mathcal{D}_Z^{k+1}$-action. Let $\mathcal{Q} \subset \mathcal{S}$ be the subset of Ricci-flat metrics. As in the compact case $\mathcal{Q}$ is an analytic subset of an analytic submanifold $Z' \subset \mathcal{S}$, defined as the zero set of the composition of the Ricci functional $\mathcal{S} \to C_\delta^{k-2,\alpha}(S^2T^*M)$ with the projection onto the image of its derivative at g. On K the derivative of the Ricci functional is the Lichnerowicz Laplacian, so T_gZ' is the space of harmonic sections of S^2T^*M, exponentially asymptotic to sections of S^2T^*X (i.e. the asymptotic limit has no dt-components). We call this the space of infinitesimal Ricci-flat EAC deformations $\varepsilon(g)$.

In general we can use by regularity and slice arguments like in §4.2.6 – 4.2.7 that Z consists of smooth EAC metrics and extend the proof of theorem 5.2.7, thus proving that $\mathcal{Q}/\mathcal{I}_g$ is homeomorphic to a neighbourhood of g in $\mathcal{W}_+$.

We have now set up the deformation theory for EAC Ricci-flat metrics that is required, together with the unobstructedness of deformations of torsion-free EAC G-structures for $G = G_2$ and $Spin(7)$ proved in §4, in order to prove theorem 5.1.6 by the same argument as for the compact case.

5.3 $\mathcal{M}$ as a fibre bundle

Let $G = Spin(7)$, G_2 or $SU(3)$, and M a compact G-manifold. We have established that the natural map

$$m : \mathcal{M} \to \mathcal{W}_G \qquad (5.9)$$

is a submersion. We now prove that it is a smooth fibre bundle. We use the correspondence between torsion-free G-structures and parallel spinors to show that the fibres are determined by the topology of M, and then deduce theorem 5.1.7. We will prove the case $G = Spin(7)$ in detail, and remark on the minor adaptions required for $G = G_2$ or $SU(3)$.

5.3.1 The case $G = Spin(7)$

Let M be a compact $Spin(7)$-manifold. We begin by observing that the fibres of m in $\Gamma(S^2 T^* M)$ and in the moduli space $\mathcal{M}$ are equivalent.

Lemma 5.3.1. *Let g be a $Spin(7)$-metric on M. Then the fibre of $m : \mathcal{M} \to \mathcal{W}_{Spin(7)}$ over $g\mathcal{D}$ is homeomorphic to the fibre of $m : \Gamma(\Lambda_{Spin(7)} T^* M) \to \Gamma(S^2 T^* M)$ over g.*

Proof. If ψ is a $Spin(7)$-structure such that $m(\psi) = \phi^* g$ for some $\phi \in \mathcal{D}$ then obviously $m((\phi^{-1})^* \psi) = g$. Thus the natural map from fibre over g to the fibre over $g\mathcal{D}$ is surjective.

Suppose that two torsion-free $Spin(7)$-structures which define the same metric are $\mathcal{D}$-equivalent. Then they represent the same class in $H^4(M)$. Since they are both harmonic (with respect to the same metric) they must be equal. Therefore the map is injective too. $\qquad\square$

Now consider the point-wise model for $Spin(7)$-structures which was described in §2.2.3. $\Lambda_{Spin(7)}(\mathbb{R}^8)^* \subseteq \Lambda^4(\mathbb{R}^8)^*$ is the $GL(\mathbb{R}^8)$-orbit of the standard $Spin(7)$-structure ψ_0 and each element defines an inner product on $\mathbb{R}^8$. The subset that defines the same inner product as ψ_0 (i.e. the standard inner product on $\mathbb{R}^8$) is the orbit $SO(8)\psi_0$. Recall that the positive real spin representation σ_8^+ of $Spin(8)$ splits into irreducible components

$$\sigma_8^+ = \mathbb{R} \oplus \mathbb{R}^7$$

under the action of $Spin(7)$. The $\mathbb{R}$ term defines an element $u \in \mathbb{P}\sigma_8^+$. Since $SO(8)$ acts transitively on $\mathbb{P}\sigma_8^+$ with stabiliser $Spin(7)$ at u it follows that there is a natural bijection $SO(8)\psi_0 \cong \mathbb{P}\sigma_8^+$.

$Spin(7)$ acts irreducibly on $\mathbb{R}^7$, so u is its unique fixed point in $\mathbb{P}\sigma_8^+$. Thus ψ_0 is the unique $Spin(7)$-structure defining the standard inner product that is fixed by $Spin(7)$. Hence if g is a $Spin(7)$-metric with $Hol(g) = Spin(7)$ then the fibre over g is a point, and $m : \mathcal{M} \to \mathcal{W}_{Spin(7)}$ is a diffeomorphism.

Let I be the set of spin structures on M^8. Given a $Spin(7)$-metric g on M, for each $i \in I$ let F_i be the space of parallel sections of the positive spinor bundle defined by i. Any torsion-free $Spin(7)$-structure inducing g determines a spin structure i and a parallel spinor in F_i. The point-wise considerations show that the fibre of m over g is homeomorphic to the disjoint union of projective spaces

$$\bigsqcup_{i \in I} \mathbb{P}F_i.$$

For each i and g either $F_i = 0$ or its dimension is given by the topological formula (2.16). The fact that $m : \mathcal{M} \to \mathcal{W}_{Spin(7)}$ is open implies that, for each i, triviality of F_i depends only on the connected component of $\mathcal{W}_{Spin(7)}$ containing $g\mathcal{D}$.

If g' is another $Spin(7)$-metric let F_i' denote the corresponding spaces of parallel spinors. If g' is close to g then we may define a linear map $F_i \to F_i'$, and because they have equal dimension it is easy to see that it is bijective for g' sufficiently close to g. If we let $\mathcal{Q}$ be the pre-moduli space of $Spin(7)$-metrics near g then this defines a local diffeomorphism

$$\mathcal{Q} \times \bigsqcup_{i \in I} \mathbb{P}F_i \to \mathcal{M}.$$

Transition functions between such maps are isomorphisms of projective varieties on the fibres. The maps can therefore be used as local trivialisations for $\mathcal{M}$ as a smooth fibre bundle over each connected component of $\mathcal{W}_{Spin(7)}$.

It remains to check that the fibres are the same over different connected components. A pair of spin structures i, j determines an element of $H^1(M, \mathbb{Z}_2)$ (the unique class whose pull-back to the total space of the frame bundle of M is the difference of the Stiefel-Whitney classes of i and j), which in turn determines a double cover $\tilde{M}$ of M (cf. [37, §II.1]). i and j lift to the same spin structure on $\tilde{M}$. If g is a $Spin(7)$-metric on M and F_i and F_j are both non-empty then the dimension of the space on parallel spinors on $\tilde{M}$ is twice that of F_i. By the topological formula (2.16) for the dimension of the space of parallel spinors on a $Spin(7)$-manifold this is a topological property of $\tilde{M}$. Hence the number of non-empty connected components $\mathbb{P}F_i$ of the fibre of m above a $Spin(7)$-metric g is determined by

the topology of M, independently of the choice of g. This concludes the proof of theorem 5.1.7 for the case $G = Spin(7)$.

Remark 5.3.2. Another way to think of this is that if we fix a torsion-free $Spin(7)$-structure ψ on M then the fibre of m containing ψ can be identified with the parallel sections of the projective spinor bundle $\mathbb{P}S^+$ determined by ψ. Any such section corresponds to a parallel spinor either on M or on a double cover of M.

5.3.2 The cases $G = G_2$ and $SU(3)$

All that needs to be changed to extend the proof of theorem 5.1.7 above to the case when M^7 is a compact G-manifold is the point-wise model used in the identification of the fibres.

Recall that the spin representation σ_7 splits into irreducible components

$$\sigma_7 \cong \mathbb{R} \oplus \mathbb{R}^7 \tag{5.10}$$

under the action of G_2. The $\mathbb{R}$ term defines an element $u \in \mathbb{P}\sigma_7$. $SO(7)$ acts transitively on $\mathbb{P}\sigma_7$ and the stabiliser of u is G_2, so there is a natural identification $SO(7)\varphi_0 \cong \mathbb{P}\sigma_7$ of the $SO(7)$-orbit of the standard G_2-structure with $\mathbb{P}\sigma_7$. For any G_2-metric g on M the fibre of m over g is identified with $\bigsqcup_{i \in I} \mathbb{P}F_i$, where F_i is the space of parallel spinors defined by g and a spin structure i. The rest of the proof of theorem 5.1.7 for $G = G_2$ is identical to the case $G = Spin(7)$.

Remark 5.3.3. Let φ be a torsion-free G_2-structure on a compact manifold M^7. The isomorphism (5.10) implies that the spinor bundle on M defined by φ is $S \cong \mathbb{R} \oplus TM$. We can therefore identify the fibre of m over φ with parallel sections of $\mathbb{P}(\mathbb{R} \oplus TM)$. This identification can be interpreted as given by 'twisting' φ.

If ψ is a G_2-structure on a vector space V and $v \in V$ is a unit vector then the 2-form $v \lrcorner \psi$ can be identified with an element $A_{v,\psi} \in \mathfrak{so}(V)$. If v is a unit vector in $\mathbb{R}^7$ and $\theta \in \mathbb{R}$ then the identification of $\mathbb{P}\sigma_7$ with $SO(7)\varphi_0$ maps the line containing $\cos\theta + v\sin\theta$ to $\varphi_\theta = \exp(2\theta A_{v,\varphi_0})\varphi_0$. Note that φ_θ satisfies a flow equation considered by Karigiannis [29],

$$\tfrac{d}{d\theta}\varphi_\theta = A_{v,\varphi_\theta}\varphi_\theta.$$

The torsion-free G_2-structure φ can in this sense be 'twisted' by parallel vector fields on M to produce other torsion-free G_2-structures defining the same metric and all elements in the connected component of φ in its fibre in $m : \mathcal{M} \to \mathcal{W}_{G_2}$ arise this way.

Finally we prove theorem 5.1.7 for $G = SU(3)$. Let M^6 be a compact Calabi-Yau 3-fold. To identify the fibres of $m : \mathcal{M} \to \mathcal{W}_{SU(3)}$ we consider the spin representation σ_6 of $Spin(6)$. This is the unique irreducible real rank 8 representation of $Spin(6)$ (it can be regarded as a rank 4 complex representation in two non-equivalent ways). As an $SU(3)$-representation

$$\sigma_6 \cong \mathbb{C} \oplus \mathbb{C}^3, \tag{5.11}$$

where $SU(3)$ acts naturally on $\mathbb{C}^3$. Let $\mathbb{P}\sigma_6$ be the space of (real) lines in σ_6. $SO(6)$ acts transitively on $\mathbb{P}\sigma_6$ and the stabiliser of an element of $\mathbb{P}\mathbb{C} \subset \mathbb{P}\sigma_6$ is $SU(3)$. If we fix once and for all an element $u \in \mathbb{P}\mathbb{C}$ then this determines an identification $SO(6)(\Omega_0, \omega_0) \cong \mathbb{P}\sigma_6$ of the $SO(6)$-orbit of the standard Calabi-Yau structure with $\mathbb{P}\sigma_6$. For any Calabi-Yau metric g on M the fibre of m over g is identified with $\bigsqcup_{i \in I} \mathbb{P}F_i$, where F_i is the space of parallel spinors defined by g and a spin structure i.

$SU(3)$ acts irreducibly on $\mathbb{C}^3$, so its fixed points in $\mathbb{P}\sigma_6$ are exactly $\mathbb{P}\mathbb{C} \cong S^1$. Thus when M is an irreducible Calabi-Yau manifold the fibres of $m : \mathcal{M} \to \mathcal{W}_{SU(3)}$ are S^1. This corresponds to the S^1-action on $\mathcal{M}$ given by multiplication of the holomorphic $(3,0)$-form $\Omega + i\hat{\Omega}$ by a unit complex scalar. The rest of the proof is identical to the case $G = Spin(7)$.

Example 5.3.4. Let X^6 a simply connected manifold with a Calabi-Yau structure (Ω, ω). $\varphi = \Omega + dt \wedge \omega$ defines a torsion-free G_2-structure on $M = S^1 \times X$. The typical fibre of $\mathcal{M} \to \mathcal{W}_{G_2}$ is S^1. The fibre containing φ corresponds exactly to the S^1 fibre of Calabi-Yau structures defining the same metric as (Ω, ω).

Example 5.3.5. In example 4.1.18 we constructed an irreducible compact Calabi-Yau manifold X^6 with structure (Ω, ω) and an involution a such that $a^*\Omega = \Omega$, $a^*\omega = -\omega$. The involution acts as a reflection on the S^1 of torsion-free G_2-structures defining the product metric on $S^1 \times X$. The pair of fixed points descend to torsion-free G_2-structures on the quotient $M = S^1 \times X/(-1, a)$. Thus the moduli space of torsion-free G_2-structures on M is a double cover of the moduli space of G_2-metrics.

Remark 5.3.6. Let M^7 be a compact G_2-manifold. Hitchin [24] shows that the Hessian of the volume functional defines a well-defined non-degenerate symmetric bilinear form on $\mathcal{M}$. If we identify $T_{\varphi\mathcal{D}}\mathcal{M} \cong H^3(M)$ then the form is positive-definite on $H^3_{1\oplus7}(M)$ and negative-definite on $H^3_{27}(M)$. In particular the signature is $(1, b^3(M) - 1)$ when M is a connected irreducible G_2-manifold.

If g is a G_2-metric and φ is a torsion-free G_2-structure inducing g then $Dm_\varphi : \mathcal{H}^3 \to \varepsilon(g)$ gives an isomorphism $\mathcal{H}^3_{1\oplus27} \cong \varepsilon(g)$ and hence defines an inner product on $\varepsilon(g)$. Because

the irreducible components of $S^2 T^* M$ under G_2 are also representations of $SO(7)$ this inner product is independent of the choice of φ. Therefore $\mathcal{W}_{G_2}$ has a well-defined Lorentzian metric (whether M is irreducible or not), such that $m : \mathcal{M} \to \mathcal{W}_{G_2}$ is a pseudo-Riemannian submersion.

5.3.3 The asymptotically cylindrical case

The only part of the proof that does not carry over directly from the compact case is the identification of the fibres of $\mathcal{M}_+ \to \mathcal{W}_G$ in terms of the topology of M. But this only requires us to determine the dimension of the space of parallel spinors on any EAC G-manifold. For $G = Spin(7)$ this is provided by proposition 4.1.7. For $G = G_2$ the spinor bundle is $\mathbb{R} \oplus TM$, and the dimension of the space of parallel 1-forms is determined by Hodge theory (corollary 2.3.40).

Remark 5.3.7. Let $G = Spin(7)$ or G_2 and $H = G_2$ or $SU(3)$ respectively, and let M be an EAC G-manifold with cross-section X. Let $B : \mathcal{M}_+ \to \mathcal{N}$ be the boundary map from the moduli space of torsion-free EAC G-structures on M to the moduli space of torsion-free H-structures on X and let $B : \mathcal{W}_G \to \mathcal{W}_H$ be the boundary map from the moduli space of G-metrics on M to the moduli space of H-metrics on X. The diagram below obviously commutes.

$$
\begin{array}{ccc}
\mathcal{M}_+ & \overset{B}{\longrightarrow} & \mathcal{N} \\
{\scriptstyle m}\big\downarrow & & \big\downarrow{\scriptstyle m} \\
\mathcal{W}_G & \overset{B}{\longrightarrow} & \mathcal{W}_H
\end{array}
$$

It is easy to see that any tangent to the image $B(\mathcal{M}_+) \subseteq \mathcal{N}$ that is also tangent to the fibre of $m : \mathcal{N} \to \mathcal{W}_H$ is actually a tangent to the image of a fibre of $m : \mathcal{M}_+ \to \mathcal{W}_G$. Therefore $m : B(\mathcal{M}_+) \to B(\mathcal{W}_G)$ is a fibre bundle, whose fibres have the same dimension as those of $m : \mathcal{M}_+ \to \mathcal{W}_G$.

Chapter 6

Deformations of glued G_2-manifolds

Let $M_\pm^7$ be a pair of EAC G_2-manifolds with the same cross-section X^6, whose G_2-structures $\varphi_\pm$ have matching asymptotic models in a sense to be made precise below. M_+ and M_- can be glued along their cylindrical ends to form a compact smooth manifold M. Topologically M can be regarded as a generalised connected sum of M_+ and M_-. In [34] Kovalev explains a gluing construction, which produces a torsion-free G_2-structure φ on M from φ_+ and φ_- together with a gluing parameter $L \in \mathbb{R}^+$, which controls the diameter of M. Kovalev constructs examples where the resulting compact G_2-manifold M has holonomy exactly G_2. Increasing the gluing parameter in the construction gives a path of G_2-structures of increasing diameter, which has no limit in the moduli space of torsion-free G_2-structures on M. Intuitively this path corresponds to a boundary point of the moduli space, where M is 'pulled apart' into the connected summands $M_\pm$.

In this chapter we study how the gluing construction behaves under deformations. In particular we try to determine whether any sufficiently small torsion-free deformation of the resulting G_2-structure φ on M arises as the gluing of some deformations of the structures on $M_\pm$. To this end we define a smooth gluing map Y which takes values in the moduli space $\mathcal{M}$ of torsion-free G_2-structures on M, and study its local properties. This leads to a precise notion of how one can add boundary points to $\mathcal{M}$, representing ways of pulling M apart.

In §6.1 we describe the gluing construction in detail, and give statements of the main results. As the domain for the gluing map we take a moduli space $\mathcal{G}$ of 'gluing data', consisting of equivalence classes of arguments $(\varphi_+, \varphi_-, L)$ for the gluing construction with

L sufficiently large. $\mathcal{G}$ is a smooth manifold, and theorem 6.1.9 states that

$$Y : \mathcal{G} \to \mathcal{M}$$

is a local diffeomorphism. In §6.2 we discuss the topology of the glued manifold M. A key result for computing the derivative of the gluing map is theorem 6.2.6, which is a Hodge theory gluing result.

The main theorem 6.1.9 is proved in §6.3. In order to show that $\mathcal{G}$ is a smooth manifold we make use of theorem 4.2.4 about the boundary map on the moduli space of torsion-free EAC G_2-structures.

In §6.4 we look at the consequences of the gluing construction for the moduli space $\mathcal{M}$. $\mathcal{G}$ can be considered as a fibre bundle with typical fibre $\mathbb{R}^+$ over a space $\mathcal{B}$ of equivalence classes of matching pairs (φ_+, φ_-). We consider how $\mathcal{M}$ can be partially compactified by attaching a quotient of $\mathcal{B}$ as a boundary, so that the path of torsion-free $\mathcal{M}$ of increasing neck length arising from gluing a matching pair (φ_+, φ_-) converges to a boundary point.

6.1 Setup

6.1.1 Gluing construction

Let $M_\pm^7$ be a pair of oriented manifolds, each with a single cylindrical end, and the same cross-section X^6. We assume that X is oriented so that its orientation agrees with that defined by M_+ on its boundary, and is the reverse of that defined by M_- on its boundary. This ensures that the connected sum of M_+ and M_- obtained by identifying their boundaries at infinity is oriented. Let $t_\pm$ be cylindrical coordinates on $M_\pm$ respectively.

Definition 6.1.1. Let $\varphi_\pm$ be torsion-free EAC G_2-structures on $M_\pm$. The pair (φ_+, φ_-) is said to *match* if their asymptotic models are $\Omega \pm dt_\pm \wedge \omega$, respectively, for some Calabi-Yau structure (Ω, ω) on X compatible with the chosen orientation. Let $\mathcal{X}_y$ be the space of such pairs.

Given $L \in \mathbb{R}^+$ let $M_\pm(L) = \{y \in M_\pm : t_\pm \leq L\}$. Identify the boundaries of $M_\pm(L)$ to form a compact smooth manifold $M(L)$, and let $j^* : X \hookrightarrow M(L)$ be the inclusion of the common boundary.

Remark 6.1.2. $M(L)$ is independent of L up to diffeomorphism, so we will often refer to it simply as M. On the other hand, sometimes we wish to emphasise that we use a particular

description $M(L)$, e.g. if we ever wish to give a representative for a de Rham cohomology class on M we consider it as a closed form on $M(L)$ for some L.

For notational convenience we suppose that the cylindrical part of $M_\pm$ is given by $t_\pm > -2$ rather than $t_\pm > 0$. By lemma 2.3.36 there is for every closed exponentially asymptotically translation-invariant m-form $\alpha_\pm$ on $M_\pm$ a decaying $(m-1)$-form $\eta_\pm(\alpha_\pm)$ which is supported on $t_\pm > L-2$, such that $\alpha_\pm + d\eta_\pm(\alpha_\pm)$ is translation-invariant on $t_\pm > L-1$.

For $(\varphi_+, \varphi_-) \in \mathcal{X}_y$ let $\tilde{\varphi}_\pm = \varphi_\pm + d\eta_\pm(\varphi_\pm)$. Then we can define a G_2-structure $\tilde{\varphi}(\varphi_+, \varphi_-, L)$ on $M(L)$ by $\tilde{\varphi}|_{M_\pm(L)} = \tilde{\varphi}_\pm|_{M_\pm(L)}$. Note that the choice of (cut-off function in the definition of) $\eta_\pm$ does not affect the cohomology class of $\tilde{\varphi}(\varphi_+, \varphi_-, L)$.

