G_2 **Geometry**
Problem sheet 2 – Classifying spaces

1. Show that $S^\infty := \cup_{n\geq 0} S^n$ (with weak topology) is contractible. (*Hint:* Consider the shift map $S^\infty \to S^\infty$ that maps $(x_1, \dots, x_n) \in S^{n+1}$ to $(0, x_1, \dots, x_n) \in S^{n+2}$.)

2. Find a pair of smooth manifolds M_1, M_2 with a homotopy equivalence $f : M_1 \to M_2$ such that $w_k(M_1) \neq f^* w_k(M_2)$ for some k.

3. Let M^n be a closed smooth manifold. For $i_1 + \cdots + i_k = n$, the $(i_1, \dots, i_k)$ Stiefel-Whitney number of M is
$$\int_M w_{i_1}(M) \cdots w_{i_k}(M) \in \mathbb{Z}_2.$$
Show that if $M = \partial X$ for some compact smooth X^{n+1} then all Stiefel-Whitney numbers of M vanish.

4. Compute the Stiefel-Whitney classes of $\mathbb{R}P^n$.
   (a) For which n is $\mathbb{R}P^n$ orientable and spin?
   (b) Show that $\mathbb{R}P^n$ is the boundary of some smooth compact $n+1$-manifold if and only if n is odd.

5. Deduce from the characterisation of the Chern and Stiefel-Whitney classes given in the lectures that if $E \to X$ is a complex vector bundle with underlying real bundle $E_\mathbb{R}$, then
$$w_{2k}(E_\mathbb{R}) = c_k(E) \mod 2 \ \in H^{2k}(M; \mathbb{Z}_2),$$
$$w_{2k+1}(E_\mathbb{R}) = 0 \ \in H^{2k+1}(M; \mathbb{Z}_2)$$

6. (a) Let $U_\pm := S^n \setminus \{(\pm 1, 0, \dots, 0)\}$, and $\pi : U_+ \cap U_- \to S^{n-1}$ the projection to the equator, $\pi(x_1, \dots, x_n) := \frac{(x_2, \dots, x_n)}{|(x_2, \dots, x_n)|}$. Given $g : S^{n-1} \to G$, define a G-bundle $P \to S^n$ with "clutching function" g by requiring transition function $U_+ \cap U_- \to G$ between trivialisations over $U_\pm$ to be $g \circ \pi$. Show that this induces a bijection
$$\pi_{n-1} G \ \to \ \{\text{isomorphism classes of } G\text{-bundles over } S^n\}.$$
   (b) Deduce the existence of such a bijection from the existence of a universal bundle EG instead.

7. (a) Show that there are exactly two isomorphism classes of $SO(3)$-bundles over S^2.
   (b) Show that $\mathrm{Bl}_p \mathbb{C}P^2$ is diffeomorphic to the total space of an S^2 fibre bundle over S^2.
   (c) Show that $\mathrm{Bl}_p \mathbb{C}P^2$ is not homotopy equivalent to $S^2 \times S^2$. (*Hint:* Compare the intersection forms.)

8. For any cochain complex C^* of free abelian groups, the short exact sequence $0 \to C^* \otimes \mathbb{Z}_2 \to C^* \otimes \mathbb{Z}_4 \to C^* \otimes \mathbb{Z}_2 \to 0$ induces a long exact sequence
$$\cdots \to H^i(C; \mathbb{Z}_2) \to H^i(C; \mathbb{Z}_4) \to H^i(C; \mathbb{Z}_2) \xrightarrow{\beta} H^{i+1}(C; \mathbb{Z}_2) \to \cdots .$$
β is called the *Bockstein map* of $0 \to \mathbb{Z}_2 \to \mathbb{Z}_4 \to \mathbb{Z}_2 \to 0$.
   (a) If C^* has a graded ring structure that descends to $H^*(C)$, show that
$$\beta(ab) = \beta(a)b + a\beta(b)$$
for any $a, b \in H^*(C; \mathbb{Z}_2)$.

(b) For a topological space X, the *first Steenrod square* $Sq^1 : H^i(X;\mathbb{Z}_2) \to H^{i+1}(X;\mathbb{Z}_2)$ is defined to equal the Bockstein map. Show that Sq^1 is a cohomology operation, *i.e.* for any $f : X \to Y$ and $a \in H^i(Y;\mathbb{Z}_2)$ we have

$$Sq^1(f^*a) = f^*(Sq^1 a) \in H^{i+1}(X;\mathbb{Z}_2).$$

(c) Show that

$$Sq^1 a = a^2 \in H^2(X;\mathbb{Z}_2)$$

for any $a \in H^1(X;\mathbb{Z}_2)$.

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November 3, 2014