G_2 **Geometry**
Problem sheet on Chern-Weil theory

1. Let E, E' be complex vector bundles on M. Deduce from the Chern-Weil definition that
   (a) $c_k(E)$ are real cohomology classes,
   (b) $c_1(E \otimes E') = c_1(E) + c_1(E')$,
   (c) $c_1(\det E) = c_1(E)$.

2. The action of $SU(2)$ on $\mathbb{C}^2$ induces an action by bundle isomorphisms on the tautological line bundle $\mathcal{O}(-1)$ over $\mathbb{C}P^1$. Show that there is a unique $SU(2)$-invariant connection on $\mathcal{O}(-1)$, and that its curvature equals $\frac{i}{2}$ of the volume form of the standard round metric ($\frac{4|dz|^2}{(1+|z|^2)^2}$ in the stereographic chart $z \mapsto (1{:}z) \in \mathbb{C}P^1$). Deduce that

$$\int_{\mathbb{C}P^1} c_1(\mathcal{O}(-1)) = -1.$$

3. Recall that $R[\lambda_1, \ldots, \lambda_n]^{S_n}$, the ring of symmetric polynomials in n variable with coefficients in a ring R, equals the free polynomial algebra $R[\sigma_1, \ldots, \sigma_n]$, where the *elementatry symmetric polynomials* σ_k are defined by

$$(t - \lambda_1) \cdots (t - \lambda_n) \; = \; t^n - t^{n-1}\sigma_1 + \cdots + (-1)^n \sigma_n.$$

   Let $\mathcal{P}_{\mathrm{GL}(n)}$ denote the ring of $\mathrm{GL}(n)$-invariant polynomials $\mathrm{Mat}_{n\times n}(\mathbb{C}) \to \mathbb{C}$, and

$$\det(t - m) \; = \; t^n - t^{n-1}\Sigma_1(m) + \cdots + (-1)^n \Sigma_n(m).$$

   for $m \in \mathrm{Mat}_{n\times n}(\mathbb{C})$. Show that

$$\begin{aligned} r : \mathcal{P}_{\mathrm{GL}(n)} &\to \mathbb{C}[\lambda_1, \ldots, \lambda_n], \\ P &\mapsto P(\mathrm{diag}(\lambda_1, \ldots, \lambda_n)) \end{aligned}$$

   is injective, with $r(\Sigma_k) = \sigma_k$. Deduce that $\mathcal{P}_{\mathrm{GL}(n)} \cong \mathbb{C}[\Sigma_1, \ldots, \Sigma_n]$.

4. Show that the direct sum of $T\mathbb{C}P^n$ with a trivial line bundle on $\mathbb{C}P^n$ is isomorphic to the direct sum of $n+1$ copies of $\mathcal{O}(-1)$. Compute the Chern classes of CP^n.

5. Let E, E' be real vector bundles over M
   (a) Show that $c_{2k+1}(E_{\mathbb{C}}) = 0$ for all k.
   (b) Show that $p_k(E \oplus E') = \sum_{i=0}^{k} p_i(E) p_{k-i}(E')$.

6. For $m \in \mathrm{Skew}_{2n}\mathbb{R}$, let $\omega(m) = \sum_{i<j} m_{ij} e^i \wedge e^j \in \Lambda^2(\mathbb{R}^{2n})^*$. Let $\mathrm{Pf} : \mathrm{Skew}_{2n}\mathbb{R} \to \mathbb{R}$ be the homogeneous degree n polynomial such that

$$\omega(m)^n = \mathrm{Pf}(m)\, e^1 \wedge \cdots \wedge e^{2n} \in \Lambda^{2n}(\mathbb{R}^{2n})^*.$$

   Show that
   (a) $\mathrm{Pf}(g^T m g) = \det(g)\mathrm{Pf}(m)$ for any $g \in GL(2n, \mathbb{R})$.
   (b) $\mathrm{Pf}(m)^2 = \det(m)$.

