G_2 Geometry
Problem sheet on classifying spaces

The following exercises take you through the computation of (most of) the cohomology of $BU(n)$, $BO(n)$ and $BSO(n)$. They rely on the following two results.

Leray-Hirsch theorem Let $\pi : E \to B$ be a fibre bundle, $i : F \to E$ the inclusion of a fibre. Let R be a ring, and suppose that $H^k(F; R)$ is free and finitely generated (as an R-module) for each k. If for each k there are classes $a_j \in H^k(E; R)$ such that i^*a_j form a basis of $H^k(F; R)$, then

$$H^*(B; R) \otimes_R H^*(F; R) \to H^*(E; R),\ b \otimes i^*a_j \mapsto \pi^*b \cup a_j$$

is an isomorphism.

Thom isomorphism theorem Let $\pi : E \to B$ be a real vector bundle of rank n, and let $\dot{E}$ be the complement of the graph of the zero section in the total space of E.

(i) There is a unique $u_2(E) \in H^n(E, \dot{E}; \mathbb{Z}_2)$ such that the restriction of $u_2(E)$ to any fibre $\mathbb{R}^n$ equals the generator of $H^n(\mathbb{R}, \mathbb{R} \setminus \{0\}; \mathbb{Z}_2)$.

$$H^k(B; \mathbb{Z}_2) \to H^{k+n}(E, \dot{E}; \mathbb{Z}_2), a \mapsto \pi^*a \cup u(E)$$

is an isomorphism.

(ii) If E is oriented then there is a unique $u(E) \in H^n(E, \dot{E}; \mathbb{Z})$ such that the restriction of $u(E)$ to any fibre $\mathbb{R}^n$ equals the positive generator of $H^n(\mathbb{R}, \mathbb{R} \setminus \{0\}; \mathbb{Z})$.

$$H^k(B; \mathbb{Z}) \to H^{k+n}(E, \dot{E}; \mathbb{Z}), a \mapsto \pi^*a \cup u(E)$$

is an isomorphism.

1. For an oriented real rank n vector bundle $\pi : E \to X$, let $u(E) \in H^n(E, \dot{E}; \mathbb{Z})$ be the Thom class, and define the *Euler class* $e(E) \in H^n(X; \mathbb{Z})$ by

$$\pi^*e(E) \cup u(E) = u(E) \cup u(E) \in H^{2n}(E, \dot{E}; \mathbb{Z}).$$

Show that

(a) e is a characteristic class.

(b) If n is odd then $2e = 0 \in H^n(X; \mathbb{Z})$.

(c) $e(E_1 \oplus E_2) = e(E_1)e(E_2) \in H^{n_1+n_2}(X)$ for any oriented $E_i \to X$ of rank n_i.

(d) $e(\mathcal{O}_{\mathbb{C}P^1}) \in H^2(\mathbb{C}P^1)$ is the negative generator.

2. *Line splitting principle for complex vector bundles*

(a) Let $E \to X$ be a complex vector bundle of rank n, and let $\pi : \mathbb{P}(E) \to X$ be the fibre bundle whose fibre over x is the projectivisation $\mathbb{P}(E_x)$. Let $e \in H^2(\mathbb{P}(E); \mathbb{Z})$ be the Euler class of the tautological bundle $\mathcal{O}_E(-1)$ over the total space of $\mathbb{P}(E)$. Show that $H^*(\mathbb{P}(E); \mathbb{Z})$ is isomorphic to the free $H^*(X; \mathbb{Z})$ module generated by $1, e, \ldots, e^{n-1}$, and that there exists a rank $n{-}1$ complex vector bundle $E_1 \to \mathbb{P}(E)$ such that $\pi^*E \cong \mathcal{O}_E(-1) \oplus E_1$.

(b) Pick an arbitrary hermitian metric on E. The "flag bundle" $\pi : \mathrm{Fl}\, E \to X$ is the bundle whose fibre over x is $\{$ordered set of orthogonal lines $\ell_1, \ldots, \ell_n \subset E_x\}$. Show that π^*E is isomorphic to a direct sum of n complex line bundles, and that $\pi^* : H^*(X; \mathbb{Z}) \to H^*(\mathrm{Fl}\, E; \mathbb{Z})$ is injective.