Proposition 6.1.3. *There is an upper semi-continuous map $L_0 : \mathcal{X}_y \to \mathbb{R}^+$ such that for any $(\varphi_+, \varphi_-) \in \mathcal{X}_y$ and $L > L_0(\varphi_+, \varphi_-)$ there is a unique diffeomorphism class of torsion-free G_2-structures on $M(L)$ in a small neighbourhood of $\tilde{\varphi}(\varphi_+, \varphi_-, L)$ in its cohomology class.*

Sketch proof. The idea is that for large L the torsion of $\tilde{\varphi}(\varphi_+, \varphi_-, L)$ is very small, and the structure can be perturbed to a torsion-free one using a contraction-mapping argument. For details see Kovalev [34, §5], in principle inspired by Floer [16].

The result can also be deduced from theorem 7.2.2 below, due to Joyce (with the uniqueness part coming from proposition 7.2.4). $\qquad\square$

Remark 6.1.4. The resulting G_2-metric on $M(L)$ has an almost cylindrical 'neck' of length roughly $2L$. The diameter is diam $M(L) \sim 2L$ as $L \to \infty$.

One might expect to be able to find examples of matching pairs of G_2-structures by taking $M_- = M_+$, with the same G_2-metric and opposite orientation and cross-sections identified trivially. In fact, this only gives trivial examples; it is only possible when M_+ has a double cover isometric to a cylinder $X \times \mathbb{R}$ (such as example 4.1.18), and the gluing of M_+ and M_- is then covered by $X \times S^1$ and cannot have holonomy exactly G_2.

Proposition 6.1.5. *Let M_- be M_+ with its orientation reversed and (φ_+, φ_-) a matching pair of G_2-structures. If φ_+ and φ_- define the same metric then M_+ has a double cover isometric to a cylinder.*

Proof. $-\varphi_-$ is a torsion-free G_2-structure on M_+ which defines the same metric as φ_+. By remark 5.3.3 this corresponds to a parallel section of $\mathbb{P}(\mathbb{R} \oplus TM_+)$. The matching condition for φ_+ and φ_- implies that the parallel section is asymptotic to $[\frac{\partial}{\partial t}]$. In other words either

M_+ or a double cover of M_+ has a parallel vector field asymptotic to $\pm\frac{\partial}{\partial t}$. By corollary 2.3.40 this is impossible for a manifold with a single end, so M_+ has a double cover which is isometric to a cylinder by theorem 4.1.3. $\qquad\square$

This does not mean that it is impossible to obtain irreducible G_2-manifolds by gluing isometric EAC G_2-manifolds, only that the identification of the boundaries must be distinct from the asymptotic limit of the isometry. In §7.3.3 we see examples where an EAC G_2-manifold is glued to a copy of itself, using an anti-holomorphic isometry of the boundary.

6.1.2 Results

We need to find an appropriate domain for the gluing map Y. Let $\mathcal{M}_\pm$ denote the moduli space of torsion-free EAC G_2-structures on $M_\pm$, and $\mathcal{N}$ the moduli space of Calabi-Yau structures on their common cross-section X. We can define a subset $\mathcal{M}_y \subseteq \mathcal{M}_+ \times \mathcal{M}_-$ consisting of pairs which have matching images in $\mathcal{N}$. While we can apply our understanding of $\mathcal{M}_\pm$ and their relationship to $\mathcal{N}$ (particularly theorem 4.2.4) to show that $\mathcal{M}_y$ is a manifold, it is not an appropriate domain. The reason is that for a matching pair of points in the moduli spaces $\mathcal{M}_+, \mathcal{M}_-$ there is some ambiguity in how to glue them. This arises both from choosing representatives of the diffeomorphism classes (this ambiguity in some sense corresponds to the quotient of the automorphisms of X by a subgroup generated by elements which extend to automorphisms of M_+ or M_-), and from the dependence of the gluing construction on a gluing parameter L. Instead we define the gluing map on a moduli space of arguments for the gluing construction.

Definition 6.1.6. A set of *gluing data* is a triple $(\varphi_+, \varphi_-, L)$ such that $(\varphi_+, \varphi_-) \in \mathcal{X}_y$ and $L > L_0(\varphi_+, \varphi_-)$. Let G_0 be the space of gluing data.

G_0 is an open subset of $\mathcal{X}_y \times \mathbb{R}$. If we let $\mathcal{M}$ be the moduli space of torsion-free G_2-structures on M then proposition 6.1.3 provides a well-defined smooth map

$$Y : G_0 \to \mathcal{M}. \tag{6.1}$$

Let $\mathcal{D}_\pm$ be the group of EAC diffeomorphisms of $M_\pm$ isotopic to the identity (cf. definition 2.3.6). Since the cross-sections of M_+ and M_- have been identified once and for all we should only expect the gluing map to be invariant under diffeomorphisms that respect this identification.

Definition 6.1.7. $(\phi_+, \phi_-) \in \mathcal{D}_+ \times \mathcal{D}_-$ is a *matching pair of EAC diffeomorphisms* if the asymptotic models of $\phi_\pm$ are $(x, t_\pm) \mapsto (\Xi_\pm(x), t_\pm + h_\pm)$ with $\Xi_+ = \Xi_-$. Let $\mathcal{D}_y$ be the identity component of the group of such pairs.

For $(\phi_+, \phi_-) \in \mathcal{D}_y$ let $h = \frac{1}{2}(h_+ + h_-)$, and define an action on $\mathcal{X}_y \times \mathbb{R}$ by

$$\phi^* : (\varphi_+, \varphi_-, L) \mapsto (\phi_+^* \varphi_+, \phi_-^* \varphi_-, L - h). \tag{6.2}$$

The map L_0 is not $\mathcal{D}_y$-invariant, so nor is the open set $G_0 \subseteq \mathcal{X}_y \times \mathbb{R}$. Nevertheless we can define

Definition 6.1.8. The *moduli space of gluing data* is $\mathcal{G}_0 = G_0 \mathcal{D}_y / \mathcal{D}_y$.

We will show that $\mathcal{G}_0$ is a smooth manifold. Moreover the gluing map (6.1) really is invariant under the action of $\mathcal{D}_y$, and therefore descends to a smooth map

$$Y : \mathcal{G}_0 \to \mathcal{M}. \tag{6.3}$$

Proposition 6.3.9 computes the derivative of the gluing map (6.3). For each matching pair (φ_+, φ_-) the derivative is invertible at $(\varphi_+, \varphi_-, L)\mathcal{D}_y \in \mathcal{G}_0$ for all but finitely many $L > L_0(\varphi_+, \varphi_-)$. Therefore Y is a local diffeomorphism on some open subset $\mathcal{G} \subseteq \mathcal{G}_0$ whose gluing parameters are sufficiently large. This gives the main result of the chapter.

Theorem 6.1.9. *The gluing map $Y : \mathcal{G} \to \mathcal{M}$ is a local diffeomorphism.*

The most important tool in the proof is that the projection $\pi_H : \mathcal{M} \to H^3(M)$ to de Rham cohomology is a local diffeomorphism. This means that we can study the local properties of the gluing map in terms of what the gluing does to the cohomology classes. This is discussed in §6.2. In particular we prove a Hodge theory gluing result.

We can restrict (6.2) to an action of $\mathcal{D}_y$ on the space $\mathcal{X}_y$ of matching pairs. The quotient $\mathcal{B}$ is smooth, and $\mathcal{G}$ is obviously a fibre bundle over $\mathcal{B}$ with typical fibre $\mathbb{R}^+$. $\mathcal{G}$ can be included in a fibre bundle $\overline{\mathcal{G}}$ over $\mathcal{B}$ with typical fibre $(0, \infty]$ (so the boundary is $\mathcal{B}$). In §6.4 we deduce from theorem 6.1.9 that $\mathcal{M}$ can be partially compactified by inclusion in a topological manifold $\overline{\mathcal{M}}$ whose boundary $\hat{\mathcal{B}}$ is covered by $\mathcal{B}$. The boundary points parametrise ways of 'pulling apart' M into a pair of EAC connected-summands.

Theorem 6.1.10. *$\mathcal{M}$ can be identified with the interior of a topological manifold $\overline{\mathcal{M}}$ with boundary $\hat{\mathcal{B}}$, so that the following holds: If $(\varphi_+, \varphi_-) \in \mathcal{X}_y$ then the path $Y(\varphi_+, \varphi_-, L)$ converges to the image of (φ_+, φ_-) in $\hat{\mathcal{B}}$ as $L \to \infty$.*

Remark 6.1.11. It may be possible to pull apart a compact G_2-manifold in more than one direction. Theorem 6.1.10 can be used to attach one 'face' to the moduli space for each topologically distinct way of writing the manifold as a union of a pair of manifolds with cylindrical ends identified at the boundary.

6.2 Gluing and topology

6.2.1 Topology of the connected sum

Let $M_\pm^n$ be oriented manifolds each with a single cylindrical end, which have common cross-section X. As above we assume that X is oriented compatibly with M_+ and reverse to M_-, and we form a generalised connected sum M. We collect here some results about the topology of M that we will use.

As we remarked before, as a smooth manifold M is independent of the choice of gluing parameter L. Up to isotopy there are natural inclusion maps
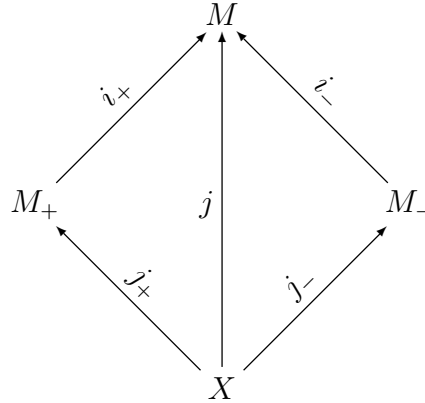
A large part of what we need to know about the topology is contained in the Mayer-Vietoris sequence for $M = M_+ \cup M_-$ and the sequence for the cohomology of $M_\pm$ relative to X.

$$\cdots \longrightarrow H^{m-1}(X) \xrightarrow{\delta} H^m(M) \xrightarrow{i_+^* \oplus i_-^*} H^m(M_+) \oplus H^m(M_-) \xrightarrow{j_+^* - j_-^*} H^m(X) \longrightarrow \cdots \quad (6.4)$$

$$\cdots \longrightarrow H^{m-1}(X) \xrightarrow{\partial_\pm} H^m_{cpt}(M_\pm) \xrightarrow{e_\pm} H^m(M_\pm) \xrightarrow{j_\pm^*} H^m(X) \longrightarrow \cdots \quad (6.5)$$

The inclusions $i_\pm : M_\pm \hookrightarrow M$ induce maps $i_{\pm *} : H^m_{cpt}(M_\pm) \to H^m(M)$. Note that

$$\delta = i_{+*} \circ \partial_+ = -i_{-*} \circ \partial_-. \qquad (6.6)$$

$j^*_\pm : H^m(M_\pm) \to H^m(X)$ is the Poincaré dual of $\pm\partial_\pm : H^{n-m-1}(X) \to H^{n-m}(M)$ (the sign difference comes from our assumption on the orientations of $M_\pm$ and X). The Poincaré dual of the Mayer-Vietoris sequence is the sequence for relative cohomology of M relative to X.

$$\cdots \; H^{m-1}(X) \xrightarrow{\partial_+ \oplus \partial_-} H^m_{cpt}(M_+) \oplus H^m_{cpt}(M_-) \xrightarrow{i_{+*}+i_{-*}} H^m(M) \xrightarrow{j^*} H^m(X) \; \cdots \qquad (6.7)$$

Lemma 6.2.1. *If $M_\pm$ are Ricci-flat EAC manifolds then $j^* : H^1(M) \to H^1(X)$ is injective.*

Proof. The kernel of j^* is the image of $H^1_{cpt}(M_+) \oplus H^1_{cpt}(M_-)$, which is 0 by corollary 2.3.42. $\qquad \square$

Denote the image of $j^*_\pm : H^m(M_\pm) \to H^m(X)$ by $A^m_\pm$, and let A^m_d be the image of $j^* : H^m(M) \to H^m(X)$. By the exactness of the Mayer-Vietoris sequence $A^m_d = A^m_+ \cap A^m_-$.

6.2.2 Gluing and cohomology

We explain how to glue a matching pair of closed forms on M_+, M_- to a well-defined cohomology class on M.

Let $\mathcal{Z}^m_y$ be the set of matching pairs of closed exponentially asymptotically translation-invariant m-forms on M_+, M_-, i.e. (ψ_+, ψ_-) such that $\psi_\pm$ is a closed exponentially asymptotically translation-invariant m-form on $M_\pm$, $B_a(\psi_+) = B_a(\psi_-)$, and $B_e(\psi_+) = -B_e(\psi_-)$.

If $(\psi_+, \psi_-) \in \mathcal{Z}^m_y$ and $L > 0$ let $\tilde{\psi}_\pm = \psi_\pm + d\eta_\pm(\psi_\pm)$. Choose the cut-off function for the cylinders in the definition (2.35) of $\eta_\pm$ to ensure that $\tilde{\psi}_\pm$ is translation-invariant on $t_\pm > 0$. Then we can define $\tilde{\psi}(\psi_+, \psi_-, L)$ on $M(L)$ by $i^*_\pm \tilde{\psi} = \tilde{\psi}_\pm$. We define a gluing map

$$Y_H : \mathcal{Z}^m_y \times \mathbb{R}^+ \to H^m(M), \;\; (\psi_+, \psi_-, L) \mapsto [\tilde{\psi}]. \qquad (6.8)$$

Y_H is independent of the choice of $\eta_\pm$, so well-defined. Furthermore, Y_H is invariant under the action of the group $\mathcal{D}_y$ of matching diffeomorphisms, defined similarly to (6.2).

Definition 6.2.2. For $(\phi_+, \phi_-) \in \mathcal{D}_y$ with asymptotic models $(x, t_\pm) \mapsto (\Xi(x), t_\pm + h_\pm)$ let $h = \frac{1}{2}(h_+ + h_-)$, and define an action on $\mathcal{Z}_y^m \times \mathbb{R}$ by

$$\phi^* : (\psi_+, \psi_-, L) \mapsto (\phi_+^* \psi_+, \phi_-^* \psi_-, L - h). \tag{6.9}$$

Proposition 6.2.3. *If* $(\psi_+, \psi_-, L) \in \mathcal{Z}_y^m \times \mathbb{R}^+$ *and* $(\phi_+, \phi_-) \in \mathcal{D}_y$ *with* $h_\pm < L$ *then*

$$Y_H(\psi_+, \psi_-, L) = Y_H(\phi_+^* \psi_+, \phi_-^* \psi_-, L - h) \in H^m(M).$$

Sketch proof. Let $\tilde{\psi} = \tilde{\psi}(\psi_+, \psi_-, L)$ and $\tilde{\psi}' = \tilde{\psi}(\phi_+^* \psi_+, \phi_-^* \psi_-, L - h)$. ϕ_+ and ϕ_- can be approximately glued to form a diffeomorphism $\tilde{\phi} : M(L - h) \to M(L)$. This pulls back $[\tilde{\psi}]$ to $[\tilde{\psi}']$. $\qquad \square$

Proposition 6.2.4. *Let* $(\psi_+, \psi_-) \in \mathcal{Z}_y^m$ *with* $B_e(\psi_\pm) = \pm \tau$. *If* $L, h \in \mathbb{R}^+$ *then*

$$Y_H(\psi_+, \psi_-, L + h) = Y_H(\psi_+, \psi_-, L) + 2h\delta([\tau]), \tag{6.10}$$

where δ *is the boundary homomorphism appearing in the Mayer-Vietoris sequence* (6.4).

Proof. It suffices to prove the result separately for the cases $B_a(\psi) = 0$ and $B_e(\psi) = 0$.

If $B_e(\psi) = 0$ pick a diffeomorphism $f : (0, L) \to (0, L + h)$ which is *id* on $(0, 1)$ and $id + h$ on $(L - 1, L)$. We can define a diffeomorphism $M(L) \to M(L + h)$ which is the identity on the images of the compact pieces $M_\pm(0)$ in $M(L)$ and $(x, t) \mapsto (x, f(t))$ on the cylindrical part. This pulls back $\tilde{\psi}(\psi_+, \psi_-, L)$ to $\tilde{\psi}(\psi_+, \psi_-, L + h)$.

If $B_a(\psi) = 0$ then let $c_\pm = \pm \tilde{\psi}_\pm - d(\rho_\pm t_\pm \tau)$, with $\rho_\pm$ chosen so that $c_\pm$ is supported in $t_\pm < 1$. By definition of the Mayer-Vietoris boundary map δ the form on $M(L)$ obtained by gluing $d(\rho_+ t_+ \tau)$ and $-d(\rho_- t_- \tau)$ is cohomologous to $\delta((t_+ + t_-)[\tau]) = 2L\delta([\tau])$ for any L. Hence for any L

$$Y_H(\psi_+, \psi_-, L) = i_{+*}([c_+]) + i_{-*}(-[c_-]) + 2L\delta([\tau]). \tag{6.11}$$

Since $i_{\pm *}$ and δ are independent of L the result follows. $\qquad \square$

It is convenient to use proposition 6.2.4 to extend Y_H to negative gluing parameters in a well-defined way.

Definition 6.2.5. Define

$$Y_H : \mathcal{Z}_y^m \times \mathbb{R} \to H^m(M)$$

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as (6.8) on $\mathcal{Z}_y \times \mathbb{R}^+$, and extend for any $L > 0$ and $h \in \mathbb{R}$ by (6.10).

6.2.3   Hodge theory

Now suppose that $M_\pm^n$ have EAC Riemannian metrics whose cylindrical models match on the cylinders. We wish to consider what the gluing of closed forms described in the previous subsection does on pairs of harmonic forms.

Let $\mathcal{H}_{\pm,0}^m$ be the space of bounded harmonic m-forms on $M_\pm$. The space of matching pairs of harmonic forms is

$$\mathcal{H}_y^m = (\mathcal{H}_{+,0}^m \times \mathcal{H}_{-,0}^m) \cap \mathcal{Z}_y^m.$$

We prove that any cohomology class on M can be obtained by gluing a matching pair of harmonic forms, except when the gluing parameter is an eigenvalue of a certain self-adjoint endomorphism which is to be defined below.

Theorem 6.2.6. *Let $M_\pm$ have EAC metrics. Considering L as a parameter, the linear map*

$$Y_H : \mathcal{H}_y^m \to H^m(M), \quad (\psi_+, \psi_-) \mapsto Y_H(\psi_+, \psi_-, L) \tag{6.12}$$

is an isomorphism except when $-2L$ is an eigenvalue of

$$\pi_E(\partial_+^{-1} C_+ + \partial_-^{-1} C_-) : E_d^{m-1} \to E_d^{m-1}. \tag{6.13}$$

We use notation for harmonic forms analogous to that in the summary in §4.1.1, and add subscripts $\pm$ to distinguish between objects associated to M_+ and M_-. We write $H^m(X)$ as orthogonal direct sums $A_\pm^m \oplus E_\pm^m$, where $A_\pm^m$ is the image of $j_\pm^* : H^m(M_\pm) \to H^m(X)$. Let $\mathcal{A}_\pm^m$ and $\mathcal{E}_\pm^m$ be the spaces of harmonic representatives.

Let $\mathcal{A}_d^m = \mathcal{A}_+^m \cap \mathcal{A}_-^m$. This is then the space of harmonic representatives for A_d^m. Similarly let $\mathcal{E}_d^m = \mathcal{E}_+^m \cap \mathcal{E}_-^m$ and denote the corresponding subspace of $H^m(X)$ by E_d^m.

Recall that $\partial_\pm$ denotes the boundary map in the long exact sequence for relative cohomology (6.5). We define an isomorphism

$$C_\pm : E_\pm^{m-1} \to \operatorname{im} \partial_\pm \subseteq H_{cpt}^m(M_\pm)$$

as follows. If $\tau \in \mathcal{E}_\pm^{m-1}$ let ψ be the unique element of $\mathcal{H}_{\pm,E}^m$ (the bounded exact harmonic forms on $M_\pm$) such that $B_e(\psi) = \tau$. Taking $\eta_\pm$ as defined in (2.35), $\psi + d\eta_\pm(\psi) - d(\rho_\pm t_\pm \tau)$

has compact support, so represents a class $C_\pm([\tau]) \in H^m_{cpt}(M_\pm)$. This class is mapped to 0 by $e_\pm$, so lies in the image of $\partial_\pm$.

Remark 6.2.7. $C_\pm$ is independent of the choice of $\rho_\pm$, but depends on both the metric and the cylindrical coordinate: replacing $t_\pm$ by $t_\pm + \lambda$ adds $\lambda \partial_\pm$ to $C_\pm$.

Composing $C_\pm$ with the inverse of $\partial_\pm : E^{m-1}_\pm \to \operatorname{im} \partial_\pm$ gives an endomorphism $\partial_\pm^{-1} C_\pm$ of $E^{m-1}_\pm$.

Lemma 6.2.8. $\partial_\pm^{-1} C_\pm : E^m_\pm \to E^m_\pm$ *is self-adjoint.*

Proof. This is really a rephrasing of proposition 2.3.38. If $\chi \in \mathcal{H}^m_{\pm,-}$ with $BD_a(\chi) = \begin{pmatrix} \beta \\ \alpha \end{pmatrix} \in (\mathcal{H}^m_X)^2$ then $d\chi$ is the unique element of $\mathcal{H}^{n+1}_{\pm,E}$ such that $B_e(d\chi) = \beta$.

$$C_\pm[\beta] = [d(\chi + \eta_\pm(d\chi) - \rho_\pm t_\pm \beta)] = \partial_\pm[\alpha] \in H^{m+1}_{cpt}(M_\pm),$$

so $\partial_\pm^{-1} C_\pm[\beta]$ is the L^2-orthogonal projection of $[\alpha]$ to $E^m_\pm$. The result thus follows from proposition 2.3.38. $\square$

It follows that the endomorphism (6.13) is self-adjoint too.

