CALIBRATED GEOMETRY

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Recommended reading:

- Salamon, Riemannian Geometry and Holonomy Groups [11]
  `http://calvino.polito.it/~salamon/G/rghg.pdf`
- Joyce, Compact Manifolds with Special Holonomy [8] (Ch 2, 3, 10)
- Besse, Einstein Manifolds [2] (Ch 10)

Lecture one

Calibrations. The notion of a calibration was introduced in 1982 by Harvey and Lawson [5].

Linear version. Let $\alpha \in \Lambda^k(\mathbb{R}^n)^*$. If $V \subset \mathbb{R}^n$ is subspace of dimension k, then the restriction

$$\alpha_{|V} \in \Lambda^k V^* \cong \mathbb{R}.$$

Given an orientation on V, the wedge product of the elements of an orthonormal basis of V^* with respect to the Euclidean metric defines a volume form

$$\mathrm{vol}_V = \eta^1 \wedge \cdots \eta^k \in \Lambda^k V^*.$$

vol_V spans $\Lambda^k V^*$, so $\alpha = \lambda \,\mathrm{vol}_V$ for some $\lambda \in \mathbb{R}$. Interpret

$$\alpha_{|V} \leq \mathrm{vol}_V \tag{1}$$

to mean that $\lambda \leq 1$. We say that α is a *calibration* if (1) holds for all oriented k-planes $V \subset \mathbb{R}^n$. Then V is called calibrated by α if equality holds in (1).

Manifold version. Let (M, g) be a Riemannian manifold. Say that $\alpha \in \Omega^k(M)$ is a *calibration* on M if

- $d\alpha = 0$, and
- $\alpha_{|V} \leq \mathrm{vol}_V$ for all oriented k-planes $V \subset T_pM$ and all $p \in M$.

A k-dimensional submanifold $N \subset M$ is *calibrated by* α if $T_pN \subset T_pM$ is a calibrated subspace for each $p \in N$.

The fundamental property of calibrated submanifolds is that they *minimise volume.*

Proposition 2. *Let $N \subset M$ be a closed submanifold calibrated by α, and $N' \subset M$ another closed submanifold in the same homology class,* i.e. $N - N' = \partial X$ *for some $k{+}1$-submanifold $X \subset M$. Then*

$$\mathrm{vol}(N) \leq \mathrm{vol}(N').$$

Proof.

$$\int_{N'} \mathrm{vol}_{N'} \geq \int_{N'} \alpha = \int_N \alpha = \int_N \mathrm{vol}_N,$$

where the middle step follows from the first property in the definition of a calibration as

$$\int_N \alpha - \int_{N'} \alpha = \int_X d\alpha = 0,$$

first and last step follow from the second property in the definition. □

Example 3. Let $z^k = x^k + iy^k$ be complex coordinates on $\mathbb{C}^n$. The Euclidean metric g_0 is the real part of the hermitian metric $dz^1 d\bar{z}^1 + \cdots dz^n d\bar{z}^n$. The imaginary part ω_0 is the Kähler form of the metric. g_0 is orthogonal with respect to the complex structure J_0 (the endomorphism of $\mathbb{C}^n$ given by multiplication by i), *i.e.* $g_0(Ju, Jv) = g_0(u, v)$, and the Kähler form can be recovered from g_0 by $\omega_0(u, v) = \omega(Ju, v)$. ω_0 is a real antisymmetric bilinear form. $\frac{\omega_0^k}{k!} \in \Lambda^{2k}_{\mathbb{R}}\mathbb{C}^n$ is a calibration, and the calibrated $2k$-planes are precisely the complex subspaces (of complex dimension k); this is a classical fact known as *Wirtinger's inequality.*

For any hermitian metric on a complex manifold (M, J) there is an associated $\omega \in \Omega^2(M)$. The metric is Kähler if J is parallel with respect to the Levi-Civita connection, or equivalently $d\omega = 0$. It follows that $\frac{\omega^k}{k!} \in \Omega^{2k}(M)$ is a calibration, and the calibrated submanifolds are precisely the complex submanifolds of M.