3. *Construction of Chern classes* Let $E \to X$ be a complex vector bundle of rank n. Q2(a) implies that, with $e \in H^2(\mathbb{P}(E); \mathbb{Z})$ as before, there exist unique classes $c_k(E) \in H^{2k}(X; \mathbb{Z})$ such that

$$e^n = -\sum_{k=1}^{n} e^{n-k} \pi^* c_k(E).$$

Define the total Chern class by

$$c(E) := \sum_{k=0}^{n} c_k \in H^*(X; \mathbb{Z}).$$

(a) Let X be the product of n copies of $\mathbb{C}P^\infty$, let $E = L_1 \oplus \cdots \oplus L_n$ be the direct sum of its n tautological line bundle, and $t_i = e(L_i) \in H^2(X; \mathbb{Z})$. Introducing a formal variable t, define $f \in H^*(X; \mathbb{Z})[t]$ by

$$f(t) = \sum_{k=0}^{n} c_k(E) t^{n-k}.$$

Show that $f(t_i) = 0 \in H^{2k}(X; \mathbb{Z})$ for each i, and deduce that the total Chern classes satisfy

$$c(E) = c(L_1) \cdots c(L_n) \in H^*(X; \mathbb{Z}).$$

(*Hint:* Define a section $s_i : X \to \mathbb{P}(E)$ such that $s_i^* e = t_i$. Use that $\mathbb{Z}[t_1, \ldots, t_n, t]$ is an integral domain.)

(b) Show that the total Chern class is exponential:

$$c(E_1 \oplus E_2) = c(E_1) c(E_2)$$

for any complex vector bundles $E_1, E_2 \to Y$.

(c) Suppose $\hat{c}_k$ are a family of characteristic classes of complex vector bundles $E \to Y$, $\hat{c}_k(E) \in H^{2k}(Y; \mathbb{Z})$, such that the total class $c(E)$ is exponential. Show that $\hat{c} = \lambda c$ for some constant $\lambda \in \mathbb{Z}$.

4. (a) Let $BU(n) = \mathrm{Gr}_n(\mathbb{C}^\infty)$, and $EU(n) = \{(v_1, \ldots, v_n) : v_i \in \mathbb{C}^\infty \text{ hermitian-orthonormal}\}$. Show that $EU(n) \to BU(n)$ is a contractible $U(n)$ bundle.

(b) Show that $H^*(\mathrm{Fl}\, EU(n); \mathbb{Z}) = \mathbb{Z}[t_1, \ldots, t_n]$, where t_i are Euler classes of n tautological line bundles. (*Hint:* Apply Leray-Hirsch, or show that $\mathrm{Fl}\, EU(n)$ is homotopy equivalent to the product of n copies of $\mathbb{C}P^\infty$.)

(c) Let $c_k := c_k(EU(n)) \in H^{2k}(BU(n); \mathbb{Z})$. Show that $\pi^* : H^*(BU(n); \mathbb{Z}) \to H^*(\mathrm{Fl}\, EU(n); \mathbb{Z})$ defines an isomorphism $H^*(BU(n); \mathbb{Z}) \cong \mathbb{Z}[t_1, \ldots, t_n]^{S_n}$, mapping c_k to the kth elementary symmetric polynomial $\sigma_k(t_1, \ldots, t_n)$.

5. Show that for any complex vector bundle $E \to X$

(a) $c_1(E) = c_1(\det E)$,

(b) $c_n(E) = e(E_{\mathbb{R}})$.

6. *Stiefel-Whitney classes*

   (a) Let $E \to X$ be a real vector bundle of rank n. Find a map $f : Y \to X$ such that f^*E is isomorphic to a direct sum of n real line bundles, and $f^* : H^*(X; \mathbb{Z}_2) \to H^*(Y; \mathbb{Z}_2)$ is injective.

   (b) Construct characteristic classes $w_k(E) \in H^k(X; \mathbb{Z}_2)$ of real vector bundle $E \to X$ such that the total classes $w(E)$ is exponential, and show that they are the unique non-zero such classes.

   (c) Show that $BO(n) = \mathrm{Gr}_n(\mathbb{R}^\infty)$. Letting $w_k := w_k(EO(n)) \in H^*(BO(n); \mathbb{Z}_2)$, show that $H^*(BO(n); \mathbb{Z}_2) = \mathbb{Z}_2[w_1, \ldots, w_n]$.