Proof of theorem 6.2.6. Consider the map $(i^*_+ \oplus i^*_-) : H^m(M) \to H^m(M_+) \oplus H^m(M_-)$ in the Mayer-Vietoris sequence. Recall that L is fixed so that Y_H is regarded as a linear map $\mathcal{H}^m_y \to H^m(M)$. In order to prove that (6.12) is an isomorphism it suffices to show that $\operatorname{im}((i^*_+ \oplus i^*_-) \circ Y_H) = \operatorname{im}(i^*_+ \oplus i^*_-)$ and that $Y_H : \ker((i^*_+ \oplus i^*_-) \circ Y_H) \to \ker(i^*_+ \oplus i^*_-)$ is an isomorphism.

$(i^*_+ \oplus i^*_-)Y_H(\psi_+, \psi_-) = ([\psi_+], [\psi_-])$ and it follows from the exactness of the Mayer-Vietoris sequence that $\operatorname{im}((i^*_+ \oplus i^*_-) \circ Y_H) = \operatorname{im}(i^*_+ \oplus i^*_-)$. Also $\ker((i^*_+ \oplus i^*_-) \circ Y_H)$ equals $\mathcal{H}^m_{y,E}$, the pairs of exact forms in $\mathcal{H}^m_y$.

Thus the problem reduces to determining whether the restriction

$$Y_H : \mathcal{H}^m_{y,E} \to \ker(i^*_+ \oplus i^*_-)$$

of (6.12) is an isomorphism. Given $\tau \in \mathcal{E}^{m-1}_d$ let (ψ_+, ψ_-) be the unique element of $\mathcal{H}^m_{y,E}$ such that $\tau = B_e(\psi_+) = -B_e(\psi_-)$. By the definition of $C_\pm$ and (6.11)

$$Y_H(\psi_+, \psi_-) = i_{+*}C_+([\tau]) + i_{-*}C_-([-\tau]) + 2L\delta([\tau]).$$

Combining with (6.6) we find

$$Y_H(\psi_+, \psi_-) = \delta \left(\partial_+^{-1} C_+([\tau]) + \partial_-^{-1} C_-([\tau]) + 2L[\tau] \right). \qquad (6.14)$$

$\delta : H^{m-1}(X) \to H^m(M)$ maps $E_d^{m-1} \to \ker(i_+^* \oplus i_-^*)$ isomorphically and vanishes on the orthogonal complement of E_d^{m-1}. Hence (6.14) is an isomorphism $\mathcal{H}_{y,E}^m \to \ker(i_+^* \oplus i_-^*)$ unless $-2L$ is an eigenvalue of the endomorphism (6.13). $\qquad \square$

6.3 The gluing map

We will now make use of the topological results of the previous subsections to study the gluing map for torsion-free G_2-structures. As in §6.1, M_+ and M_- are EAC G_2-manifolds with a common cross-section X and M is their connected sum. $\mathcal{M}$ denotes the moduli space of torsion-free G_2-structures on M and G_0 the space of gluing data.

In order to prove theorem 6.1.9 we need to show that the gluing map is $\mathcal{D}_y$-invariant so that it is well-defined on $\mathcal{G}_0 = G_0 \mathcal{D}_y / \mathcal{D}_y$, to show that $\mathcal{G}_0$ is a smooth manifold and to compute the derivative of the gluing map.

6.3.1 Diffeomorphism invariance

Note that the composition $\pi_H \circ Y : G_0 \to H^3(M)$ of the gluing map (6.1) with the local diffeomorphism $\pi_H : \mathcal{M} \to H^3(M)$ is simply the restriction to G_0 of the map Y_H given by definition 6.2.5. We will use this first to show that Y induces a well-defined map on $\mathcal{G}_0$, i.e. the quotient of G_0 by the group $\mathcal{D}_y$ of matching EAC diffeomorphisms. Later we will determine the local properties of $Y : \mathcal{G}_0 \to \mathcal{M}$ from those of $Y_H : (\mathcal{X}_y \times \mathbb{R}) / \mathcal{D}_y \to H^3(M)$.

Proposition 6.3.1. *The map Y is $\mathcal{D}_y$-invariant, so descends to a well-defined continuous function*

$$Y : \mathcal{G}_0 \to \mathcal{M}. \qquad (6.15)$$

Proof. We need to show that if $\phi \in \mathcal{D}_y$ and $(\varphi_+, \varphi_-, L) \in G_0$ such that $\phi^*(\varphi_+, \varphi_-, L) \in G_0$ then

$$Y(\varphi_+, \varphi_-, L) = Y(\phi^*(\varphi_+, \varphi_-, L)).$$

The idea of the proof is to connect $(\varphi_+, \varphi_-, L)$ and $\phi^*(\varphi_+, \varphi_-, L)$ by a path in G_0. The image under Y of this path is the lift by the local diffeomorphism $\pi_H : \mathcal{M} \to H^3(M)$ of a path in $H^3(M)$, which is determined by propositions 6.2.3 and 6.2.4.

Let $[0,1] \to \mathcal{D}_y$, $s \mapsto \phi_s$ be a path connecting the identity to ϕ, and take k sufficiently large that $\phi_s^*(\varphi_+, \varphi_-, L + k) \in G_0$ for all s. By proposition 6.2.3 the path $[0,1] \to \mathcal{M}$, $s \mapsto Y(\phi_s^*(\varphi_+, \varphi_-, L + k))$ is a lift of a constant path in $H^3(M)$, so

$$Y(\phi^*(\varphi_+, \varphi_-, L + k)) = Y((\varphi_+, \varphi_-, L + k)).$$

By proposition 6.2.4 the paths $[0, k] \to \mathcal{M}$

$$s \mapsto Y(\phi^*(\varphi_+, \varphi_-, L + k - s)),$$
$$s \mapsto Y(\varphi_+, \varphi_-, L + k - s)$$

are both lifts of $s \mapsto Y_H(\varphi_+, \varphi_-, L + k) - 2s\delta([\omega])$, so in particular they have the same value at $s = 1$, which gives the result. $\qquad \square$

6.3.2 A coordinate chart

Next we describe coordinate charts for $(\mathcal{X}_y \times \mathbb{R})/\mathcal{D}_y$, which contains $\mathcal{G}_0$ as an open subset. This mainly relies on the pre-moduli spaces of torsion-free EAC G_2-structures that were constructed in §4.2. The definition (6.2) of the action of $\mathcal{D}_y$ on $\mathcal{X}_y \times \mathbb{R}$ can be restricted to give an action on $\mathcal{X}_y$, and $(\mathcal{X}_y \times \mathbb{R})/\mathcal{D}_y$ is obviously a principal $\mathbb{R}$-bundle over $\mathcal{B} = \mathcal{X}_y/\mathcal{D}_y$. It therefore suffices to find charts for $\mathcal{B}$.

Let $\mathcal{M}_\pm$ be the moduli space of torsion-free EAC G_2-structures on $M_\pm$, and $\mathcal{N}$ the moduli space of Calabi-Yau structures on X. Let $\mathcal{M}_y \subseteq \mathcal{M}_+ \times \mathcal{M}_-$ be the pairs of diffeomorphism classes of EAC torsion-free G_2-structures whose boundary images in $\mathcal{N}$ match. There is a natural projection

$$\mathcal{B} \to \mathcal{M}_y, \quad (\varphi_+, \varphi_-)\mathcal{D}_y \mapsto (\varphi_+ \mathcal{D}_+, \varphi_- \mathcal{D}_-). \tag{6.16}$$

If $b^1(M) = 0$ then we will see that this projection is a covering map. We begin by finding charts for $\mathcal{M}_y$.

Proposition 6.3.2. *$\mathcal{M}_y$ is a submanifold of $\mathcal{M}_+ \times \mathcal{M}_-$.*

Each point in $\mathcal{M}_y$ can be represented by a matching pair of torsion-free G_2-structures (φ_+, φ_-), asymptotic to a Calabi-Yau structure (Ω, ω) on X. Let $\mathcal{R}_\pm$ be the pre-moduli space of torsion-free EAC G_2-structures near $\varphi_\pm$.

Definition 6.3.3. The pre-moduli space of matching pairs of torsion-free EAC G_2-structures near (φ_+, φ_-) is a neighbourhood $\mathcal{R}_y$ of (φ_+, φ_-) in $\mathcal{X}_y \cap (\mathcal{R}_+ \times \mathcal{R}_-)$.

To use $\mathcal{R}_y$ as a coordinate chart we first need to show that its boundary values form a manifold. Let $\mathcal{Q}$ be the pre-moduli space of Calabi-Yau structures near (Ω, ω). By theorem 4.2.4 the boundary map

$$B_\pm : \mathcal{R}_\pm \to \mathcal{Q}$$

is a submersion onto its image, which is an open subset of a submanifold $\mathcal{Q}_{\pm,A}$ defined by the equations $[\Omega'] \in A_\pm^3, [\omega'^2] \in A_\pm^4$. Thus the intersection

$$\mathcal{Q}_{d,A} = \mathcal{Q}_{+,A} \cap \mathcal{Q}_{-,A}$$

is defined as a subset of $\mathcal{Q}$ by $[\Omega'] \in A_d^3, [\omega'^2] \in A_d^4$.

Lemma 6.3.4. $\mathcal{Q}_{d,A} \subseteq \mathcal{Q}$ is a submanifold.

Proof. We can recycle the proof of proposition 4.2.9. □

Proof of proposition 6.3.2. Let $\mathcal{D}_X$ be the group of diffeomorphisms of X isotopic to the identity. If $(\psi_+, \psi_-) \in \mathcal{R}_+ \times \mathcal{R}_-$ then the boundary values $B_\pm(\psi_\pm)$ both lie in the pre-moduli space $\mathcal{Q}$. Therefore proposition 3.1.10 implies that they are $\mathcal{D}_X$-equivalent if and only if they are equal. Hence $\mathcal{R}_y$ is homeomorphic to a neighbourhood of $(\varphi_+ \mathcal{D}_+, \varphi_- \mathcal{D}_-)$ in $\mathcal{M}_y$, and it suffices to prove that $\mathcal{R}_y$ is a submanifold of $\mathcal{R}_+ \times \mathcal{R}_-$.

By lemma 6.3.4 the image of $\mathcal{Q}_{d,A}$ in $\mathcal{Q}_{+,A} \times \mathcal{Q}_{-,A}$ under the diagonal map is a submanifold. $\mathcal{R}_y \subseteq \mathcal{R}_+ \times \mathcal{R}_-$ is the inverse image of $\mathcal{Q}_{d,A} \subseteq \mathcal{Q}_{+,A} \times \mathcal{Q}_{-,A}$ under the submersion $B_+ \times B_- : \mathcal{R}_+ \times \mathcal{R}_- \to \mathcal{Q}_{+,A} \times \mathcal{Q}_{-,A}$, so it is a submanifold. □

Remark 6.3.5. This argument shows also that for $(\varphi_+, \varphi_-) \in \mathcal{X}_y$ the tangent spaces to the corresponding pre-moduli spaces $\mathcal{Q}_{d,A}$ and $\mathcal{R}_y$ are what one would naively expect. In particular

$$T_\varphi \mathcal{R}_y = \mathcal{H}_{y,cyl}^3, \tag{6.17}$$

the subspace of matching pairs in $\mathcal{H}_{+,cyl}^3 \times \mathcal{H}_{-,cyl}^3$.

Now consider the fibres of the projection (6.16). Given $(\psi_+, \psi_-) \in \mathcal{R}_y$ any point in the fibre above $(\psi_+, \psi_-)\mathcal{D}_y$ is represented by a matching pair $(\phi_+^* \psi_+, \phi_-^* \psi_-)$ with $\phi_\pm \in \mathcal{D}_\pm$. The pair (ϕ_+, ϕ_-) need not match, and even if it does it need not lie in the identity component $\mathcal{D}_y$ of the group of matching pairs of EAC diffeomorphisms. However, if $(x, t) \mapsto (\Xi_\pm(x), t + h_\pm)$

denotes the the asymptotic limit of $\phi_\pm$ then $\chi = \Xi_-^{-1} \circ \Xi_+ \in \mathcal{I}_X$, where $\mathcal{I}_X \subset \mathcal{D}_X$ is the group of automorphisms of the Calabi-Yau manifold X that are isotopic to the identity. Because ϕ_+ and ϕ_- are isotopic to the identity there is a path $h : \mathbb{R} \to \mathcal{D}_X$ which is id for $t \leq -1$ and ϕ for $t \geq 0$ such that $(\phi_+, \phi_- h) \in \mathcal{D}_y$ (regarding h as an EAC diffeomorphism of M_-).

Definition 6.3.6. Let $\tilde{\mathcal{I}}_X$ be the group of paths in $\mathcal{D}_X$ from the identity to elements of $\mathcal{I}_X$ modulo homotopies relative to the endpoints.

Remark 6.3.7. Proposition 3.1.10 implies that $\mathcal{I}_X$ and $\tilde{\mathcal{I}}_X$ are independent of of the Calabi-Yau structure in $\mathcal{Q}$. So are $\mathcal{A}_d^1$ and $\mathcal{A}_\pm^1$, the harmonic representatives of A_d^1 and $A_\pm^1$. Similarly the automorphism group $\mathcal{I}_{M_\pm}$ of the EAC G_2-manifold $M_\pm$ is independent of the choice of G_2-structure in $\mathcal{R}_\pm$.

If $\tilde{\chi} \in \tilde{\mathcal{I}}_X$ then a representative for $\tilde{\chi}$ can be considered as a diffeomorphism $h_\pm \in \mathcal{D}_\pm$. This is determined by $\tilde{\chi}$ up to multiplication by diffeomorphisms asymptotic to the identity. For $(\psi_+, \psi_-) \in \mathcal{R}_y$ let

$$\tilde{\chi}^*((\psi_+, \psi_-)\mathcal{D}_y) = (\psi_+, h_-^*\psi_-)\mathcal{D}_y \in \mathcal{B}. \tag{6.18}$$

This is well-defined, independent of the choice of h_-. Any point in the pre-image of (ψ_+, ψ_-) in $\mathcal{B}$ can be written in this form, so essentially the points of the fibre correspond to different ways of identifying the cylinders of the G_2-structures ψ_+ and ψ_-.

The identity component of $\tilde{\mathcal{I}}_X$ is the universal cover of the identity component of $\mathcal{I}_X$. Proposition 5.2.4 implies that $T_{id}\mathcal{I}_X$ is the Abelian Lie algebra $(\mathcal{H}_X^1)^\sharp$. That is therefore the universal cover of the identity component of $\mathcal{I}_X$, so we can identify the identity component of $\tilde{\mathcal{I}}_X$ with $H^1(X)$.

For $(\psi_+, \psi_-) \in \mathcal{R}_y$ let $S \subseteq \tilde{\mathcal{I}}_X$ be the stabiliser of $(\psi_+, \psi_-)\mathcal{D}_y$ under the action (6.18). Then

$$\tilde{\mathcal{I}}_X/S \times \mathcal{R}_y \to \mathcal{B}, \quad (\tilde{\chi}S, \psi_+, \psi_-) \mapsto \tilde{\chi}^*((\psi_+, \psi_-)\mathcal{D}_y) \tag{6.19}$$

is a homeomorphism onto a neighbourhood. $\tilde{\chi} \in S$ means that $\tilde{\chi} = \tilde{\chi}_+\tilde{\chi}_-$ for some $\tilde{\chi}_\pm \in \tilde{\mathcal{I}}_X$ such that the corresponding EAC diffeomorphisms $h_\pm \in \mathcal{D}_\pm$ are equivalent to automorphisms of $M_\pm$ modulo diffeomorphisms asymptotic to the identity. The Lie algebra of $\mathcal{I}_{M_\pm}$ corresponds to the bounded harmonic 1-forms $\mathcal{H}_{\pm,0}^1$, whose image under the boundary map $B_\pm$ is $\mathcal{A}_\pm^1 \subseteq \mathcal{H}_X^1$. Therefore the identity component of S is $A_+^1 \oplus A_-^1$. Each $A_\pm^1 \subseteq H^1(X)$ is a half-dimensional subspace according to proposition 4.1.6, while their intersection is $A_d^1 \cong H^1(M)$. Thus the components of $\tilde{\mathcal{I}}_X/S$ are planes of dimension $b^1(M)$.

The most interesting case is when M has holonomy exactly G_2. Then $b^1(M) = 0$, so $\mathcal{B} \to \mathcal{M}_y$ is a covering map. In general one can still show that transition function between the maps of the form (6.19) are smooth, and give $\mathcal{B}$ the structure of a smooth manifold.

Finally, let us write down a convenient coordinate chart for $(\mathcal{X}_y \times \mathbb{R})/\mathcal{D}_y$ based on (6.19). First, for $\tilde{\chi} \in \tilde{\mathcal{I}}_X$ we define

$$\tilde{\chi}^*((\psi_+, \psi_-, L)\mathcal{D}_y) = (\psi_+, h_-^* \psi_-, L)\mathcal{D}_y \in (\mathcal{X}_y \times \mathbb{R})/\mathcal{D}_y. \tag{6.20}$$

like (6.18). Note that if $\tilde{\chi}$ is a loop in $\mathcal{D}_X$ based at the identity then the glued G_2-structures $Y(\psi_+, \psi_-, L)$ and $Y(\tilde{\chi}^*(\psi_+, \psi_-, L))$ on M are certainly diffeomorphic, but not necessarily by a diffeomorphism isotopic to the identity.

Let $E_{d,\psi}^1$ be the orthogonal complement of $A_+^1 + A_-^1$ in $H^1(X)$ with respect to the Kähler metric on X defined by the boundary value of $(\psi_+, \psi_-) \in \mathcal{R}_y$. By lemma 6.2.1, $j^* : H^1(M) \to H^1(X)$ is injective and the image is precisely A_d^1. It follows from proposition 4.1.6 that the complex structure J_ψ maps $A_d^1 \to E_{d,\psi}^1$ bijectively. Composition gives a homomorphism

$$\beta_\psi : H^1(M) \to \tilde{\mathcal{I}}_X/S, \tag{6.21}$$

mapping onto the identity component. Hence

$$\mathcal{R}_y \times H^1(M) \times \mathbb{R} \to (\mathcal{X}_y \times \mathbb{R})/\mathcal{D}_y, \quad (\psi_+, \psi_-, a, L) \mapsto \beta_\psi(a)^*(\psi_+, \psi_-, L)\mathcal{D}_y \tag{6.22}$$

parametrises a neighbourhood of $(\varphi_+, \varphi_-, L)\mathcal{D}_y$ in $(\mathcal{X}_y \times \mathbb{R})/\mathcal{D}_y$.

6.3.3 The derivative of the gluing map

Since $\pi_H : \mathcal{M} \to H^3(M)$ is a local diffeomorphism the local behaviour of the gluing map $Y : \mathcal{G}_0 \to \mathcal{M}$ is determined by that of $Y_H = \pi_H \circ Y$. Y_H is just the gluing map for cohomology of definition 6.2.5, so can be defined on all of $(\mathcal{X}_y \times \mathbb{R})/\mathcal{D}_y$. We will compute the derivative.

Let $(\varphi_+, \varphi_-) \in \mathcal{X}_y$, and let $\mathcal{R}_y$ be the pre-moduli space of matching pairs of torsion-free EAC G_2-structures near it. In the chart (6.22), Y_H is affine linear in the $\mathbb{R}$ factor by proposition 6.2.4. The same is true for the $H^1(M)$ factor.

Lemma 6.3.8. *Let $(\psi_+, \psi_-) \in \mathcal{R}_y$ with common boundary (Ω, ω), and let δ be the boundary map in the Mayer-Vietoris sequence (6.4). Let $\alpha \in \mathcal{H}_X^1$, and let $\tilde{\chi}$ be the corresponding*

element of the identity component of $\tilde{\mathcal{I}}_X$. Then

$$Y_H(\tilde{\chi}^*(\psi_+, \psi_-, L)) = Y_H(\psi_+, \psi_-, L) - \delta([\alpha^\sharp \lrcorner \Omega]).$$

Proof. Let $\rho : \mathbb{R} \to [0,1]$ be a smooth function such that $\rho(t) = 0$ for $t \leq -1$ and $\rho(t) = 1$ for $t \geq 0$. Then $h : \mathbb{R} \to \mathcal{D}_X$, $t \mapsto \exp(\rho(t)\alpha^\sharp)$ is a representative for $\tilde{\chi}$. h can be considered as a diffeomorphism of M_-, and

$$Y_H(\tilde{\chi}^*(\psi_+, \psi_-, L)) = Y_H((\psi_+, h^*\psi_-, L)).$$

For $s \in [0,1]$

$$\frac{d}{ds}Y_H\left(\varphi_+, (\exp(\rho s \alpha^\sharp))^*\varphi_-, L\right) = Y_H\left(0, d(\rho\alpha^\sharp \lrcorner(\exp(\rho s \alpha^\sharp))^*\varphi_-), L\right) = -\delta\left([\alpha^\sharp \lrcorner \Omega]\right). \quad \square$$

Recall that $E_d^2 = *A_d^4$, the Hodge dual of the image of $j^* : H^4(M) \to H^4(X)$. Proposition 4.1.5 implies that E_d^2 splits by type:

$$E_d^2 = E_{d,1}^2 \oplus E_{d,6}^2 \oplus E_{d,8}^2.$$

Let $\pi_E : H^2(X) \to E_d^2$ denote the orthogonal projection. In §6.2.3 we described the self-adjoint endomorphism $\pi_E(\partial_+^{-1}C_+ + \partial_-^{-1}C_-)$ of E_d^2. By an argument similar to the proof of theorem 6.2.6 we show that the gluing map has invertible derivative at $(\varphi_+, \varphi_-, L)\mathcal{D}_y \in \mathcal{G}_0$ unless $-2L$ is an eigenvalue of $\pi_E(\partial_+^{-1}C_+ + \partial_-^{-1}C_-)$ for some eigenvector in $E_{d,8}^2$. In particular the derivative is invertible for large L.