7. (a) For $a_1, \ldots, a_n \in \mathbb{R}$, let $\mathrm{Block}(a_1, \ldots, a_n) \in \mathrm{Skew}_{2n}\mathbb{R}$ be the matrix with blocks $\left(\begin{smallmatrix} 0 & a_k \\ -a_k & 0 \end{smallmatrix}\right)$ along the diagonal. Show that

$$\begin{aligned} r : \mathcal{P}_{O(2n)} &\to \mathbb{R}[a_1, \ldots, a_n], \\ P &\mapsto P(\mathrm{Block}(a_1, \ldots, a_n)) \end{aligned}$$

   is injective, with image $\mathbb{R}[a_1^2, \ldots, a_n^2]^{S_n}$. Deduce that $\mathcal{P}_{O(2n)} = \mathbb{R}[\Sigma_2, \Sigma_4, \ldots, \Sigma_{2n}]$.

(b) Show that $\mathcal{P}_{SO(2n)} = \mathbb{R}[\Sigma_2, \Sigma_4, \ldots, \Sigma_{2n-2}, \mathrm{Pf}]$.

(c) Show that $\mathcal{P}_{O(2n+1)} = \mathcal{P}_{SO(2n+1)} = \mathbb{R}[\Sigma_2, \Sigma_4, \ldots, \Sigma_{2n}]$.

8. Let E be a complex vector bundle of rank n, and $E_\mathbb{R}$ the underlying real vector bundle of rank $2n$.

   (a) Express $p_k(E_\mathbb{R})$ in terms of the Chern classes of E.

   (b) Show that $e(E_\mathbb{R}) = c_n(E)$.

9. (a) Let E be a real vector bundle over M, and $s \in \Gamma(E)$ a transverse section, *i.e.* if s vanishes at $p \in M$ then the composition $D_p s : T_pM \to T_pE \cong T_pM \oplus E_p \to E_p$ is surjective. Then $X := s^{-1}(0)$ is a smooth submanifold. Show that the normal bundle of X in M is isomorphic to the restriction $E_{|X}$.

   (b) Let s be a transverse holomorphic section of a holomorphic vector bundle E over a complex manifold M. Then $X := s^{-1}(0)$ is a complex submanifold of M. Show that the normal bundle of X in M is isomorphic (as a complex vector bundle) to $E_{|X}$.

   (c) Let M be a complex manifold and $X \subset M$ a complex submanifold. Let U_i be an open cover of M with holomorphic functions $f_i : U_i \to \mathbb{C}$ such that $X \cap U_i = f_i^{-1}(0)$, and f_i vanishes to order 1 along X. Let $[X]$ be the holomorphic line bundle with trivialisations over U_i such that the transition function from U_i to U_j is multiplication by $\frac{f_j}{f_i}$. Explain why this is well-defined, and show that $[X]$ has a global holomorphic section vanishing to order 1 along X. (Then $c_1([X]) = e([X]) = (\text{Poincaré dual to } X) \in H^2(M)$).

10. Let $p : \mathbb{C}^{n+1} \to \mathbb{C}$ be a homogeneous polynomial such that $D_z p \neq 0$ for all $z \in \mathbb{C}^n$ where $p(z) = 0$. Describe the Chern classes of $X := \{p = 0\} \subset \mathbb{C}P^n$, and calculate the Euler characteristic $\chi(X)$.

11. (a) For an open set $U \subset \mathbb{C}^n$ containing 0, let $\mathrm{Bl}_0 U := \{(z, p) \in \mathbb{C}^n \times \mathbb{C}P^{n-1} : z \in p\}$. For a complex manifold M of dimension n and $x \in M$, show that there is a manifold $\mathrm{Bl}_x M$ with a holomorphic map $\pi : \mathrm{Bl}_x M \to M$ such that $E := \pi^{-1}(x) \cong \mathbb{C}P^{-1}$, $\pi_{|\mathrm{Bl}_x M \setminus E}$ is a biholomorphism $\mathrm{Bl}_x M \setminus E \to M \setminus \{x\}$, and E has a neighbourhood biholomorphic to $\mathrm{Bl}_0 \Delta^n$.

    (b) Show that $\mathrm{Bl}_p M$ is diffeomorphic to $M \# (-\mathbb{C}P^n)$, the connected sum of M and $\mathbb{C}P^n$ with its orientation reversed.

    (c) Show that $H^*(\mathrm{Bl}_p M)$ is a direct sum of $\pi^* H^*(M)$ and the subspace generated by $c_1([E])$, $\ldots, c_1([E])^{n-1}$.

    (d) The canonical line bundle of M is the holomorphic line bundle $K_M := \Lambda^n T^* M$. Show that $K_{\mathrm{Bl}_p M} \cong \pi^* K_M \otimes [E]^{\otimes (n-1)}$.

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