   (d) For a real vector bundle $E \to X$ of rank n, show that the image of $w_n(E) \in H^n(X; \mathbb{Z}_2)$ under the Thom isomorphism $H^n(X; \mathbb{Z}_2) \to H^n(E, \dot{E}; \mathbb{Z}_2)$ is the square of the Thom class.

7. (a) Show that $BSO(n) = \widetilde{\mathrm{Gr}}(\mathbb{R}^\infty)$, the Grassmannian of oriented n-dimensional subspace in $\mathbb{R}^\infty$ (a double cover of $\mathrm{Gr}_n(\infty)$).

   (b) For $k \geq 2$, write w_k also for $w_k(ESO(n)) \in H^k(BSO(n); \mathbb{Z}_2)$. Show that $H^k(BSO(n); \mathbb{Z}_2) = \mathbb{Z}_2[w_2, \ldots, w_n]$. (*Hint:* Consider classifying maps $BSO(n) \to BO(n)$ for $ESO(n)$ as an unoriented real vector bundle, and $BO(n) \to BSO(n{+}1)$ of $EO(n) \oplus \det(EO(n))$.)

8. *Real plane splitting principle*

   (a) Let $\widetilde{\mathrm{Gr}}_2(\mathbb{R}^n) := \{\text{oriented planes in } \mathbb{R}^n\}$. Given $\Pi \in \widetilde{\mathrm{Gr}}_2(\mathbb{R}^n)$, pick an oriented orthonormal basis $v_1, v_2 \in \Pi$ and set $f(\Pi) := [v_1 + iv_2] \in \mathbb{C}P^{n-1}$. Show that $f : \widetilde{\mathrm{Gr}}_2(\mathbb{R}^n) \to \mathbb{C}P^n$ is well-defined, and maps $\widetilde{\mathrm{Gr}}_2(\mathbb{R}^n)$ diffeomorphically onto $Q := \{[z_1 : \cdots : z_n] \in \mathbb{C}P^n : z_1^2 + \cdots z_n^2 = 0\}$.

   (b) Let $E \to X$ be a real vector bundle of rank ≥ 3. Pick a metric g on E, extend to a complex bilinear form on $E_\mathbb{C}$, and and let $Q(E) := \{[w] \in \mathbb{P}(E_\mathbb{C}) : g(w, w) = 0\}$, a fibre bundle $\pi : Q(E) \to X$ with typical fibre Q. Show that any element in the kernel of $\pi^* : H^*(X; \mathbb{Z}) \to H^*(Q(E); \mathbb{Z})$ is 2-torsion (*i.e.* $\pi^*x = 0 \Rightarrow 2x = 0 \in H^*(X; \mathbb{Z})$). Show that π^*E is as a direct sum of an oriented bundle of rank 2 and a bundle of rank $n - 2$. (*Hint:* Consider the composition of π^* with the Thom isomorphism of the normal bundle of Q in $\mathbb{P}(E_\mathbb{C})$ and a map to $H^*(\mathbb{P}(E_\mathbb{C}); \mathbb{Z})$.)

   (c) Show that for any real vector bundle $E \to X$ there is a map $f : Y \to X$ such that f^*E is a direct sum of oriented bundles of rank ≤ 2, which can all be taken to be oriented if E is, while all the kernel of $f : H^*(X; \mathbb{Z}) \to H^*(Y; \mathbb{Z})$ is 2-primary torsion ($f^*x = 0 \Rightarrow 2^m x = 0$ for some m.)

9. For any real vector bundle $E \to X$ let $p_k(E) := c_{2k}(E) \in H^{4k}(X; \mathbb{Z})$.

   (a) Show that the space $\{\text{ordered set of orthogonal oriented 2-planes } \Pi_1, \ldots, \Pi_n \subset \mathbb{R}^\infty\}$ is homotopy equivalent to the product of n copies of $\mathbb{C}P^\infty$.

   (b) Show that the quotient of $H^*(BO(n); \mathbb{Z})$ by its 2-primary torsion is $\mathbb{Z}[p_k : 2k \leq n]$.

   (c) Show that the quotient of $H^*(BSO(2n{+}1); \mathbb{Z})$ by its 2-primary torsion is $\mathbb{Z}[p_1, \ldots, p_n]$.

   (d) Show that the quotient of $H^*(BSO(2n); \mathbb{Z})$ by its 2-primary torsion is $\mathbb{Z}[p_1, \ldots, p_{n-1}, e]$.