Proposition 6.3.9. *Given $(\varphi_+, \varphi_-) \in \mathcal{X}_y$ the derivative of*

$$Y_H : (\mathcal{X}_y \times \mathbb{R})/\mathcal{D}_y \to H^3(M)$$

at $(\varphi_+, \varphi_-, L)\mathcal{D}_y$ is bijective for all sufficiently large L.

Proof. Let

$$Y_H' : \mathcal{R}_y \times H^1(M) \times \mathbb{R} \to H^3(M)$$

be the representation of Y_H in the coordinate chart (6.22). By (6.17) the tangent space of $\mathcal{R}_y$ at (φ_+, φ_-) is a space of matching pairs of harmonic forms $\mathcal{H}_{y,cyl}^3$. It can be described as the subspace of $\mathcal{H}_y^3$ consisting of pairs (ψ_+, ψ_-) such that the corresponding cylindrical form $B(\psi) = \sigma + dt \wedge \tau$ is in $\mathcal{H}_{SU}^3$ (i.e. satisfies (3.25)).

Now consider the derivative

$$(DY'_H)_{(\varphi_+, \varphi_-, 0, L)} : \mathcal{H}^3_{y,cyl} \times H^1(M) \times \mathbb{R} \to H^3(M).$$

As in theorem 6.2.6 we consider the map $(i^*_+ \oplus i^*_-) : H^3(M) \to H^3(M_+) \oplus H^3(M_-)$ in the Mayer-Vietoris sequence. To show that DY'_H is an isomorphism it suffices to show that $\mathrm{im}\,((i^*_+ \oplus i^*_-) \circ DY'_H) = \mathrm{im}\,(i^*_+ \oplus i^*_-)$, and that $DY'_H : \ker((i^*_+ \oplus i^*_-) \circ DY'_H) \to \ker(i^*_+ \oplus i^*_-)$ is an isomorphism.

The restriction of DY'_H to $\mathcal{H}^3_{y,cyl} \times 0 \times 0$ is just Y_H. We first show that

$$(i^*_+ \oplus i^*_-) \circ Y_H : \mathcal{H}^3_{y,cyl} \to \mathrm{im}\,(i^*_+ \oplus i^*_-)$$

is surjective. If $[\psi]$ is a class in $H^3(M)$, let $\psi_\pm$ be the unique representative of $i^*_\pm[\psi]$ in $\mathcal{H}^3_{\pm,cyl}$ such that $\pi_8 B_e(\psi_\pm) = 0$ (cf. corollary 4.2.10). Then (ψ_+, ψ_-) lies in $\mathcal{H}^3_{y,cyl}$ and satisfies $(i^*_+ \oplus i^*_-)Y_H(\psi_+, \psi_-) = (i^*_+[\psi], i^*_-[\psi])$.

It remains to determine whether

$$DY'_H : \ker((i^*_+ \oplus i^*_-) \circ DY'_H) \to \ker(i^*_+ \oplus i^*_-)$$

is an isomorphism. $\delta : H^2(X) \to H^3(M)$ restricts to an isomorphism $E^2_d \to \ker(i^*_+ \oplus i^*_-)$. The derivative DY'_H maps $\mathbb{R}$ isomorphically to $\delta(E^2_{d,1})$ by proposition 6.2.4. Also $\alpha \mapsto *[\alpha \wedge \Omega]$ defines an isomorphism $A^1_d \to E^2_{d,6}$ by proposition 4.1.5, so lemma 6.3.8 implies that DY'_H maps $H^1(M)$ isomorphically to $\delta(E^2_{d,6})$.

The kernel of $(i^*_+ \oplus i^*_-) \circ Y_H$ in $\mathcal{H}^3_{y,cyl}$ is $\mathcal{H}^3_{y,E,cyl} = \mathcal{H}^3_{y,cyl} \cap \mathcal{H}^3_{y,E}$. If we identify $\mathcal{H}^3_{y,E}$ with E^2_d by $(\psi_+, \psi_-) \mapsto B(\psi)$ then $\mathcal{H}^3_{y,E,cyl}$ is identified with $E^2_{d,8}$ (cf. corollary 4.2.10). By (6.14), DY'_H restricted to $\mathcal{H}^3_{y,E,cyl} \times 0 \times 0$ is identified with

$$E^2_{d,8} \to H^3(M), \ \ [\alpha] \mapsto \delta\left(\pi_E(\partial_+^{-1}C_+ + \partial_-^{-1}C_-)[\alpha] + 2L[\alpha]\right).$$

Hence the derivative is bijective unless $\pi_E(\partial_+^{-1}C_+ + \partial_-^{-1}C_-)$ has a non-zero eigenvector in $E^2_{d,8}$ with eigenvalue $-2L$. $\qquad\square$

Remark 6.3.10. It is possible to show that $\pi_E(\partial_+^{-1}C_+ + \partial_-^{-1}C_-)$ maps $E^2_{d,8}$ to itself.

Lemma 6.2.8 implies that $\pi_E(\partial_+^{-1}C_+ + \partial_-^{-1}C_-)$ has only real eigenvalues. Therefore

$$\{(\varphi_+, \varphi_-, L) \in \mathcal{X}_y \times \mathbb{R} : -2L \text{ is an eigenvalue of } \pi_E(\partial_+^{-1}C_+ + \partial_-^{-1}C_-)\}$$

is locally a union of graphs of smooth functions $\mathcal{X}_y \to \mathbb{R}$. The image of this set in the quotient $(\mathcal{X}_y \times \mathbb{R})/\mathcal{D}_y$ is locally a union of graphs of smooth sections over $\mathcal{B}$. In view of proposition 6.3.9, $Y_H : (\mathcal{X}_y \times \mathbb{R})/\mathcal{D}_y \to H^3(M)$ is a local diffeomorphism on the complement of this subset. We therefore define

$$G = \{(\varphi_+, \varphi_-, L) \in G_0 : -2L < \text{all eigenvalues of } \pi_E(\partial_+^{-1} C_+ + \partial_-^{-1} C_-)\}.$$

$G \subseteq G_0$ is the subset of gluing data for which the gluing parameter is large enough to guarantee that the derivative of the gluing map is surjective. The quotient $\mathcal{G} = G\mathcal{D}_y/\mathcal{D}_y$ is an open subset of $\mathcal{G}_0$, and $Y : \mathcal{G} \to \mathcal{M}$ is a local diffeomorphism. This completes the proof of theorem 6.1.9.

6.4   Boundary points of the moduli space

Let the compact manifold M^7 be the gluing of two EAC G_2-manifolds $M_\pm^7$ as before and

$$Y : \mathcal{G} \to \mathcal{M}$$

the gluing map for G_2-structures. The gluing space $\mathcal{G}$ is a fibre bundle over $\mathcal{B}$ with typical fibre $\mathbb{R}^+$. It can be considered as the interior of a topological manifold $\overline{\mathcal{G}}$ with boundary $\mathcal{B}$ 'at infinity' by adding a limit point to each of the fibres. Proposition 6.3.9 means that each boundary point in $\overline{\mathcal{G}}$ has an open neighbourhood $\bar{U}$ such that Y is a diffeomorphism on the interior of $\bar{U}$ (i.e. $\bar{U} \cap \mathcal{G}$). Therefore we can use $Y : \bar{U} \to \mathcal{M} \cup \mathcal{B}$ as coordinate charts for the disjoint union of $\mathcal{M}$ and $\mathcal{B}$ as a topological manifold with boundary. The only problem is that the induced topology on $\mathcal{M} \cup \mathcal{B}$ is not Hausdorff, because points of $\mathcal{B}$ need not be separated by open sets; when we attach $\mathcal{B}$ as a boundary of $\mathcal{M}$ distinct points of $\mathcal{B}$ may become limits of the same path in $\mathcal{M}$. We then say that they *define the same boundary point* for $\mathcal{M}$.

To get around this problem we will instead attach a quotient of $\mathcal{B}$ as a boundary of $\mathcal{M}$. To do this we need to check that the relation of defining the same boundary point is an equivalence relation on $\mathcal{B}$, and let $\hat{\mathcal{B}}$ be the quotient. We then verify that the quotient map $\mathcal{B} \to \hat{\mathcal{B}}$ is a covering map, so that $\hat{\mathcal{B}}$ is a smooth manifold. Let $\overline{\mathcal{M}} = \mathcal{M} \cup \hat{\mathcal{B}}$, and use coordinate charts

$$Y : \bar{U} \to \overline{\mathcal{M}} \tag{6.23}$$

to give $\overline{\mathcal{M}}$ the structure of a (Hausdorff) topological manifold with boundary, and thus prove theorem 6.1.10.

As a first step we show that elements of $\mathcal{B}$ that define the same boundary point must have the same image in $\mathcal{M}_y$. If $b^1(M) = 0$ then $\mathcal{B} \to \mathcal{M}_y$ is a covering map, and we will find that $\hat{\mathcal{B}}$ is an intermediate cover. As usual, the case $b^1(M) = 0$ is most interesting since it is a necessary condition for the glued manifold M to have holonomy exactly G_2. The details of the proof are more complicated in general, so in the $b^1(M) > 0$ case we will only point out what the correct statements are. One still finds that $\hat{\mathcal{B}}$ is obtained by identifying some connected components of the fibres of the submersion $\mathcal{B} \to \mathcal{M}_y$. The upshot is that whether $b^1(M) = 0$ or not, $\hat{\mathcal{B}}$ is smooth manifold covered by $\mathcal{B}$, there is a natural submersion $\hat{\mathcal{B}} \to \mathcal{M}_y$, the connected components of the fibres of the submersion look like those for $\mathcal{B} \to \mathcal{M}_y$, and theorem 6.1.10 holds.

Proposition 6.4.1. *Suppose that* $(\varphi_+, \varphi_-), (\psi_+, \psi_-) \in \mathcal{X}_y$ *define the same boundary point for* $\mathcal{M}$. *Then* $\psi_\pm$ *is* $\mathcal{D}_\pm$-*equivalent to* $\varphi_\pm$.

Proof. Pick some R greater than the diameter of both of the compact pieces $M_\pm(0)$. For any L the set of points $x \in M(L)$ for which the complement of the ball $B(x, 2R)$ is connected is contained in the compact subset $M_+(3R) \cup M_-(3R) \subset M(L)$.

Let $\mathcal{R}_y$ be the pre-moduli space of matching pairs of torsion-free G_2-structures near φ. The hypothesis means that there are sequences $L_i, L_i' > 0$, and $(\varphi_{i,+}, \varphi_{i,-}), (\psi_{i,+}, \psi_{i,-}) \in \mathcal{R}_y$ such that $L_i, L_i' \to \infty$, $\varphi_{i,\pm} \to \varphi_\pm$ and $\psi_{i,\pm} \to \psi_\pm$, together with a sequence of diffeomorphisms $\phi_i : M(L_i) \to M(L_i')$ which pull back $Y(\psi_{i,+}, \psi_{i,-}, L_i)$ to $Y(\varphi_{i,+}, \varphi_{i,-}, L_i')$.

Pick a pair of points $x_\pm \in M_\pm(0)$. Then $\phi_i(x_\pm) \in M_\pm(3R)$ for all i. ($\phi_i(x_\pm)$ cannot lie in $M_\mp(3R)$ because ϕ_i is isotopic to the identity.) By passing to a subsequence we may assume that $\phi_i(x_\pm)$ converges to some $x_\pm' \in M_\pm(3R)$. Moreover, the derivatives $D\phi_i|_{T_{x_\pm}M}$ can be regarded as maps defined on $T_{x_\pm}M_\pm$, and we may assume that they converge to isometries $A_\pm : (T_{x_\pm}M_\pm, g_{\varphi_\pm}) \to (T_{x_\pm'}M_\pm, g_{\psi_\pm})$. Now for any $v \in T_{x_\pm}M$

$$\phi_i(\exp_\varphi v) \to \exp_\psi A_\pm v$$

as $i \to \infty$. We can therefore define diffeomorphisms $\phi_\pm$ of $M_\pm$ by $\phi_\pm(\exp_\varphi v) = \exp_\psi(A_\pm v)$ for all $v \in T_{x_\pm}M_\pm$. The convergence of $\phi_i \to \phi_\pm$ is uniform on $M_\pm(L)$ for each L (regarded as a compact subset of either $M_\pm$ or $M(L')$ for all $L' > L$), and $\phi_\pm$ pulls back $\psi_\pm$ to $\varphi_\pm$. It is and is smooth and EAC by proposition 2.3.7. $\square$

We now make the simplifying assumption that $b^1(M) = 0$.

Lemma 6.4.2. *Suppose that $(\varphi_+, \varphi_-), (\psi_+, \psi_-) \in \mathcal{X}_y$ define the same boundary point for $\overline{\mathcal{M}}$. Then the path $Y(\psi_+, \psi_-, L)$ converges to $(\varphi_+, \varphi_-)\mathcal{D}_y$ in $\mathcal{M} \cup \mathcal{B}$ as $L \to \infty$.*

Proof. We only prove the lemma for $b^1(M) = 0$.

Proposition 6.4.1 implies that $(\psi_+, \psi_-) = \tilde{\chi}^*(\varphi_+, \varphi_-)$ for some $\tilde{\chi} \in \tilde{\mathcal{I}}_X$, in the sense of (6.18). In particular $i^*_\pm[\psi_\pm] = i^*_\pm[\varphi_\pm] \in H^3(M_\pm)$, and the paths $Y_H(\psi_+, \psi_-, L)$ and $Y_H(\varphi_+, \varphi_-, L)$ both lie in the affine space $K = [\varphi] + \delta(H^2(X))$. They are both affine rays with slope $2\delta([\omega])$.

Now consider the image in $H^3(M)$ of the interior U of a small neighbourhood of $(\varphi_+, \varphi_-)\mathcal{D}_y$ in $\overline{\mathcal{G}}$. When $b^1(M) = 0$ the image contains an affine cone in K. In particular it contains all but a finite part of the ray $Y_H(\psi_+, \psi_-, L)$, and from this we will deduce the result.

If $b^1(M) = 0$ then we may take U to be represented by $\mathcal{R}_y \times (L_1, \infty)$, where $\mathcal{R}_y$ is a pre-moduli space of matching pairs of torsion-free EAC G_2-structures near (φ_+, φ_-) and $L_1 \in \mathbb{R}$ is large. The projection of the gluing to $H^3(M)$ can be written as

$$Y_H : \mathcal{R}_y \times (L_1, \infty) \to H^3(M), \quad (\varphi'_+, \varphi'_-, L) = f(\varphi'_+, \varphi'_-) + 2L\delta([\omega']), \qquad (6.24)$$

where $i^*_\pm f(\varphi'_+, \varphi'_-) = [\varphi'_\pm]$, and ω' is the Kähler form of the common boundary value of $\varphi'_\pm$. If we let

$$\mathcal{R}'_y = \{(\varphi'_+, \varphi'_-) \in \mathcal{R}_y : i^*_\pm \varphi'_\pm = i^*_\pm \varphi_\pm\}$$

then the restriction of Y_H maps $\mathcal{R}'_y \times (L_1, \infty)$ locally diffeomorphically to K. The computation of the derivative in proposition 6.3.9 shows that the map

$$\mathcal{R}'_y \to \delta(H^2(X)), \quad (\varphi'_+, \varphi'_-) \mapsto \delta([\omega'])$$

is an embedding, and that the tangent space to the image is a direct complement to the radial line. The image of $\mathcal{R}'_y \times (L_1, \infty)$ under Y_H therefore contains an open affine cone in K, and in particular it contains the ray $Y_H(\psi_+, \psi_-, L)$.

$\tilde{\chi}^* U$ is the interior of a neighbourhood of $(\psi_+, \psi_-)\mathcal{D}_y$ in $\overline{\mathcal{G}}$. By lemma 6.3.8, $Y_H(\tilde{\chi}^* U)$ is a translate of $Y_H(U)$, so their intersection also contains an open affine cone in K. In particular the intersection is simply-connected. Therefore if $Y(U)$ and $Y_H(\tilde{\chi}^* U)$ intersect in $\mathcal{M}$ then any pair of points in $Y(U)$ and $Y_H(\tilde{\chi}^* U)$ which represent the same cohomology class must be equal. Hence $Y(U)$ contains $Y(\psi_+, \psi_-, L)$ for all sufficiently large L. As this holds for the interior of any small neighbourhood of $(\varphi_+, \varphi_-)\mathcal{D}_y$ in $\overline{\mathcal{G}}$ the result follows. $\square$

The lemma implies that the relation on $\mathcal{B}$ of defining the same boundary point for $\mathcal{M}$ is transitive. Let $\hat{\mathcal{B}}$ be the quotient. Proposition 6.4.1 implies that there is a natural projection $\hat{\mathcal{B}} \to \mathcal{M}_y$. It is clear that the quotient map $\mathcal{B} \to \hat{\mathcal{B}}$ has discrete fibres, because points in a coordinate neighbourhood of $\mathcal{B}$ can always be separated by open sets in $\mathcal{M} \cup \mathcal{B}$. Next we check (when $b^1(M) = 0$) that the quotient map is a covering map.

Lemma 6.4.3. *Assume $b^1(M) = 0$. Let $(\varphi_+, \varphi_-) \in \mathcal{X}_y$, and let $\mathcal{R}_y$ be the pre-moduli space of matching pairs of torsion-free EAC G_2-structures near φ. Suppose that φ and $\tilde{\chi}^* \varphi$ define the same boundary point for some $\tilde{\chi} \in \tilde{\mathcal{I}}_X$. Then φ' and $\tilde{\chi}^* \varphi'$ define the same boundary point for each $\varphi' \in \mathcal{R}_y$.*

Proof. The path $Y_H(\varphi', L)$ is a translation of $Y_H(\tilde{\chi}^* \varphi', L)$ by an element in $\delta(H^2(X))$, so the result follows by the argument of lemma 6.4.2. $\square$

Thus $\hat{\mathcal{B}}$ is an intermediate cover between $\mathcal{B}$ and $\mathcal{M}_y$. In particular it is a manifold, so if we let $\overline{\mathcal{M}} = \mathcal{M} \cup \hat{\mathcal{B}}$ then we can use the gluing maps (6.23) to give $\overline{\mathcal{M}}$ the structure of a topological manifold with boundary. This completes the proof of theorem 6.1.10 for the case when $b^1(M) = 0$.

Remark 6.4.4. We could give $\overline{\mathcal{G}}$, and hence $\overline{\mathcal{M}}$, a smooth structure by choosing a homeomorphism of the fibre $(0, \infty]$ with a half-open interval $[0, 1)$ that is a diffeomorphism on the interior, but it is not clear if there is a natural choice.

Finally let us say something about the case $b^1(M) > 0$, starting with a sketch proof of lemma 6.4.2. $(\varphi_+, \varphi_-) \in \mathcal{X}_y$ determines a splitting $E^2 = E_1^2 \oplus E_6^2 \oplus E_8^2$, which in turn gives

$$\delta(H^2(X)) = \delta(E_1^2) \oplus \delta(E_6^2) \oplus \delta(E_8^2). \tag{6.25}$$

The image in $H^3(M)$ of the interior of a neighbourhood of (φ_+, φ_-) in $\overline{\mathcal{G}}$ contains a cone in K, whose tangent space is $\delta(E_{1 \oplus 8}^2)$ at points of the ray $Y_H(\varphi_+, \varphi_-, L)$. If $\psi = \tilde{\chi}^* \varphi$ determines the same boundary point as φ for some $\tilde{\chi} \in \tilde{\mathcal{I}}_X$ then necessarily

$$Y_H(\varphi, L) - Y_H(\psi, L) \in \delta(E_{1 \oplus 8}^2).$$

We can then deduce that the image in $H^3(M)$ of the interior of any small neighbourhood of $\varphi \mathcal{D}_y$ in $\overline{\mathcal{G}}$ contains $Y_H(\psi_+, \psi_-, L)$ for all sufficiently large L, and proceed similarly to the $b^1(M) = 0$ case.

The quotient map $\hat{\mathcal{B}} \to \mathcal{B}$ has discrete fibres as before. To check that it is a covering map one proves a more general version of lemma 6.4.3. Let $(\varphi_+, \varphi_-) \in \mathcal{X}_y$ and $\mathcal{R}_y$ the

pre-moduli space of matching pairs of torsion-free EAC G_2-structures near φ, and suppose that φ and $\tilde{\chi}^*\varphi$ define the same boundary point for some $\tilde{\chi} \in \tilde{\mathcal{I}}_X$. What we can say when $b^1(M) > 0$ is that there is a smooth map $\tilde{\chi}' : \mathcal{R}_y \to \tilde{\mathcal{I}}_X$ such that $\tilde{\xi}^*\varphi'$ and $(\tilde{\xi}\tilde{\chi}')^*\varphi'$ define the same boundary point for each $\varphi' \in \mathcal{R}_y$ and $\tilde{\xi}$ in the identity component of $\tilde{\mathcal{I}}_X$. Thus $\hat{\mathcal{B}}$ is smooth manifold covered by $\mathcal{B}$, and theorem 6.1.10 holds.

Chapter 7

Pulling apart G_2-manifolds

In this chapter we attempt to reverse the gluing construction for compact G_2-manifolds from [34] summarised in §6.1, and show that some of the compact G_2-manifolds produced by Joyce's Kummer-type construction can be deformed to the result of gluing a matching pair of EAC G_2-manifolds. A key technical result that is needed is a generalisation to the EAC case of Joyce's theorems for perturbing G_2-structures with small torsion to torsion-free G_2-structures. We are able to cut some of Joyce's compact examples into two parts such that we can find torsion-free EAC G_2-structures on each half, and apply deformation results from §6 to show that the gluing of the two halves can be deformed to the original manifold.

In the process we find some examples of irreducible EAC G_2-manifolds. We compute their Betti numbers and exhibit some examples of asymptotically cylindrical coassociative submanifolds.

7.1 Results

In §7.2.1 we review how Joyce [27] constructs examples of compact G_2-manifolds M by desingularising the quotient of a flat torus T^7 by a finite group Γ preserving the standard flat G_2-structure. The construction has two main steps. First, the singularities of T^7/Γ are resolved to obtain a smooth manifold M with a family of G_2-structures with closed defining 3-form $\tilde{\varphi}$. These have arbitrarily small torsion in a suitable sense; one can estimate norms of a 3-form ψ such that $d\Theta(\tilde{\varphi}) = d*\psi$. When the torsion is sufficiently small $\tilde{\varphi}$ can then be perturbed to a torsion-free G_2-structure φ in the same cohomology class.

Suppose that the torsion-free G_2-structure φ on M can be obtained by perturbing a closed G_2-structure $\tilde{\varphi}$ with small torsion that has a cylindrical neck, in the following sense. There is a Calabi-Yau 3-fold X^6 with structure (Ω, ω) and an interval $I = (-\epsilon, \epsilon)$ such that M has an open subset $N \cong X \times I$ with

$$\tilde{\varphi}|_N = \Omega + dt \wedge \omega, \tag{7.1}$$

such that the complement of N in M has exactly 2 connected components. Furthermore we require that the 3-form ψ controlling the torsion vanishes on the neck. Let $M_\pm(0)$ denote the closures of the connected components of the complement in M of the hypersurface $X \times \{0\} \subset N$.

Let $M_\pm$ be the manifolds with cylindrical ends obtained by gluing $X \times [0, \infty)$ to $M_\pm(0)$ along the common boundary X. $\tilde{\varphi}|_{M_\pm(0)}$ and $\Omega + dt \wedge \omega$ can obviously be glued to define a closed G_2-structure $\tilde{\varphi}_\pm$ on M which is exactly cylindrical and torsion-free on the cylindrical end. This has small torsion in the same sense as $\tilde{\varphi}$. Joyce's perturbation result for G_2-structures (cf. theorem 7.2.2) is carefully phrased so that it does not intrinsically rely on the underlying manifold being compact (e.g. it does not involve any volume estimates). It is therefore relatively straight-forward to generalise it to the EAC setting.

Theorem 7.1.1. *Let μ, ν, λ positive constants. Then there exist positive constants κ, K such that whenever $0 < t < \kappa$ the following is true.*

Let M^7 be a manifold with cylindrical ends and cross-section X^6, and let $\tilde{\varphi}$ be a closed G_2-structure on M that is exactly cylindrical and torsion-free on the cylindrical end. Suppose that ψ is a smooth compactly supported 3-form on M satisfying $d^*\psi = d^*\tilde{\varphi}$, and let $r(\tilde{\varphi})$ and $R(\tilde{\varphi})$ be the injectivity radius and Riemannian curvature of the EAC metric $g_{\tilde{\varphi}}$ on M. If

(i) $\|\psi\|_{L^2} < \lambda t^4$, $\|\psi\|_{C^0} < \lambda t^{1/2}$, $\|d^*\psi\|_{L^{14}} < \lambda$,

(ii) $r(\tilde{\varphi}) > \mu t$,

(iii) $\|R(\tilde{\varphi})\|_{C^0} < \nu t^{-2}$,

then there is a smooth exact 3-form $d\eta$ on M, exponentially decaying with all derivatives, such that

$$\|d\eta\|_{L^2} < Kt^4, \quad \|d\eta\|_{C^0} < Kt^{1/2}, \quad \|\nabla d\eta\|_{L^{14}} < K, \tag{7.2}$$

and $\varphi = \tilde{\varphi} + d\eta$ is a torsion-free G_2-structure.

Remark 7.1.2. The fact that $d\eta$ is exponentially decaying is more important than its precise rate of decay. We will need to choose the rate $\delta > 0$ so that δ^2 is smaller than any non-zero eigenvalue of the Hodge Laplacian on X. It should be easy to modify the proof of the theorem to allow $\tilde{\varphi}$ to be EAC and ψ to be exponentially decaying. In that case one would also need δ to be smaller than the decay rates of $\tilde{\varphi}$ and ψ.

Applying theorem 7.1.1 to $\tilde{\varphi}_\pm$ gives a matching pair of torsion-free EAC G_2-structures (φ_+, φ_-) on $M_\pm$. As discussed in §6, these can be glued with an additional sufficiently large parameter L to give a torsion-free G_2-structure $Y(\varphi_+, \varphi_-, L)$ on M. Varying the parameter L defines a path in the moduli space $\mathcal{M}$ whose image in $H^3(M)$ is an affine line with slope $\delta([\omega])$. In the partial compactification $\overline{\mathcal{M}}$ defined in §6.4 the path converges to a point in the boundary $\hat{\mathcal{B}}$.

Restricting the glued structures to $M_\pm$ defines the same cohomology class as the original G_2-structure φ, but it is hard to relate them directly beyond that. We can, however, define another path of G_2-structures on M by 'stretching' φ. Let $M(L)$ denote the manifold obtained by increasing the length of the cylindrical neck of M by $2L$. $\tilde{\varphi}$ can be stretched to define a G_2-structure $\tilde{\varphi}(L)$ on $M(L)$. This too has small torsion, so can be perturbed to a torsion-free G_2-structure $\varphi(L)$ by Joyce's results. As $L \to \infty$ the path $\varphi(L)$ converges to a boundary point in the partial compactification $\overline{\mathcal{M}}$.

Theorem 7.1.3. *Let M^7 be a compact manifold with a closed G_2-structure $\tilde{\varphi}$. Suppose that ψ is a 3-form such that $d^*\tilde{\varphi} = d^*\psi$, that the estimates (i)-(iii) in theorem 7.1.1 are satisfied, and that t is sufficiently small. Suppose further that M has a cylindrical neck in the sense of (7.1), and that ψ vanishes on the neck. Define the path $\varphi(L)$ as above. Let $\varphi_\pm$ be torsion-free EAC G_2-structures on $M_\pm$ that are close to $\tilde{\varphi}_\pm$ in the sense of (7.2), which exist by theorem 7.1.1.*

Then $\varphi(L)$ is in the image of the gluing map for sufficiently large L. In the sense of theorem 6.1.10, the path $L \mapsto \varphi(L)\mathcal{D}$ converges to a point in the boundary $\hat{\mathcal{B}}$ of $\overline{\mathcal{M}}$, whose image under the projection $\hat{\mathcal{B}} \to \mathcal{M}_y$ is $(\varphi_+\mathcal{D}_+, \varphi_-\mathcal{D}_-)$.

We prove the results in §7.2. For them to be meaningful we also need examples to which they can be applied. In §7.3 we show how some of Joyce's examples can be obtained by performing the resolution of orbifold singularities in two steps so that the intermediate resolution has a cylindrical neck in the sense of (7.1), and theorem 7.1.3 applies. Some of the EAC G_2-manifolds obtained this way have holonomy exactly G_2. We explain how to compute their Betti numbers and find some examples of asymptotically cylindrical coassociative submanifolds.

Joyce's construction actually gives a path of torsion-free G_2-structures, whose limit can be thought of as a boundary point of the moduli space $\mathcal{M}$ of torsion-free G_2-structures, defined by the orbifold G_2-structure the construction started from. When theorem 7.1.3 can be applied we therefore find that a component of $\mathcal{M}$ has boundary points of two different types: both orbifold and connected-sum.

7.2 Main arguments

7.2.1 Summary of Joyce's construction

We sketch the setup used by Joyce [27, §11] to construct examples of compact G_2-manifolds. Let T^7 be the quotient of $\mathbb{R}^7$ by a lattice, equipped with the standard flat G_2-structure, and let Γ be a finite group of automorphisms of T^7. The quotient T^7/Γ is a G_2-orbifold.

T^7/Γ has singularities coming from the fixed point sets of elements of Γ. For each component of the fixed point sets we choose a topological resolution of the singularity and a family of G_2-structures with small torsion on the resolution. Since different parts of the fixed point set may intersect or map to the same singularity in T^7/Γ these choices have to satisfy some compatibility conditions if they are to be used to resolve the singularities of T^7/Γ. Joyce calls a suitable choice of resolutions a set of *R-data*. For simplicity we only sketch the definition of R-data and the resolution process for the case when the fixed point sets do not intersect.

For concreteness we assume also that the elements of Γ are all of order 2, since those are the only examples we consider. Then each component F_α of the fixed point set is a flat torus T^3. A tubular neighbourhood of F_α is isomorphic to a neighbourhood of $T^3 \times \{0\}$ in $T^3 \times \mathbb{C}^2$, equipped with a flat product G_2-structure. In suitable coordinates (θ^i on T^3, z^j on $\mathbb{C}^2$) the product structure is of the form

$$d\theta^1 \wedge d\theta^2 \wedge d\theta^3 + d\theta^1 \wedge \omega_0 + d\theta^2 \wedge \beta_0 + d\theta^3 \wedge \beta_0',$$

where $\omega_0 = -\frac{i}{2}(dz^1 \wedge d\bar{z}^1 + dz^2 \wedge d\bar{z}^2)$ and $\beta_0 + i\beta_0' = dz^1 \wedge dz^2$.

Let γ_α be the element of Γ that fixes F_α. The action of γ_α on the tubular neighbourhood of F_α corresponds to multiplication by -1 on $\mathbb{C}^2$. Choose a resolution $\pi_\alpha : W_\alpha \to \mathbb{C}^2/\{\pm 1\}$, together with an asymptotically locally Euclidean (ALE) Calabi-Yau metric with Kähler form ω_α. This means that the restriction $\pi_\alpha : W_\alpha \setminus \pi_\alpha^{-1}(0) \to (\mathbb{C}^2 \setminus \{0\})/\{\pm 1\}$ is an isomorphism, and that the difference between ω_0 and the push-forward of ω_α is of order

$O(r^{-4})$, where r is the distance to the origin in $\mathbb{C}^2$. Then W_α has a parallel $(2,0)$-form $\beta + i\beta'$, and

$$\varphi_\alpha = d\theta^1 \wedge d\theta^2 \wedge d\theta^3 + d\theta^1 \wedge \omega_\alpha + d\theta^2 \wedge \beta + d\theta^3 \wedge \beta'$$

is a torsion-free G_2-structure on $T^3 \times W_\alpha$. We require that there is an equivariance condition with respect to Γ: if $\gamma \in \Gamma$ maps F_α to F_β then there is an identification $W_\alpha \leftrightarrow W_\beta$ such that the corresponding map $T^3 \times W_\alpha \to T^3 \times W_\beta$ pulls back φ_β to φ_α. In particular the stabiliser of F_α acts by automorphisms on $T^3 \times W_\alpha$. The R-data consists of choosing the resolutions W_α for each fixed set component F_α and the isomorphisms $W_\alpha \leftrightarrow W_\beta$ for each γ.

Once the R-data is fixed, we obtain a compact resolution M of T^7/Γ. For each component of the singular set in T^7/Γ we cut out a neighbourhood, pick a pre-image F_α and glue in a neighbourhood of $T^3 \times W_\alpha$ divided by the stabiliser of F_α (the stabiliser acts freely on $T^3 \times W_\alpha$ because of the simplifying assumption that different pieces of the fixed point set do not intersect). We can use a (fixed) smooth cut-off function to interpolate between the flat G_2-structure φ_0 on T^7 and the torsion-free G_2-structures φ_α on the resolving neighbourhoods, and define a G_2-structure $\tilde{\varphi}$ on M with $d\tilde{\varphi} = 0$. The equivariance condition for the R-data with respect to Γ ensures that both M and $\tilde{\varphi}$ are independent of the choice of pre-images F_α.

Although both φ_0 and the φ_α are torsion-free, $\tilde{\varphi}$ has some torsion in the region where the cut-off function used is non-constant. The magnitude of the torsion depends on the derivatives of φ_α in the cutting-off region. This can be reduced by composing the resolutions $W_\alpha \to \mathbb{C}^2/\{\pm 1\}$ with $\mathbb{C}^2 \to \mathbb{C}^2, z \mapsto tz$ for small $t > 0$. We then need to rescale the metric ω_α by t^2 to keep it ALE. This increases the Riemannian curvature of ω_α by a factor t^{-2}. Using these resolutions instead will thus decrease the torsion of $\tilde{\varphi}$ at the cost of increasing the curvature of the metric it defines.

Theorem 7.2.1 ([27, Theorem 11.5.7]). *Let T^7/Γ be an orbifold of T^7 with flat G_2-structure φ_0. Suppose we are given a set of R-data for T^7/Γ, and M is the corresponding resolution. Then we can write down the following data on M explicitly:*

- *constants $\epsilon \in (0,1]$, and $\lambda, \mu, \nu > 0$,*

- *a G_2-structure $\tilde{\varphi}$ on M with $d\tilde{\varphi} = 0$ for each $t \in (0, \epsilon]$,*

- *a smooth 3-form ψ on M with $d^*\psi = d^*\tilde{\varphi}$ for each $t \in (0, \epsilon]$.*

These satisfy the three conditions

(i) $\|\psi\|_{L^2} < \lambda t^4$, $\|\psi\|_{C^0} < \lambda t^3$, $\|d^*\psi\|_{L^{14}} < \lambda t^{16/7}$,

(ii) *the injectivity radius is $r(\tilde{\varphi}) > \mu t$,*

(iii) *the Riemannian curvature $R(\tilde{\varphi})$ satisfies $\|R\|_{C^0} < \nu t^{-2}$.*

These conditions ensure that for sufficiently small t the G_2-structure $\tilde{\varphi}$ can be perturbed within its cohomology class to a torsion-free G_2-structure on M.

Theorem 7.2.2 (cf. [27, Theorem 11.6.1])**.** *Let μ, ν, λ positive constants. Then there exist positive constants κ, K such that whenever $0 < t < \kappa$ the following is true.*

Let M^7 be a compact manifold, and $\tilde{\varphi}$ a closed G_2-structure on M. Suppose ψ is a smooth 3-form on M satisfying $d^*\psi = d^*\tilde{\varphi}$, and

(i) $\|\psi\|_{L^2} < \lambda t^4$, $\|\psi\|_{C^0} < \lambda t^{1/2}$, $\|d^*\psi\|_{L^{14}} < \lambda$,

(ii) *the injectivity radius is $> \mu t$,*

(iii) *the Riemannian curvature R satisfies $\|R\|_{C^0} < \nu t^{-2}$.*

Then there is a smooth exact 3-form $d\eta$ on M with

$$\|d\eta\|_{L^2} < K t^4, \quad \|d\eta\|_{C^0} < K t^{1/2}, \quad \|\nabla d\eta\|_{L^{14}} < K, \tag{7.3}$$

such that $\varphi = \tilde{\varphi} + d\eta$ is torsion-free.

Thus we obtain a family of torsion-free G_2-structures on M. Note that as $t \to 0$ both $\tilde{\varphi}$ and φ resemble the singular G_2-structure on T^7/Γ. We can think of the family φ as converging to a boundary point of the moduli space of torsion-free G_2-structures on M, defined by the singular G_2-structure.

Also note that there may be many different choices of R-data for a single orbifold T^7/Γ, which can give rise to many topologically inequivalent resolutions $\pi : M \to T^7/\Gamma$ and many different diffeomorphism classes for M.

We will often informally refer to a form satisfying estimates like (7.3) as *small*. For future reference we point out that the uniform estimate in (7.3) follows from the L^2 and L_1^{14} estimates by a version of Sobolev embedding.

Theorem 7.2.3 ([27, Theorem G1])**.** *Let μ, ν and t be positive constants, and suppose M is a complete Riemannian 7-manifold, whose injectivity radius δ and Riemannian curvature*

R satisfy $\delta \geq \mu t$ and $\|R\|_{C^0} \leq \nu t^{-2}$. Then there exists $C > 0$ depending only on μ and ν, such that if $\chi \in L_1^{14}(\Lambda^3) \cap L^2(\Lambda^3)$ then

$$\|\chi\|_{C^0} \leq C(t^{1/2}\|\nabla\chi\|_{L^{14}} + t^{-7/2}\|\chi\|_{L^2}).$$

The torsion-free G_2-structure φ given by theorem 7.2.2 depends continuously on the input $\tilde{\varphi}$ and ψ. In a certain sense it represents a unique diffeomorphism class of torsion-free G_2-structures near $\tilde{\varphi}$. In particular, the resolution φ of $(\tilde{\varphi}, \psi)$ is independent of ψ up to diffeomorphism, and two families of torsion-free G_2-structures which are cohomologous and "close" must be diffeomorphic.

Proposition 7.2.4. *For $i = 0, 1$ let $\tilde{\varphi}_i$ be a closed G_2-structure and ψ_i a smooth 3-form satisfying the hypotheses of theorem 7.2.2. Suppose moreover that the difference between $\tilde{\varphi}_0$ and $\tilde{\varphi}_1$ is exact, and that it satisfies*

$$\|\tilde{\varphi}_0 - \tilde{\varphi}_1\|_{L^2} < \lambda t^4, \quad \|\tilde{\varphi}_0 - \tilde{\varphi}_1\|_{C^0} < \lambda t^{1/2}, \quad \|\tilde{\varphi}_0 - \tilde{\varphi}_1\|_{L_1^{14}} < \lambda$$

(measured in the metric defined by $\tilde{\varphi}_0$). If t is sufficiently small then the torsion-free G_2-structures φ_i produced from $(\tilde{\varphi}_i, \psi_i)$ by theorem 7.2.2 are diffeomorphic.

Proof. Connect $\tilde{\varphi}_0$ and $\tilde{\varphi}_1$ by an affine line segment of closed G_2-structures $\tilde{\varphi}_s$, $s \in [0, 1]$. Let $\psi_s' = \tilde{\varphi}_s + *_s(*_0\psi_0 - \Theta(\tilde{\varphi}_0))$, where $*_s$ denotes the Hodge star of the metric defined by $\tilde{\varphi}_s$. Then $(\tilde{\varphi}_s, \psi_s')$ satisfies the hypotheses of theorem 7.2.2, so when t is sufficiently small it defines a torsion-free perturbation φ_s' of $\tilde{\varphi}_s$ for each $s \in [0, 1]$.

Then define a line segment ψ_s'' connecting ψ_1 to ψ_1'. Theorem 7.2.2 gives a torsion-free perturbation φ_s'' of $\tilde{\varphi}_1$ for each input $(\tilde{\varphi}_1, \psi_s'')$. The concatenation of the paths φ_s' and φ_s'' is a continuous path connecting φ_0 and φ_1, staying in a single cohomology class. By theorem 3.2.1 the image of the path in the moduli space of torsion-free G_2-structures is constant, so the end-points are diffeomorphic. $\square$

7.2.2 Proof of theorem 7.1.1

Theorem 7.1.1 is an extension of theorem 7.2.2 to the EAC setting. We wish to find an exact exponentially asymptotically decaying 3-form $d\eta$ such that $\tilde{\varphi} + d\eta$ is torsion-free. The proof consists of three parts. First we show that for $\tilde{\varphi} + d\eta$ to be torsion-free it suffices that η is a solution of a certain elliptic equation, the same that Joyce used in the proof of theorem 7.2.2. Then we find a solution for this equation by a contraction-mapping argument. The

details of this are complicated, but largely carry over from the compact case as solved by Joyce. Elliptic regularity shows that solutions are smooth and uniformly decaying. In the next subsection we prove that the solutions decay exponentially, which completes the proof.

Proposition 7.2.5. *There is an absolute constant ϵ_1 such that the following holds. Let M^7 be an EAC manifold, $\tilde{\varphi}$ a closed EAC G_2-structure on M and ψ an exponentially decaying 3-form such that $\|\psi\|_{C^0} < \epsilon_1$ and $d^*\psi = d^*\tilde{\varphi}$. Suppose that η is 2-form asymptotic to a translation-invariant harmonic form, and that $\|d\eta\|_{C^0} < \epsilon_1$. Suppose further that*

$$\triangle\eta = d^*\psi + d^*(f\psi) + *dF(d\eta), \tag{7.4}$$

where the function f is defined by $f\tilde{\varphi} = \frac{7}{3}\pi_1(d\eta)$ and F is the quadratic part of the non-linear fibre-wise map $\Theta : \Lambda^3_{G_2} \to \Lambda^4$, $\varphi \mapsto *_\varphi\varphi$. Then $\tilde{\varphi} + d\eta$ is a torsion-free EAC G_2-structure on M.