10. Show that a characteristic class $\hat{e}$ that assigns to an oriented rank n bundle $E \to X$ an element $\hat{e}(E) \in H^n(X; \mathbb{Z})$, for all n, such that

$$\hat{e}(E_1 \oplus E_2) = \hat{e}(E_1)\hat{e}(E_2) \text{ for any } E_1, E_2 \to X$$

must be a constant multiple of the Euler class.

11. Recall that $Sq^1 : H^k(X;\mathbb{Z}_2) \to H^{k+1}(X;\mathbb{Z}_2)$ is the snake map of the long exact sequence induced by the short exact sequence of coefficients $0 \to \mathbb{Z}_2 \to \mathbb{Z}_4 \to \mathbb{Z}_2 \to 0$. Similarly, let $\beta_2 : H^k(X;\mathbb{Z}_2) \to H^{k+1}(X;\mathbb{Z})$ and $\beta_4 : H^k(X;\mathbb{Z}_4) \to H^{k+1}(X;\mathbb{Z})$ denote the snake maps induced by $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_2 \to 0$ and $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_4 \to 0$ respectively. The image of β_n is the n-torsion subgroup $T_nH^{k+1}(X;\mathbb{Z})$.

    (a) Show that $T_4H^k(X;\mathbb{Z})/T_2H^k(X;\mathbb{Z})$ is isomorphic to the quotient of the kernel of $Sq^1 : H^{k-1}(X;\mathbb{Z}_2) \to H^k(X;\mathbb{Z}_2)$ by the image of $\rho_2 : H^{k-1}(X;\mathbb{Z}) \to H^{k-1}(X;\mathbb{Z}_2)$.

    (b) Show that

$$\begin{aligned} Sq^1 w_{2k+1} &= w_1 w_{2k+1} \\ Sq^1 w_{2k} &= w_1 w_{2k} + w_{2k+1} \end{aligned}$$

    (c) Show that the torsion subgroup of $H^k(BSO(n);\mathbb{Z})$ and $H^k(BO(n);\mathbb{Z})$ is always 2-torsion. (*Hint:* The image of ρ_2 contains that of Sq^1. For $BO(n)$ case, think of elements of $\ker Sq^1$ as polynomials in w_1, and consider the top coefficient.)

12. (a) Recall that for any fibre bundle $E \to B$ with fibre F, there is a long exact sequence of homotopy groups $\pi_k F \to \pi_k E \to \pi_k B \to \pi_{k-1} F \to \cdots$. Deduce that $\pi_k Spin(n) \cong \pi_k Spin(n+1)$ for $k \leq n-1$, and $\pi_k SU(n) \cong \pi_k SU(n+1)$ for $k \leq 2n$.

    (b) For $m \geq 2n$, consider the lift $i : SU(n) \hookrightarrow Spin(m)$ of $SU(n) \hookrightarrow SO(2n) \hookrightarrow SO(m)$. Show that $i_* : \pi_k SU(2) \hookrightarrow \pi_k Spin(5)$ is an isomorphism for $k \leq 5$. (*Hint:* $Spin(5)$ acts on its spin representation $\mathbb{H}^2$ by quaternionic linear maps. The restriction of the spin representation of $Spin(4)$ to $SU(2)$ is isomorphic to $\Lambda^*\mathbb{C}^2$ (complex rank 4).)

    (c) For $m \geq 2n$, let $f : BSU(n) \to BSpin(m)$ be the classifying map for $ESU(n)$ considered as a $Spin(m)$ bundle. By considering a commuting diagram

$$\begin{array}{ccccc} SU(n) & \longrightarrow & ESU(n) & \longrightarrow & BSU(n) \\ \downarrow{\scriptstyle i} & & \downarrow & & \downarrow{\scriptstyle f} \\ Spin(m) & \longrightarrow & ESpin(m) & \longrightarrow & BSpin(m) \end{array}$$

    show that $f_* : \pi_k BSU(n) \to \pi_k BSpin(m)$ is an isomorphism for $k \leq 4$, $m \geq 5$, $n \geq 2$.

    (d) Let $g : BSpin(m) \to BSO(m)$ be the classifying map for $ESpin(m)$ considered as an $SO(m)$-bundle. Show that for $m \geq 5$, $H^4(BSpin(m);\mathbb{Z}) \cong \mathbb{Z}$ has a generator q such that $2q = g^*p_1$, and $f^*q = -c_2$.

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November 3, 2014