Proof. This is an asymptotically cylindrical version of [27, Theorem 10.3.7]. The proof relies on integrating by parts. It is easy to check that, in the asymptotically cylindrical setting, the necessary integrals still converge provided that η is bounded and $d\eta$ decays, so we can still use (7.4) as a sufficient condition for the torsion to vanish. $\square$

A key part in the proof of theorem 7.2.2 is the contraction-mapping result [27, Proposition 11.8.1]. We observe that this can easily be adapted to the EAC case.

Proposition 7.2.6. *Let (Ω, ω) be a Calabi-Yau structure on a compact manifold X^6 and μ, ν, λ be positive constants. Then there exist positive constants κ, K, C_1 such that whenever $0 < t < \kappa$ the following is true.*

Let M^7 be a manifold with cylindrical ends and cross-section X, and $\tilde{\varphi}$ a closed EAC G_2-structure on M with asymptotic limit $\Omega + dt \wedge \omega$. Suppose that ψ is a smooth exponentially decaying 3-form on M satisfying $d^*\psi = d^*\tilde{\varphi}$, and that

(i) $\|\psi\|_{L^2} < \lambda t^4$, $\|\psi\|_{C^0} < \lambda t^{1/2}$, $\|d^*\psi\|_{L^{14}} < \lambda$,

(ii) *the injectivity radius is $> \mu t$,*

(iii) *the Riemannian curvature R satisfies $\|R\|_{C^0} < \nu t^{-2}$.*

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Then there is a sequence η_j of smooth exponentially asymptotically translation-invariant 2-forms with $\eta_0 = 0$ satisfying the equation

$$\triangle \eta_j = d^* \psi + d^*(f_{j-1}\psi) + *dF(d\eta_{j-1}), \tag{7.5}$$

where $f_j \tilde{\varphi} = \frac{7}{3}\pi_1(d\eta_j)$ for each $j > 0$. The solutions satisfy the inequalities

(i) $\|d\eta_j\|_{L^2} < 2\lambda t^4$,

(ii) $\|\nabla d\eta_j\|_{L^{14}} < 4C_1\lambda$,

(iii) $\|d\eta_j\|_{C^0} < Kt^{1/2}$,

(iv) $\|d\eta_{j+1} - d\eta_j\|_{L^2} < 2^{-j}\lambda t^4$,

(v) $\|\nabla(d\eta_{j+1} - d\eta_j)\|_{L^{14}} < 4 \cdot 2^{-j}C_1\lambda$,

(vi) $\|d\eta_{j+1} - d\eta_j\|_{C^0} < 2^{-j}Kt^{1/2}$.

Proof. We prove that the sequence exists using the Hodge theory in §2.3. As before we let $\mathcal{H}^m_\infty$ denote the space of translation-invariant harmonic m-forms on the cylinder $X \times \mathbb{R}$ and ρ a cut-off function for the cylinder on M. Take $\delta > 0$ smaller than the decay rates of $\tilde{\varphi}$ and ψ such that δ^2 is smaller than any positive eigenvalue of the Hodge Laplacian on X. Inductively, the RHS of (7.5) is d^* of a 3-form that decays exponentially with rate δ. The EAC Hodge decomposition theorem 2.3.27 implies that there is a unique coexact solution $\eta_j \in C^{k,\alpha}_\delta(\Lambda^2) \oplus \rho\mathcal{H}^2_\infty$ for all $k \geq 2$.

The inequalities can be proved inductively, using exactly the same argument as in [27, Theorem 11.8.1]. (i) and (iv) are proved using an integration by parts argument, and since each $d\eta_j$ decays exponentially this is still justified when M has cylindrical ends.

(ii), (iii), (v) and (vi) are proved using interior estimates, which do not require compactness. $\qquad\square$

It follows that $d\eta_j$ is a Cauchy sequence and has a limit χ with

$$\|\chi\|_{L^2} < Kt^4, \quad \|\chi\|_{C^0} < Kt^{1/2}, \quad \|\nabla\chi\|_{L^{14}} < K, \tag{7.6}$$

for some $K > 0$. χ is closed, L^2-orthogonal to the decaying harmonic forms $\mathcal{H}^3_+$ and satisfies the equation

$$d^*\chi = d^*\psi + d^*(f\psi) + *dF(\chi), \tag{7.7}$$

where $f\tilde{\varphi} = \frac{7}{3}\pi_1(\chi)$. We do not know *a priori* that χ is the exterior derivative of a bounded form, so we cannot yet apply proposition 7.2.5 to show that $\tilde{\varphi} + \chi$ is torsion-free. We first show by elliptic regularity that χ is smooth and uniformly decaying.

Proposition 7.2.7. *If t is sufficiently small then $\chi \in L_k^{14}(\Lambda^3)$ for all $k \geq 0$.*

Proof. Since $F(\chi)$ depends only point-wise on χ and is of quadratic order we can write

$$*dF(\chi) = P(\chi, \nabla\chi) + Q(\chi), \tag{7.8}$$

where $P(u, v)$ is linear in v and smooth of linear order in u, while $Q(u)$ is smooth of quadratic order in u for u small. We can then rephrase (7.7) as stating that $\beta = \chi$ is a solution of

$$d^*\beta - P(\chi, \beta) - d^*(f(\beta)\psi) = d^*\psi + Q(\chi),$$
$$d\beta = 0, \tag{7.9}$$

where $f(\beta)\tilde{\varphi} = \frac{7}{3}\pi_1(\beta)$. The LHS is a linear partial differential operator acting on β. Its symbol depends on χ and ψ, but not on their derivatives. By taking t small we can ensure that χ and ψ are both small in the uniform norm (see (7.6) and hypothesis (i) in proposition 7.2.6) so that the equation is elliptic.

Now suppose that χ has regularity L_k^{14}. Then so do the coefficients and the RHS of (7.9). Because $\beta = \chi \in L_1^{14}(\Lambda^3)$ is a solution of (7.9) standard interior estimates (a Sobolev version of theorem 4.2.20) imply that it must have regularity L_{k+1}^{14} locally. Moreover, because the metric is asymptotically cylindrical the local bounds are actually uniform (cf. theorem 4.2.22), so in fact χ is globally L_{k+1}^{14}. The result follows by induction on k. $\square$

Corollary 7.2.8. *If t is sufficiently small then χ decays uniformly with all derivatives.*

Proof. Because M is EAC, standard Sobolev embedding results imply that we can pick $r > 0$ such that M is covered by balls $B(x_i, r)$ with the following property:

$$\|\chi|_{B(x_i,r)}\|_{C^k} < C\|\chi|_{B(x_i,2r)}\|_{L_{k+1}^{14}},$$

where the constant $C > 0$ is independent of $x_i \in M$. If we ensure that each point of M is contained in no more than N of the balls $B(x_i, 2r)$ then

$$\sum_i \|\chi|_{B(x_i,r)}\|_{C^k}^{14} < NC^{14}\|\chi\|_{L_{k+1}^{14}}^{14}.$$

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As the sum is convergent the terms tend to 0, i.e. the kth derivatives of χ decay uniformly.

$\square$

If we can show that χ decays exponentially then, because χ is closed and L^2-orthogonal to the decaying harmonic forms $\mathcal{H}^3_+$, $\chi = d\eta$ for some exponentially asymptotically translation-invariant η by the Hodge decomposition theorem 2.3.27. Proposition 7.2.5 then implies that $\tilde{\varphi} + \chi$ is torsion-free, and the proof of theorem 7.1.1 would be complete. In the next subsection we prove the required exponential decay result.

7.2.3   Exponential decay

By hypothesis $\tilde{\varphi}$ is exactly cylindrical on the cylindrical end $t > 0$ of M, and ψ is supported in $t \leq 0$. Thus on the cylindrical end the equation (7.7) for χ simplifies to

$$d^*\chi = *dF(\chi). \tag{7.10}$$

On the cylindrical end $t > 0$ we can write

$$\chi = \sigma + dt \wedge \tau,$$
$$F(\chi) = \beta + dt \wedge \gamma,$$

where $\tau \in \Omega^2(X)$, $\sigma, \gamma \in \Omega^3(X)$ and $\beta \in \Omega^4(X)$ are forms on the cross-section X depending on the parameter t. Let d_X denote the exterior derivative on X. Then the conditions $d\chi = 0$ and (7.10) are equivalent to

$$d_X \sigma = 0, \tag{7.11a}$$
$$\tfrac{\partial}{\partial t}\sigma = d_X \tau, \tag{7.11b}$$
$$d_X *\tau = -d_X \beta, \tag{7.11c}$$
$$\tfrac{\partial}{\partial t}*\tau = -d_X *\sigma - \tfrac{\partial}{\partial t}\beta + d_X \gamma. \tag{7.11d}$$

(7.11b) implies that $\sigma(t_1) - \sigma(t_2)$ is exact for any $t_1, t_2 > 0$. Since the exact forms form a closed subspace of the space of 3-forms on X (in the L^2 norm) and $\sigma \to 0$ as $t \to \infty$ it follows that σ is exact for all $t > 0$. Similarly (7.11d) implies that $*\tau - \beta$ is exact for all $t > 0$. (The equations (7.11a) and (7.11c) are thus redundant.) The path (σ, τ) is therefore

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constrained to lie in the space

$$\mathcal{F} = \{(\sigma, \tau) \in d_X L_1^2(\Lambda^2 T^* X) \times L^2(\Lambda^2 T^* X) : *\tau - \beta \text{ is exact}\}.$$

Remark 7.2.9. We have not assumed that χ is L^1 on M.

β is a function of σ and τ, and it is of quadratic order. The implicit function theorem applies to show that if we replace $\mathcal{F}$ with a small neighbourhood of 0 then it is a Banach manifold with tangent space

$$T_0\mathcal{F} = B = d_X L_1^2(\Lambda^2 T^* X) \times d_X^* L_1^2(\Lambda^3 T^* X).$$

We can now interpret (7.11b) and (7.11d) as a flow on $\mathcal{F}$, or equivalently near the origin in B. By the chain rule we can write $\frac{\partial}{\partial t}\beta$ as

$$\frac{\partial}{\partial t}\beta = A_2 \frac{\partial}{\partial t}\tau + A_3 \frac{\partial}{\partial t}\sigma + \beta',$$

where A_m is a linear map from $\Lambda^m T^* X$ to $\Lambda^2 T^* X$, determined point-wise by σ and τ and of linear order, while β' is a 2-form determined point-wise by σ and τ and of quadratic order. In particular, for large t the norm of A_2 is small, and (7.11b) and (7.11d) are equivalent to

$$\begin{aligned} \frac{\partial}{\partial t}\sigma &= d_X\tau, \\ \frac{\partial}{\partial t}\tau &= (id + A_2)^{-1}(d_X^*\sigma - *A_3 d_X\tau - *\beta' + *d_X\gamma). \end{aligned} \tag{7.12}$$

The origin is a stationary point for the flow, and the linearisation of the flow near the origin is given by the linear operator $L = \begin{pmatrix} 0 & d_X \\ d_X^* & 0 \end{pmatrix}$ on B. Because L is formally self-adjoint B has an orthonormal basis of eigenvectors. L is injective on B, so B can be written as a direct sum of subspaces with positive and negative eigenvalues,

$$B = B_+ \oplus B_-.$$

$\{e^{\mp tL} : t \geq 0\}$ defines a continuous semi-group of bounded operators on $B_\pm$. If we let μ denote the smallest absolute value of the eigenvalues of L then $e^{t\mu}e^{\mp tL}$ is uniformly bounded on $B_\pm$ for $t \geq 0$, so the origin is a hyperbolic fixed point. By analogy with the finite-dimensional case we expect that any flow line approaching the origin must do so at an exponential rate.

A similar problem of exponential convergence for an infinite-dimensional flow is considered by Mrowka, Morgan and Ruberman [44, Lemma 5.4.1]. Their problem is more general in that the linearisation of their flow has non-trivial kernel, so that they need to consider convergence to a 'centre manifold' rather than to a well-behaved isolated fixed point. As a simple special case we can prove L^2 exponential decay for χ.

Proposition 7.2.10. *Let $\delta > 0$ such that δ^2 is smaller than any positive eigenvalue of the Hodge Laplacian on X. Then χ is L_δ^2.*

Proof. Identify $\mathcal{F}$ with a neighbourhood of the origin in the tangent space B, and let x be the path in B corresponding to (σ, τ) in $\mathcal{F}$. Then (7.12) transforms to a differential equation for x,

$$\frac{dx}{dt} = Lx + Q(x),$$

where L is the linearisation of (7.12) as above, and Q is the remaining quadratic part. Let $x = x_+ + x_-$ with $x_\pm \in B_\pm$. If we let μ denote the smallest absolute value of the eigenvalues of L then

$$\|Lx_+\|_{L^2} \geq \mu\|x_+\|_{L^2}, \quad \|Lx_-\|_{L^2} \leq -\mu\|x_-\|_{L^2}.$$

Applying the chain rule to the quadratic part gives

$$\|Q(x)\|_{L^2} < O(\|x\|_{L^2})\|x\|_{L_1^2} + O(\|x\|_{L^2}^2).$$

By corollary 7.2.8, x converges uniformly to 0 with all derivatives as $t \to \infty$. Therefore for any fixed $k > 0$ we can find t_0 such that

$$\|Q(x)\|_{L^2} < k\|x\|_{L^2}$$

for any $t > t_0$. As μ^2 is an eigenvalue for the Hodge Laplacian on X we may fix k so that $\mu - 2k > \delta$.

We thus obtain that for $t > t_0$

$$\frac{d}{dt}\|x_+\|_{L^2} \geq \quad \mu\|x_+\|_{L^2} - k\|x\|_{L^2}, \tag{7.13a}$$

$$\frac{d}{dt}\|x_-\|_{L^2} \leq -\mu\|x_-\|_{L^2} + k\|x\|_{L^2}. \tag{7.13b}$$

In particular $\|x_+\|_{L^2} - \|x_-\|_{L^2}$ is an increasing function of t. Because it converges to 0 as

$t \to \infty$,

$$\|x_+\|_{L^2} \le \|x_-\|_{L^2}$$

for all $t > t_0$. Substituting into (7.13b)

$$\frac{d}{dt}\|x_-\|_{L^2} \le -\mu\|x_-\|_{L^2} + 2k\|x_-\|_{L^2},$$

so $\|x_-\|_{L^2}$ is of order $e^{(-\mu+2k)t}$. Hence so is $\|x\|_{L^2}$, so $e^{\delta t}\chi$ is L^2-integrable on M. $\qquad\square$

Corollary 7.2.11. χ *decays exponentially with rate* δ.

Proof. We prove by induction that χ is $L^2_{k,\delta}$ for all $k \ge 0$. Interior estimates for the elliptic operator $d + d^*$ on M imply that we can fix some $r > 0$ and cover the cylindrical part of M with open balls $U = B(x,r)$ such that

$$\|\chi\|_{L^2_{k+1}(U)} < C_1(\|d\chi\|_{L^2_k(U)} + \|d^*\chi\|_{L^2_k(U)}) + C_2\|\chi\|_{L^2(U)}.$$

The constants C_1 and C_2 depend on the local properties of the metric and the volume of U. Since M is EAC we can take the constants to be independent of U. Recall that, on the cylinder, $d\chi = 0$ and $d^*\chi = *dF(\chi)$. In view of the chain rule expression (7.8) there is a constant $C_3 > 0$ such that

$$\|dF(\chi)\|_{L^2_k(U)} < C_3\|\chi\|_{C^k(U)}\left(\|\nabla\chi\|_{L^2_k(U)} + \|\chi\|_{L^2_k(U)}\right).$$

As χ decays uniformly we can ensure that $\|\chi\|_{C^k(U)} < 1/2C_1C_3$ by taking U to be sufficiently far along the cylindrical end. Then

$$\|\chi\|_{L^2_{k+1}(U)} < \|\chi\|_{L^2_k(U)} + 2C_2\|\chi\|_{L^2(U)}.$$

Hence χ is $L^2_{k,\delta}$ for all $k \ge 0$. $\qquad\square$

This completes the proof of theorem 7.1.1.

Remark 7.2.12. We did not obtain any bounds on the weighted norms in terms of t. Thus, while we have estimates for the non-weighted norms of $d\eta$ and know that $d\eta$ decays exponentially, we have no idea how far down the cylinder we need to go before the exponential decay kicks in.

7.2.4 Proof of theorem 7.1.3

Recall that the torsion-free G_2-structures $\varphi(L)$ are obtained by perturbing the closed G_2-structures $\tilde{\varphi}(L)$ with small torsion, which are in turn defined by stretching the cylindrical neck $X \times I$ of $\tilde{\varphi}$ by a length $2L$. The cohomology class $[\varphi(L)] = [\varphi] + 2L\delta([\omega])$, where ω is the Kähler form on X, so the image of the path $\varphi(L)$ in $H^3(M)$ is an affine line with slope $2\delta([\omega])$.

We also defined torsion-free EAC G_2-structures $\varphi_\pm$ on $M_\pm$ by perturbing the G_2-structures $\tilde{\varphi}_\pm$ obtained by cutting $\tilde{\varphi}$ in half. Gluing φ_+ and φ_- defines a path $Y(\varphi_+, \varphi_-, L)$ of torsion-free G_2-structures on M. The restrictions $i_\pm^*[Y(\varphi_+, \varphi_-, L)] = i_\pm^*[\varphi]$, so the image of the path in $H^3(M)$ lies in the affine space $K = [\varphi] + \delta(H^2(X))$. It too is an affine line with slope $2\delta([\omega])$.

We first prove the result using the simplifying assumption that $b^1(M) = 0$. Let $\mathcal{R}_y$ be the pre-moduli space of matching pairs of torsion-free G_2-structures near (φ_+, φ_-), and

$$\mathcal{R}_y' = \{(\psi_+, \psi_-) \in \mathcal{R}_y : i_\pm^*\psi_\pm = i_\pm^*\varphi_\pm\}.$$

As in the proof of lemma 6.4.2, for large $L_1 \in \mathbb{R}$ the image of $Y_H : \mathcal{R}_y' \times (L_1, \infty) \to K$ contains an open affine cone in K.

The difference between $[\varphi(L)]$ and $[Y(\varphi_+, \varphi_-, L)]$ is constant in L. Therefore for sufficiently large L there is an $L'(L)$ close to L and a matching pair $(\varphi_+(L), \varphi_-(L)) \in \mathcal{R}_y'$ such that $\varphi(L)$ is cohomologous to the glued structure $Y(\varphi_+(L), \varphi_-(L), L'(L))$. In fact, because the RHS of (6.24) is dominated by the $2L\delta([\omega])$ term for large L, the distance between $(\varphi_+(L), \varphi_-(L))$ and (φ_+, φ_-) is of order $1/L$, measured in the C^1 norm (since $\mathcal{R}_y$ has finite dimension all sensible norms are Lipschitz equivalent). Hence the difference between $Y(\varphi_+, \varphi_-, L)$ and $Y(\varphi_+(L), \varphi_-(L), L)$ is of order $1/L$ in C^0 norm. As the volume is of order L it follows also that the difference is of order $L^{-1/2}$ in L^2-norm, and order $L^{-13/14}$ in L_1^{14}-norm.

Now $\varphi(L)$ and $Y(\varphi_+(L), \varphi_-(L), L'(L))$ are both torsion-free perturbations of $\tilde{\varphi}(L)$ within its cohomology class, so we can try to use proposition 7.2.4 to show that they are diffeomorphic. For large L the difference between $Y(\varphi_+(L), \varphi_-(L), L'(L))$ and $\tilde{\varphi}(L)$ is dominated by the difference between $\tilde{\varphi}_\pm$ and $\varphi_\pm$, which is estimated in terms of t in (7.2). Therefore if t is sufficiently small then for all sufficiently large L the estimates required to

apply proposition 7.2.4 are satisfied, and

$$Y(\varphi_+(L), \varphi_-(L), L'(L)) \cong \varphi(L).$$

This proves theorem 7.1.3.

Remark 7.2.13. Looking more closely at the contraction-mapping argument in the proof of theorem 7.2.2, one can use $Y(\varphi_+(L), \varphi_-(L), L'(L))$ as an ansatz for $\varphi(L)$ and deduce that the distance between the two is in fact controlled by L alone, and not by t. Therefore the representatives $\varphi(L)$ converge in 'geometric' sense. Let $M_\pm(L)$ denote the gluing of $M_\pm(0)$ and $X \times [0, L]$, which can be regarded as a compact subset of either $M_\pm$ or $M(L')$ for $L' > L$. Then $\varphi(L')|_{M_\pm(L)} \to \varphi_\pm|_{M_\pm(L)}$ uniformly as $L' \to \infty$ with L fixed. This way torsion-free EAC G_2-structures can be recovered from compact G_2-structures.

Some minor changes to the above argument are needed to deal with the case $b^1(M) > 0$. Then (cf. (6.25)) $\delta(H^2(X))$ splits as

$$\delta(H^2(X)) = \delta(E_1^2) \oplus \delta(E_6^2) \oplus \delta(E_8^2).$$

The affine space K is modelled on $\delta(H^2(X))$ so we may consider the quotient $K/\delta(E_6^2)$. The image of $Y_H : \mathcal{R}'_y \times (L_1, \infty) \to K/\delta(E_6^2)$ contains an open cone. Therefore for sufficiently large L there is an $L'(L)$ close to L and a matching pair $(\varphi_+(L), \varphi_-(L)) \in \mathcal{R}'_y$ such that

$$Y_H(\varphi_+(L), \varphi_-(L), L'(L)) - [\varphi(L)] \in \delta(E_6^2).$$

Lemma 6.3.8 implies that there is $\tilde{\chi}(L)$ in the identity component of $\tilde{\mathcal{I}}_X$ such that $\varphi(L)$ is cohomologous to the glued structure $Y(\tilde{\chi}^*(\varphi_+(L), \varphi_-(L), L'(L)))$. As $L \to \infty$, we have the same bounds as before for the difference between $\tilde{\varphi}(L)$ and $Y(\varphi_+(L), \varphi_-(L), L)$.

Meanwhile, $\tilde{\chi}(L)$ converges as $L \to \infty$. We can therefore choose the representative for $Y(\tilde{\chi}^*(\varphi_+(L), \varphi_-(L), L'(L)))$ so that it is close to $Y(\varphi_+(L), \varphi_-(L), L'(L))$ in C^0, L^2 and L_1^{14}-norm. For one is obtained from the other by 'twisting' the cylindrical part by a path of automorphisms of the cross-section. If the twisting is stretched out to take place over a length L of the cylinder then the difference in L^1-norm is kept approximately constant, but the C^0 norm is of order $1/L$, the L^2 norm of order $L^{-1/2}$, and the L_1^{14}-norm of order $L^{-13/14}$. One can then complete the proof as in the $b^1(M) = 0$ case.

7.3 Examples

We now show how theorem 7.1.3 can be applied to pull apart some of Joyce's examples of compact G_2-manifolds into pairs of EAC G_2-manifolds. In the process we find some examples of EAC manifolds with holonomy exactly G_2. We compute the Betti numbers of the manifolds, and find some examples of EAC coassociative submanifolds.

7.3.1 A preliminary example

We begin with an example which does not require any preparation (but for which the glued G_2-manifold is reducible). Let $\gamma : T^7 \to T^7$ be the involution acting by

$$(x_1, x_2, x_3, x_4, x_5, x_6, x_7) \mapsto (-x_1, x_2, -x_3, x_4, -x_5, x_6, -x_7).$$

The singularities of $T^7/\langle\gamma\rangle$ can be resolved with the classical Kummer construction to give $M = T^3 \times K3$. If we resolve the singularities using Joyce's results then we find that the conditions of theorem 7.1.3 are satisfied, so that we can pull apart M.

γ preserves the standard flat G_2-structure on T^7 and theorem 7.2.1 provides a family of closed G_2-structures φ with small torsion on the resolution M.

Now let $I \subset (0, \frac{1}{2})$ be an open interval and X the flat torus T^6 with the standard Calabi-Yau structure. $I \times X \subset T^7$ maps injectively to its image $N \subseteq T^7/\langle\gamma\rangle$. If I is not too close to 0 or $\frac{1}{2}$ then N is not affected when the singularities are resolved, so can be regarded as a subset of M.

$$\varphi|_N = \Omega + dt \wedge \omega,$$

so $N \subset M$ can be taken as a cylindrical neck. The family of torsion-free G_2-structures on M together with the neck N satisfies the hypotheses of theorem 7.1.3 (with an appropriate choice of λ). The resulting EAC G_2-manifolds are of the form $T^3 \times S_\pm$, where $S_\pm$ are EAC manifolds of holonomy $SU(2)$.

Remark 7.3.1. S_+ and S_- are equivalent as complex surfaces, and can be obtained as the complement of an anti-canonical divisor of the blow-up of $\mathbb{C}P^2$ at 9 points. Since S_+ and S_- can be glued to form a $K3$ we will refer to them as *half-K3 surfaces*.

7.3.2 Preparing the examples

We describe how to cast some of Joyce's examples into a form to which theorem 7.1.3 applies by resolving only some orbifold singularities in a first step. This idea is similar to an outline argument for how to construct irreducible quasi-asymptotically locally Euclidean G_2-manifolds in Joyce [27, p. 277].

Assume that the flat torus from which the construction starts has the form $S^1 \times T^6$, where T^6 is a torus with a flat Calabi-Yau structure. Let Γ be a finite group of G_2 automorphisms of $S^1 \times T^6$. Choose R-data and form the resolution M of $(S^1 \times T^6)/\Gamma$. Theorem 7.2.1 provides a family of closed G_2-structures $\tilde{\varphi}$ with small torsion on M, which can be perturbed to torsion-free G_2-structures. We would like to claim that these torsion-free G_2-structures can also be obtained by perturbing a G_2-structure with small torsion that satisfies the hypotheses of theorem 7.1.3.

Suppose that the elements of Γ act by products of isometries of S^1 and T^6 and let Γ' be the subgroup acting trivially on the S^1 factor. We assume that some elements of Γ act by reflection on S^1 (as has to be the case if $b^1(M) = 0$). If $\theta \in S^1$ is not a fixed point of any of the reflections then $\{\theta\} \times T^6$ does not meet the fixed point set of $\Gamma \setminus \Gamma'$, and its image divides $(S^1 \times T^6)/\Gamma$ into exactly two connected components. In order to make things work we assume furthermore that the fixed point set of $\Gamma \setminus \Gamma'$ does not meet that of Γ'.

The R-data for $(S^1 \times T^6)/\Gamma$ can be restricted to give R-data for $S^1 \times T^6/\Gamma'$. Let M' be the corresponding resolution. Theorem 7.2.1 gives closed G_2-structures $\tilde{\varphi}'$ with small torsion and by theorem 7.2.2 they can be perturbed to torsion-free G_2-structures φ'. M' is homeomorphic to $S^1 \times X^6$ for a compact manifold X.

The lemma below can be thought of as a simple version of the Cheeger-Gromoll line splitting theorem (cf. lemma 4.1.9) and ensures that there is a Calabi-Yau structure on X such that M' is isomorphic to $S^1 \times X$ as a G_2-manifold (cf. Chan [11, p. 15]).

Lemma 7.3.2. *Let T^m be a torus and X a compact manifold with $b^1(X) = 0$. If g is a Ricci-flat metric on $T^n \times X$ that is invariant under translations of the torus factor then there is a function $f : X \to \mathbb{R}^n$ such that the graph diffeomorphism*

$$T^n \times X \to T^n \times X, \ (t, x) \mapsto (t + f(x), x)$$

pulls g back to a product metric.

Sketch proof. Let $\frac{\partial}{\partial \theta^1}, \ldots, \frac{\partial}{\partial \theta^n}$ be the unit coordinate vector fields on T^n, and set $\alpha_i = \frac{\partial}{\partial \theta^i}^\flat$. $\frac{\partial}{\partial \theta^i}$ is a Killing vector field, so α_i is harmonic by proposition 5.2.4. Since $b^1(X) = 0$ the

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closed forms $\alpha_i|_X$ are exact. Define $f : X \to \mathbb{R}^n$ by picking f_i such that $\alpha_i = df_i$. $\qquad\square$

The compatibility conditions for the R-data ensure that the quotient group $\Psi = \Gamma/\Gamma'$ acts in a well-defined way on M'. Moreover $\tilde{\varphi}'$ is invariant under this action and hence φ' is too. We can use the R-data to resolve the singularities of M'/Ψ and topologically this gives M. If $I \subset S^1$ is an interval not containing any fixed points of the reflections then $I \times X$ maps homeomorphically onto its image N in M and is a candidate for a cylindrical neck. We wish to to define a closed G_2-structure on M with small torsion, whose restriction to N is φ'.

The issue is that theorem 7.2.2 relies on the orbifold singularities that are to be resolved with small torsion being modelled on a quotient of the flat G_2-structure. But φ' need not be flat near the fixed point set F of Ψ.

Let S be a tubular neighbourhood of F. Recall that $\varphi' = \tilde{\varphi}' + d\eta'$ for some $d\eta'$ satisfying the estimates (7.3). The assumption that the fixed point sets of Γ' and $\Gamma \setminus \Gamma'$ are disjoint ensures that $\tilde{\varphi}'|_S$ is flat. In order to 'restore' the flatness near F we wish to define an exact form that is supported on S, equal to $d\eta'$ near F, and small in the same sense that $d\eta'$ is small. To do this we use a version of the classical Poincaré lemma.

Lemma 7.3.3. *Let F be a compact Riemannian manifold and I a bounded open interval. For any $n \geq 0$, $k \geq 0$ and $p \geq 1$ there is a constant $C > 0$ such that for any exact L_k^p m-form $d\eta$ on the Riemannian product $X = F \times I^n$ there is an $(m-1)$-form χ with $d\chi = d\eta$ and*

$$\|\chi\|_{L_{k+1}^p} < C\|d\eta\|_{L_k^p}. \tag{7.14}$$

Proof. The proof is by induction on n. The result holds for $n = 0$ by usual Hodge theory. For the inductive step, we show that if a manifold X satisfies the conclusion of the theorem, then so does $X \times I$ (with the product metric).

For $s \in I$ let X_s denote the hypersurface $X \times \{s\}$. Let t denote the coordinate on I, and write

$$d\eta = \alpha + dt \wedge \beta,$$

with α and β sections of the pull-back of $\Lambda^* T^* X$ to $X \times I$. Write $\alpha(s)$, $\beta(s)$ for the corresponding forms on X_s. Fix $s_0 \in I$ and let

$$\chi_1(s) = \int_{s_0}^{s} \beta(t)dt.$$

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Let ∇ denote the covariant derivative on $X \times I$, and consider χ_1 as a form on $X \times I$. For any $0 \le i \le k$ and $s \in I$

$$\|(\nabla^i \chi_1)(s)\|^p_{L^p(X_s)} = \int_X \left\| \int_{s_0}^s (\nabla^i \beta)(t) dt \right\|^p vol_X$$
$$\le V^{p-1} \int_X \int_{s_0}^s \|(\nabla^i \beta)(t) dt\|^p dt \, vol_X \le V^{p-1} \|\nabla^i \beta\|^p_{L^p(X \times I)},$$

where V is the length of I. Hence

$$\|\nabla^i \chi_1\|^p_{L^p(X \times I)} \le \int_I \|(\nabla^i \chi_1)(s)\|^p_{L^p(X_s)} ds \le V^p \|\nabla^i \beta\|^p_{L^p(X \times I)},$$

and

$$\|\chi_1\|_{L^p_k(X \times I)} \le V \|d\eta\|_{L^p_k(X \times I)}.$$

$d(\eta - \chi_1)$ has no dt-component, so the dt-component of $d^2(\eta - \chi_1)$ is $\frac{\partial}{\partial t} d(\eta - \chi_1) = 0$. Hence $d(\eta - \chi_1)$ is the pull-back to $X \times I$ of an exact form on X. By the inductive hypothesis there is a form χ_2 such that $d\chi_2 = d(\eta - \chi_1)$ and $\chi = \chi_1 + \chi_2$ satisfies (7.14) for some C independent of $d\eta$. $\qquad\square$

Therefore there is a 2-form χ on S such that

$$d\chi = d\eta'|_S$$

and χ satisfies estimates proportional to (7.3) (the uniform estimate comes from theorem 7.2.3). If ρ is a cut-off function which is 1 near F and 0 outside S then

$$\|d(\rho\chi)\|_{L^2} < K't^4, \quad \|d(\rho\chi)\|_{C^0} < K't^{1/2}, \quad \|\nabla d(\rho\chi)\|_{L^{14}} < K', \qquad (7.15)$$

with K' independent of t. $\tilde{\varphi}' + d(\eta' - \rho\chi)$ is a family of closed G_2-structures which are flat near F. It is clear from the chain rule that torsion is small, but we need to take care to choose the small form ψ' such that $d\ast\psi' = d\Theta(\tilde{\varphi} + d(\eta' - \rho\chi))$ in such a way that it vanishes not only on the cylindrical neck region, but also near F. Because F has dimension 3 any closed 4-form on the tubular neighbourhood S is exact. By lemma 7.3.3 we can write

$$(\Theta(\tilde{\varphi}' + d\eta') - \Theta(\tilde{\varphi}'))|_S = d\chi'$$

for some 3-form χ' on S, so that $d(\rho\chi')$ satisfies estimates of the form (7.15). We can then

take

$$\psi' = *(\Theta(\tilde{\varphi}' + d(\eta' - \rho\chi)) - \Theta(\tilde{\varphi}' + d\eta') + d(\rho\chi')),$$

which is supported in S but 0 near F.

We can ensure that all forms are invariant under Ψ, so $d(\eta' - \rho\chi)$ descends to an exact 3-form β on the orbifold M'/Ψ. As it is supported away from the singular set, β is also defined on the resolution M. In the same way ψ' descends to a small 3-form ψ_1 on M.

Recall that we denoted by $\tilde{\varphi}$ the G_2-structure on M with small torsion obtained by resolving $(S^1 \times T^6)/\Gamma$. $\tilde{\varphi}' + d(\eta' - \rho\chi)$ descends to an orbifold G_2-structure on M'/Ψ with small torsion. Its orbifold singularities are modelled on quotients of the flat G_2-structure, so the singularities can be resolved like in theorem 7.2.1 to define a closed G_2-structure on M. This is precisely $\tilde{\varphi} + \beta$. The torsion introduced by the resolution is small, in the sense that there is a smooth 3-form ψ_2 on M, supported near the pre-image F' of the singular set, such that $d^*\psi_2 = d^*\tilde{\varphi}$ near F' and ψ_2 satisfies the estimate (i) in theorem 7.2.1. Now $(\tilde{\varphi} + \beta, \psi_1 + \psi_2)$ satisfies the hypotheses of theorem 7.1.3 (for t sufficiently small). Moreover, proposition 7.2.4 shows that the torsion-free G_2-structures obtained by perturbing the 'one-step' resolution $\tilde{\varphi}$ and the 'two-step' resolution $\tilde{\varphi}+\beta$ are diffeomorphic.

7.3.3 A simple example

We now discuss the example of [27, §12.2]. This is a resolution of the quotient of T^7 by the group $\Gamma \cong \mathbb{Z}_2^3$ generated by

$$\begin{aligned}
\alpha &: (x_1, \ldots, x_7) \mapsto (x_1, x_2, x_3, -x_4, -x_5, -x_6, -x_7), \\
\beta &: (x_1, \ldots, x_7) \mapsto (x_1, -x_2, -x_3, x_4, x_5, \tfrac{1}{2} - x_6, -x_7), \\
\gamma &: (x_1, \ldots, x_7) \mapsto (-x_1, x_2, -x_3, x_4, \tfrac{1}{2} - x_5, x_6, \tfrac{1}{2} - x_7).
\end{aligned} \tag{7.16}$$

The fixed point set of each of α, β and γ consists of 16 copies of T^3 and these are all disjoint. $\alpha\beta$, $\beta\gamma$, $\gamma\alpha$ and $\alpha\beta\gamma$ act freely on T^7. Furthermore $\langle\beta, \gamma\rangle$ acts freely on the set of 16 3-tori fixed by α, so they map to 4 copies of T^3 in the singular set of T^7/Γ. Similarly $\langle\alpha, \gamma\rangle$ and $\langle\alpha, \beta\rangle$ acts freely on the 16 3-tori fixed by β and γ, respectively. Thus the singular set of T^7/Γ consists of 12 copies of T^3.

At each T^3 in the singular set the normal space is $\mathbb{C}^2/\{\pm 1\}$. To form a set of R-data we need to choose a resolution of $\mathbb{C}^2/\{\pm 1\}$ for each component of the fixed point set. Topologically the essentially unique resolution is the blow-up Y of $\mathbb{C}^2/\{\pm 1\}$ at the origin. This has a family of ALE $SU(2)$-metrics called Eguchi-Hansen metrics. So for each of

the 48 copies of T^3 in the fixed point set we use Y with an Eguchi-Hansen metric as our resolution of $\mathbb{C}^2/\{\pm 1\}$. The equivariance condition on the R-data then means that we must actually choose the same Eguchi-Hansen metric for each quadruple of 3-tori which map to the same component of the singular set.

Once R-data have been chosen we can apply theorems 7.2.1 and 7.2.2 to give torsion-free G_2-structures on a resolution M^7 of T^7/Γ. This G_2-manifold can be pulled apart in several different ways. In order to say something about the topology of the resulting EAC manifolds let us first recall the technique for computing the Betti numbers of the resolutions from [27, §12.1]. We consider first the compact G_2-manifold M.

The cohomology of T^7/Γ is just the Γ-invariant part of the cohomology of T^7, so $b^2(T^7/\Gamma) = 0$ while $b^3(T^7/\Gamma) = 7$. For each of the 12 copies of T^3 in the singular set we cut out a tubular neighbourhood, which deformation retracts to T^3, and glue in a resolution $T^3 \times Y$. Y is the blow-up of $\mathbb{C}^2/\{\pm 1\}$ at the singular point, which is biholomorphic to $T^*\mathbb{C}P^1$. $T^3 \times Y$ therefore deformation retracts to $T^3 \times S^2$. Each of the operations increases the Betti numbers of M by the difference between the Betti numbers of $T^3 \times Y$ and T^3. This is justified using the long exact sequences for the cohomology of T^7/Γ relative to its singular set and M relative to the resolving neighbourhoods. Hence

$$b^2(M) = 12 \cdot 1 = 12,$$
$$b^3(M) = 7 + 12 \cdot 3 = 43.$$

To compute the Betti numbers of the EAC G_2-manifolds we consider them as resolutions of $(T^6 \times \mathbb{R})/\Gamma$. In the cases where the holonomy is exactly G_2 we find that the Betti numbers of $(T^6 \times \mathbb{R})/\Gamma$ are $b^2 = 0$, $b^3 = 4$, $b^4 = 3$, $b^5 = 0$. There are singular sets not only of the form T^3, but also $T^2 \times \mathbb{R}$. Resolving these adds 1, 2 and 1 to b^2, b^3 and b^4, respectively.

(i) To begin with, choose Γ' to be the stabiliser of the S^1 factor corresponding to the x_1 coordinate, i.e. $\Gamma' = \langle \alpha, \beta \rangle$. The quotient $S^1 \times T^6/\Gamma'$ is isomorphic to $S^1 \times X_{19}$, for a simply-connected Calabi-Yau 3-fold X_{19}. The fixed points of the reflections on S^1 are 0 and $\frac{1}{2}$, so if we take $I \subset (0, \frac{1}{2})$ then the image N in M of $I \times X_{19}$ can be used as a cylindrical neck for some closed G_2-structure $\tilde{\varphi}$ with small torsion. The manifolds $M_\pm$ with cylindrical ends that result from pulling apart along N are simply connected with a single end. Therefore by theorem 4.1.11 they have holonomy exactly G_2.

The G_2-manifolds M_+ and M_- are isometric. Their boundaries are identified by the anti-holomorphic involution of X_{19} induced by the action of γ on T^6.

The singular set in $(\mathbb{R} \times T^6)/\Gamma$ consists of 8 copies of $T^2 \times \mathbb{R}$ and 2 copies of T^3. Therefore

$$b^2(M_\pm) = 8 \cdot 1 + 2 \cdot 1 = 10,$$
$$b^3(M_\pm) = 4 + 8 \cdot 2 + 2 \cdot 3 = 26,$$
$$b^4(M_\pm) = 3 + 8 \cdot 1 + 2 \cdot 3 = 17,$$
$$b^5(M_\pm) = 2 \cdot 1 = 2.$$

We can also compute the Betti numbers of the cross-section X_{19}, and find that $b^2(X_{19}) = 19$, $b^3(X_{19}) = 40$. Therefore its Hodge numbers are

$$h^{1,1}(X_{19}) = h^{1,2}(X_{19}) = 19.$$

If instead we pull M apart in the x_2 or x_4 direction we get essentially the same result. We just need to use $\langle \gamma, \alpha \rangle$ or $\langle \beta, \gamma \rangle$ as Γ' to define the intermediate resolution.

(ii) If we pull apart along the x_3 direction we get a slightly different result. In this case $\Gamma' = \langle \alpha, \beta\gamma \rangle$, which only contains one element with fixed points. The cross-section of the neck is a resolution X_{11} of T^6/Γ'. The first Betti number $b^1(X_{11})$ vanishes, but X_{11} is doubly covered by $T^2 \times K3$, so $Hol(X_{11}) = \mathbb{Z}_2 \ltimes SU(2)$. The EAC G_2-manifolds $M_\pm$ are however simply-connected without cylindrical double covers. They are thus examples of irreducible EAC G_2-manifolds with locally reducible cross-section.

In this case the singular set in each half is 4 copies of T^3 and 4 copies of $T^2 \times \mathbb{R}$. The Betti numbers are therefore

$$b^2(M_\pm) = 4 \cdot 1 + 4 \cdot 1 = 8,$$
$$b^3(M_\pm) = 4 + 4 \cdot 2 + 4 \cdot 3 = 24,$$
$$b^4(M_\pm) = 3 + 4 \cdot 1 + 4 \cdot 3 = 19,$$
$$b^5(M_\pm) = 4 \cdot 1 = 4.$$

The Hodge numbers of $X_{11} = (T^2 \times K3)/\mathbb{Z}_2$ are

$$h^{1,1}(X_{11}) = h^{1,2}(X_{11}) = 11.$$

(iii) If we pull apart along the x_5 or x_6 direction then Γ' is $\langle \beta \rangle$ or $\langle \gamma \rangle$ respectively. The resulting EAC G_2-manifolds are of the form $S^1 \times M'_\pm$, where $M'_\pm$ is a simply-connected EAC Calabi-Yau 3-fold and the cross-section is $T^2 \times K3$. This case therefore looks

similar to the twisted connected sums of reducible EAC G_2-manifolds used by Kovalev in [34]. However, this example cannot arise from Kovalev's method for generating matching pairs of reducible EAC G_2-manifolds, since the connected sum of such pairs always has $b^2 \leq 9$.

(iv) Finally, if we pull apart along the x_7 direction then $\Gamma' = \langle \alpha\beta \rangle$. This has no fixed points, so the cross-section X of the neck is flat. It has $b^1(X) = 2$. Like in §7.3.1 there is no need for an intermediate resolution in this case. Of the two EAC G_2-manifolds one is a product of S^1 with a simply-connected EAC Calabi-Yau 3-fold, and the other is a quotient of the product of T^3 with a half-$K3$ surface, as described in remark 7.3.1.

7.3.4 EAC coassociative submanifolds

Let us also give some examples of EAC coassociative submanifolds of EAC manifolds with holonomy exactly G_2. The deformation problem for such submanifolds has been consider by Joyce and Salur [28].

Coassociative submanifolds are instances of calibrated submanifolds as introduced by Harvey and Lawson [23]. They call a closed m-form α on a Riemannian manifold a calibration if $\alpha|_V \leq vol_V$ for any oriented m-dimensional tangent space V. An oriented m-dimensional submanifold C is said to be calibrated by α if $\alpha|_C = vol_C$. It is elementary to deduce from the definition that any calibrated submanifold is volume-minimising in its homology class. If M^7 is a G_2-manifold with G_2-structure φ then $*\varphi$ is a calibration.

Definition 7.3.4. A 4-dimensional submanifold $C \subset M$ is called *coassociative* if it is calibrated by $*\varphi$.

The next two propositions are elementary to prove by working in the point-wise model for the G_2-structure.

Proposition 7.3.5 ([27, Lemma 10.8.2])**.** *A 4-dimensional submanifold $C \subset M$ is orientable and coassociative (with respect to one of its orientations) if and only if $\varphi|_C = 0$.*

Proposition 7.3.6 ([27, Proposition 10.8.5])**.** *Let $\sigma : M \to M$ be an involution such that $\sigma^*\varphi = \varphi$. Then each connected component of the fixed point set of σ is either a coassociative 4-manifold or a single point.*

Let M^7 be the compact G_2-manifold constructed in §7.3.3, and denote its torsion-free G_2-structure by φ. Joyce [27, §12.6] applies proposition 7.3.6 to find examples of

coassociative submanifolds of M. We consider just the two simplest examples, where the coassociative 4-manifolds are diffeomorphic to T^4 or $K3$.

Example 7.3.7. Define an orientation-reversing isometry of T^7 as in [27, Example 12.6.4].

$$\sigma : (x_1, \ldots, x_7) \mapsto (\tfrac{1}{2} - x_1, x_2, x_3, x_4, x_5, \tfrac{1}{2} - x_6, \tfrac{1}{2} - x_7). \tag{7.17}$$

Then σ commutes with the action of Γ defined by (7.16) and pulls back φ_0 to $-\varphi_0$. When the singularities of T^7/Γ are resolved to form the compact G_2-manifold M one can ensure that σ lifts to an involution of M such that $\sigma^*\varphi = -\varphi$. It is easy to see that the fixed point set of σ in M consists of 16 isolated points and one copy of T^4, which is a coassociative submanifold of M.

We can also consider (7.17) to define an involution σ of $T^6 \times \mathbb{R}$. Provided that the $\mathbb{R}$ factor corresponds to the x_2, x_3 or x_4 coordinate this again commutes with the action of Γ. When we pull apart M in the x_2, x_3 or x_4 direction the resulting irreducible EAC G_2-manifolds $M_\pm$ are resolutions of $(T^6 \times \mathbb{R})/\Gamma$, so σ lifts to an involution of $M_\pm$ that reverses the torsion-free G_2-structure. The fixed point set in each half $M_\pm$ consists of 8 isolated points and one copy of $T^3 \times \mathbb{R}$, which is an asymptotically cylindrical coassociative submanifold of $M_\pm$.

Example 7.3.8. Define an orientation-reversing isometry of T^7 as in [27, Example 12.6.4].

$$\sigma : (x_1, \ldots, x_7) \mapsto (\tfrac{1}{2} - x_1, \tfrac{1}{2} - x_2, \tfrac{1}{2} - x_3, x_4, x_5, x_6, x_7).$$

Again, σ lifts to an involution of M such that $\sigma^*\varphi = -\varphi$ and whose fixed point set in T^7/Γ consists of 16 isolated points and two copies of $T^4/\{\pm 1\}$. The corresponding coassociative submanifolds in M are two copies of the usual Kummer resolution of $T^4/\{\pm 1\}$, which is a $K3$ surface.

If we pull apart M in the x_4 direction then σ defines also involutions of the resulting irreducible EAC G_2-manifolds $M_\pm$. In each half the fixed point set has two 4-dimensional components, which are resolutions of $(T^3 \times \mathbb{R})/\{\pm 1\}$. These are asymptotically cylindrical coassociative submanifolds of M, diffeomorphic to the half-$K3$ from remark 7.3.1.

Remark 7.3.9. Compact coassociative submanifolds have a well-behaved deformation theory. For any coassociative submanifold $C \subset M$ the normal bundle of C is isomorphic to $\Lambda_+^2 T^*C$. McLean [41] shows that if C is compact then the moduli space of coassociative submanifolds isotopic to C is a smooth manifold of dimension $b_+^2(C)$. In particular, if C is diffeomorphic to T^4 or $K3$ then the dimension of the deformation space equals the codi-

mension of C in M. They also have trivial self-dual bundle, so they are candidates for fibres of fibrations by coassociative submanifolds.

Joyce and Salur [28] prove an EAC analogue of McLean's result. Let M^7 be an EAC G_2-manifold with cross-section X^6 and $C \subset M$ an asymptotically cylindrical coassociative submanifold with cross-section $L \subset X$ (L is then a special Lagrangian submanifold of the Calabi-Yau 3-fold X, i.e. L is calibrated by the real part Ω of the holomorphic volume form on X). Then the space of deformations of C asymptotic to the fixed boundary L is a smooth manifold and its dimension is $b_+^2(C)$, the dimension of the positive part of the compactly supported subspace $H_0^2(C) \subseteq H^2(C)$. For $T^3 \times \mathbb{R}$ or the half-$K3$ surface this vanishes, so the coassociative submanifolds in example 7.3.7 and 7.3.8 are rigid if the boundary is kept fixed.

7.3.5 Another example

Now let us consider the example of [27, §12.3]. This is a slight modification of the previous example. The components of the singular set still do not intersect each other, but some of the components have two different possible resolutions.

Let T^7 be the standard torus with the standard flat G_2-structure φ_0. Let $\Gamma \cong \mathbb{Z}_2^3$ be the group of automorphisms of T^7 generated by

$$\alpha : (x_1, \ldots, x_7) \mapsto (x_1, x_2, x_3, -x_4, -x_5, -x_6, -x_7),$$
$$\beta : (x_1, \ldots, x_7) \mapsto (x_1, -x_2, -x_3, x_4, x_5, \tfrac{1}{2} - x_6, -x_7),$$
$$\gamma : (x_1, \ldots, x_7) \mapsto (-x_1, x_2, -x_3, x_4, -x_5, x_6, \tfrac{1}{2} - x_7).$$

The fixed point set of each of α, β and γ consists of 16 copies of T^3 and these are all disjoint. $\alpha\beta$, $\beta\gamma$, $\gamma\alpha$ and $\alpha\beta\gamma$ act freely on T^7. Furthermore $\langle \beta, \gamma \rangle$ and $\langle \alpha, \gamma \rangle$ act freely on the set of 16 3-tori fixed by α and β, respectively. However, $\alpha\beta$ maps each component of the fixed point set of γ to itself. Therefore the singular set of T^7/Γ consists of 8 copies of T^3 and 8 copies of $T^3/\mathbb{Z}_2$. The T^3 fixed by γ has a natural choice of coordinates x_2, x_4, x_6. In these coordinates the action of $\alpha\beta$ is

$$\alpha\beta : T^3 \to T^3, \quad (x_2, x_4, x_6) \mapsto (-x_2, -x_4, \tfrac{1}{2} + x_6).$$

To form R-data we can first use the blow-up Y of $\mathbb{C}^2/\{\pm 1\}$ with an Eguchi-Hansen metric for each of the 8 3-tori in the singular set, just as in §7.3.3. For each of the 8

copies of $T^3/\mathbb{Z}_2$ there are, however, two non-equivalent choices Y_+ and Y_- of resolution of $\mathbb{C}/\{\pm 1\}$. This is because $\alpha\beta$ acts on the normal space $\mathbb{C}^2$ to the T^3 fixed by γ, and the R-data must include a compatible automorphism of the resolution. Both Y_+ and Y_- are diffeomorphic to Y, but $\alpha\beta$ acts differently on them. In fact the induced action on the second cohomology $H^2(Y_\pm) \cong \mathbb{R}$ is $(\alpha\beta)^* = \pm 1$. Y_+ and Y_- both have ALE $SU(2)$-metrics which are $\alpha\beta$-invariant.

The upshot is that for each of the 8 copies of $T^3/\mathbb{Z}_2$ there are two topologically distinct choices of resolutions. The different R-data will therefore give 2^8 topologically distinct resolutions $\pi : M \to T^7/\Gamma$ (however some of the manifolds M are diffeomorphic).

We can pull the resulting G_2-manifold M apart in several ways as in §7.3.3. To help compute the Betti numbers of the parts let us tabulate the contributions from the different types of resolutions used.

	Singularity	Resolution	b^2	b^3	b^4	b^5
A	$T^3 \times (\mathbb{C}^2/\{\pm 1\})$	$T^3 \times Y$	1	3	3	1
B	$(T^3 \times (\mathbb{C}^2/\{\pm 1\}))/\mathbb{Z}_2$	$(T^3 \times Y_+)/\mathbb{Z}_2$	1	1	1	1
C	$(T^3 \times (\mathbb{C}^2/\{\pm 1\}))/\mathbb{Z}_2$	$(T^3 \times Y_-)/\mathbb{Z}_2$	0	2	2	0
D	$T^2 \times \mathbb{R} \times (\mathbb{C}^2/\{\pm 1\})$	$T^2 \times \mathbb{R} \times Y$	1	2	1	0
E	$(T^2 \times \mathbb{R} \times (\mathbb{C}^2/\{\pm 1\}))/\mathbb{Z}_2$	$(T^2 \times \mathbb{R} \times Y_+)/\mathbb{Z}_2$	1	1	0	0
F	$(T^2 \times \mathbb{R} \times (\mathbb{C}^2/\{\pm 1\}))/\mathbb{Z}_2$	$(T^2 \times \mathbb{R} \times Y_-)/\mathbb{Z}_2$	0	1	1	0

For the compact G_2-manifold M one uses 8 A-resolutions, and a total of 8 B- or C-resolutions. If $0 \le k \le 8$ is the number of B-resolutions used then

$$b^2(M) = 8 \cdot 1 + k \cdot 1 = 8 + k,$$
$$b^3(M) = 7 + 8 \cdot 3 + k \cdot 1 + (8 - k)2 = 47 - k.$$

(i) Pulling apart M in the x_1 direction gives a pair of simply-connected EAC manifolds $M_\pm$ with holonomy exactly G_2 and cross-section X_{19}, similar to §7.3.3(i). The resolved singular set in each half consists of 8 copies of $T^2 \times \mathbb{R}$ and 4 copies of $T^3/\mathbb{Z}_2$. If we

use $k_\pm$ B-resolutions and $4 - k_\pm$ C-resolutions (so $k = k_+ + k_-$) then

$$b^2(M_\pm) = 8 \cdot 1 + k_\pm \cdot 1 = 8 + k_\pm,$$
$$b^3(M_\pm) = 4 + 8 \cdot 2 + k_\pm \cdot 1 + (4 - k_\pm)2 = 28 - k_\pm,$$
$$b^4(M_\pm) = 3 + 8 \cdot 1 + k_\pm \cdot 1 + (4 - k_\pm)2 = 19 - k_\pm,$$
$$b^5(M_\pm) = k_\pm \cdot 1 = k_\pm.$$

(ii) Pulling apart M in the x_2 or x_4 directions still gives simply-connected EAC manifolds $M_\pm$ with holonomy exactly G_2, and cross-section X_{19}, but the topology is different. The resolutions used are $2 \times A$, $4 \times D$, $k \times E$ and $(8 - k) \times F$.

$$b^2(M_\pm) = 4 \cdot 1 + 2 \cdot 1 + k \cdot 1 = 6 + k,$$
$$b^3(M_\pm) = 4 + 4 \cdot 2 + 2 \cdot 3 + k \cdot 1 + (8 - k)1 = 26,$$
$$b^4(M_\pm) = 3 + 4 \cdot 1 + 2 \cdot 3 + (8 - k)1 = 21 - k,$$
$$b^5(M_\pm) = 2 \cdot 1 = 2.$$

(iii) Pulling apart M in the x_3 or x_5 direction gives a pair of simply-connected EAC manifolds $M_\pm$ with holonomy exactly G_2 and cross-section X_{11}, similar to §7.3.3(ii). Now one uses resolutions $2 \times A$, $4 \times D$, $k_\pm \times B$ and $(4 - k_\pm) \times C$.

$$b^2(M_\pm) = 4 \cdot 1 + 2 \cdot 1 + k_\pm \cdot 1 = 6 + k,$$
$$b^3(M_\pm) = 4 + 4 \cdot 2 + 2 \cdot 3 + k_\pm \cdot 1 + (4 - k_\pm)2 = 26 - k_\pm,$$
$$b^4(M_\pm) = 3 + 4 \cdot 1 + 2 \cdot 3 + k_\pm \cdot 1 + (4 - k_\pm)2 = 21 - k_\pm,$$
$$b^5(M_\pm) = 2 \cdot 1 + k_\pm \cdot 1 = 2 + k_\pm.$$

(iv) Pulling apart M in the x_6 direction one obtains EAC G_2-manifolds which are products of S^1 with simply-connected EAC Calabi-Yau 3-folds, topologically the same as those in §7.3.3(iii).

(v) Pulling apart M in the x_6 direction produces one EAC G_2-manifolds which is a product of S^1 with a simply-connected EAC Calabi-Yau 3-fold and one which is doubly covered by the product of T^3 and half-$K3$, similar to §7.3.3(iv).

7.4 Concluding remarks

We have explained the pulling-apart argument for G_2-manifolds only in some simple cases and many natural questions remain. One obvious question is how to deal with those examples obtained by desingularising quotients of tori where different parts of the singular set intersect. In these cases it is more difficult to write down an intermediate resolution which has both a torsion-free cylindrical neck and suitably small torsion.

One way to deal with this difficulty would be to implement a method for constructing G_2-structures with small torsion proposed by Joyce in [27, p. 304]. Let X^6 be a Calabi-Yau 3-fold with Calabi-Yau structure (Ω, ω) and suppose that a is an anti-holomorphic involution of X with $a^*\Omega = \Omega$, $a^*\omega = -\omega$. Then $\Omega + dt \wedge \omega$ is a torsion-free G_2-structure on $S^1 \times X$, which descends to a well-defined G_2-structure on the quotient $(S^1 \times X)/(-1, a)$ (cf. example 4.1.18). If the fixed point set L of a is non-empty then the quotient is singular. In this case L is a real 3-dimensional submanifold of X (in fact it is special Lagrangian). The singular set of $(S^1 \times X)/(-1, a)$ is $\{0, \frac{1}{2}\} \times L$ and locally each point should be resolved like $\mathbb{R}^3 \times Y$, where Y is an Eguchi-Hansen space. One would need to vary the choice of metric on Y smoothly. Joyce suggests that a suitable choice can be made to obtain a G_2-structure with small torsion on the resolution provided that L has a non-vanishing harmonic 1-form. The main purpose of this argument would be to produce new examples of compact G_2-manifolds. However, finding Calabi-Yau 3-folds to which it can be applied is a non-trivial task.

Examples produced by this method would automatically satisfy the conditions of theorem 7.1.3, so they could be pulled apart. If the construction were made rigorous then it could almost certainly be applied to give intermediate resolutions for many of Joyce's examples.

For many cases it would suffice to carry the argument through for the special case when $X = T^2 \times K3$, with a acting by $(x, y) \mapsto (x, -y)$ on T^2 and by an anti-holomorphic involution on $K3$. By applying a hyper-Kähler rotation to $K3$ this problem is equivalent to resolving the singularities of a quotient $(T^2 \times K3)/\langle b \rangle$ within holonomy $SU(3)$, where b acts by -1 on T^2 and by a holomorphic involution on $K3$. The quotient is then a singular analytic variety, and can be resolved by blowing up the singular set. One would then need to find a Calabi-Yau metric on the blow-up and give estimates for its distance to the $T^2 \times K3$ metric away from the singularities.

Resolving quotients of $T^2 \times K3$ within holonomy $SU(3)$ should also be sufficient to construct some examples of EAC manifolds with holonomy exactly $Spin(7)$. One reason

why the method in §7.3 for constructing EAC manifolds with holonomy exactly G_2 does not carry over directly is that in the Kummer-type construction for compact $Spin(7)$-manifolds there are *always* intersecting singular sets. The simplest example Joyce gives is in [27, §14.2]. This starts from the singular quotient of T^8 by a group $\Gamma \cong \mathbb{Z}_2^4$ generated by

$$\alpha : (x_1, \ldots, x_8) \mapsto (-x_1, -x_2, -x_3, -x_4, x_5, x_6, x_7, x_8),$$
$$\beta : (x_1, \ldots, x_8) \mapsto (x_1, x_2, x_3, x_4, -x_5, -x_6, -x_7, -x_8),$$
$$\gamma : (x_1, \ldots, x_8) \mapsto (\tfrac{1}{2} - x_1, \tfrac{1}{2} - x_2, x_3, x_4, \tfrac{1}{2} - x_5, \tfrac{1}{2} - x_6, x_7, x_8),$$
$$\delta : (x_1, \ldots, x_8) \mapsto (-x_1, x_2, \tfrac{1}{2} - x_3, x_4, -x_5, x_6, \tfrac{1}{2} - x_7, x_8).$$

The quotient can be resolved in an essentially unique way to produce a compact $Spin(7)$-manifold M^8. Pulling apart M in the x_4 (or x_8) direction yields halves which have the right topology to be EAC manifolds with holonomy exactly $Spin(7)$, but the argument from §7.3.2 cannot be applied directly to produce an intermediate resolution with a cylindrical neck because the fixed point sets of α and β intersect at the 256 fixed points of $\alpha\beta$. One could get around this problem by performing the resolution in three steps. First resolve the quotient $T^8/\langle\beta\rangle$ to obtain $T^4 \times K3$. Then resolve $(T^4 \times K3)/\langle\gamma, \delta\rangle$ to get $S^1 \times Y^7$, where Y is a G_2-manifold. Hopefully the second resolution can be made in such a way that a neighbourhood of the singular set of α remains very close to a product $\mathbb{R}^4 \times K3$, with α acting as -1 on $\mathbb{R}^4$ and trivially on the $K3$ factor. Finally the singularities of $(S^1 \times Y)/\langle\alpha\rangle$ could then be resolved to give a $Spin(7)$-structure with small torsion and a neck region isomorphic to the product of Y with an interval. An analogue of theorem 7.1.1 should then provide torsion-free EAC $Spin(7)$-structures on the two halves.

If one succeeds in constructing EAC $Spin(7)$-manifolds this way one could ask if the compact $Spin(7)$-manifold can be deformed to a glued manifold. This would require a set-up for a gluing construction for EAC $Spin(7)$-manifolds and a study of its deformation properties, similar to that for G_2-manifolds in §6. The technical details are likely to be more complicated. The fact that the image in $H^4(M)$ of the moduli space of torsion-free $Spin(7)$-structures is a submanifold rather than an open subset makes it harder to write down a local expression for the gluing map.

Another remaining question concerns the boundary of the moduli space $\mathcal{M}$ of torsion-free G_2-structures. We have exhibited some examples of compact G_2-manifolds M^7 where a single component of $\mathcal{M}$ has boundary points of both orbifold and connected-sum type. In fact, in these examples there are 7 different kinds of connected-sum boundary points,

corresponding to the different directions in which M can be pulled apart. One can ask whether $\mathcal{M}$ can be compactified in a natural way, so that different types of boundary points lie on the same connected component. Naively, one would expect that a face of orbifold type and a face of connected-sum type would be joined by a codimension 2 edge, parametrised by classes of matching pairs of EAC orbifolds. Similarly, two different faces of connected-sum type boundary point may meet at a codimension 2 edge defined by quarters of M. Defining the boundary conditions for such quarters would be rather more complicated than the definition for EAC manifolds.

Finally, one possible application of "pulling-apart" is to provide a way to study Joyce's examples *complex analytically*. While we highlighted the cases where the connected components have holonomy exactly G_2 for their novelty, the cases when the connected components are reducible may give more information about the decomposed manifold. When the connected components are of the form $S^1 \times X^6_\pm$, where $X_\pm$ are EAC Calabi-Yau 3-folds, then $X_\pm$ can be considered as the complement of an anti-canonical divisor in a compact complex 3-fold $\overline{X}_\pm$. This is very similar to the matching pairs of reducible EAC G_2-manifolds found in [34]. In [35] Kovalev proposes a method to produce coassociative fibrations for compact G_2-manifolds obtained as the gluing of such pairs, by patching together fibrations of each half by complex submanifolds of the $X_\pm$-factor. This way one may hope to find coassociative fibrations of those of Joyce's examples which can be pulled apart into reducible connected components.

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