Some of the motivations for studying calibrated geometry are:

- To find examples of volume-minimising submanifolds.
- That it can be viewed as a generalisation of Kähler geometry.
- That regularity theory of volume-minimisers together with the topological constraint on the volume of a calibrated submanifold make compactifications of the moduli space of calibrated submanifolds plausible, and one can hope to use this to gain understanding of the ambient manifold (*cf.* Gromov-Witten theory of symplectic manifolds, Donaldson invariants in gauge theory of 4-manifolds).

For (M, g, α) to have a rich calibrated geometry, we need "enough" calibrated k-planes at each $p \in M$ that we can at least find some calibrated submanifolds in neighbourhoods of points. Note that for any $\alpha_0 \in \Lambda^k(\mathbb{R}^n)^*$, there is a μ such that $\nu\alpha_0$ is a calibration if and only $\nu \leq \mu$ (because the Grassmannian of k-planes in $\mathbb{R}^n$ is compact). If $\nu < \mu$, then there are no k-planes calibrated by $\nu\alpha_0$; when $\nu = \mu$, there might be just a single calibrated k-plane. So for any closed k-form α on (M, g), we can always multiply it by a constant to make it a calibration, but it might leave it with just a single calibrated plane at a single point. If we want to find a calibration on a given Riemannian manifold, with symmetries to force there to be a large number of calibrated planes, then it helps if the given Riemannian metric has special holonomy. (Another way to search for calibrations would be to start with a closed form, and then look for a Riemannian metric that makes it a calibration.)

Riemannian holonomy. Let (M^n, g) be a connected Riemannian manifold, and ∇ the Levi-Civita connection. For a (piecewise smooth) path $\gamma\colon [0,1] \to M$ from $p = \gamma(0)$ to $q = \gamma(1)$, let

$$P_\gamma\colon T_pM \to T_qM$$

be the linear map defined by parallel transport of ∇ along γ.

Definition 4. For $p \in M$, the *holonomy group* is the subgroup

$$\mathrm{Hol}_p(g) = \{P_\gamma : \gamma \text{ is a path from } p \text{ to itself}\} \subseteq GL(T_pM).$$

The *restricted holonomy group* is the subgroup

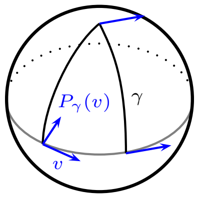
$$\mathrm{Hol}^0_p(g) = \{P_\gamma : \gamma \text{ is null-homotopic}\} \subseteq \mathrm{Hol}_p(g).$$

If M is simply-connected, then $\mathrm{Hol}^0_p(g) = \mathrm{Hol}_p(g)$. Since ∇ preserves g, P_γ is an isometry for each γ, and $\mathrm{Hol}_p(g) \subseteq O(T_pM)$. Since it is connected, $\mathrm{Hol}^0_p(g) \subseteq SO(T_pM)$.

Choosing a frame for T_pM identifies $\mathrm{Hol}_p(g)$ with a subgroup of $O(n)$. If γ is any path from p to q, then

$$\mathrm{Hol}_q(g) = P_\gamma \circ \mathrm{Hol}_p(g) \circ P_\gamma^{-1}.$$

Since M is connected, $\mathrm{Hol}(g) \subseteq O(n)$ is well-defined up to conjugation, independently of the choice of p (and of the frame). In other words, there is a representation of $\mathrm{Hol}(g)$ on $\mathbb{R}^n$, well-defined up to isomorphism. If the choice of metric g is implicit, we will call this simply the holonomy group $\mathrm{Hol}(M)$ of M.



Example 5. Let $M = S^2$ with the round metric. $P_\gamma \in O(2)$ is a rotation by an angle θ, which by the Gauss-Bonnet formula is equal to the area enclosed by γ. So $\mathrm{Hol}(S^2) = SO(2)$.

More generally, if γ bounds a region D in a Riemannian surface Σ, then P_γ is rotation by $\theta = \int_D \kappa \, dA$, where κ is the Gauss curvature. Therefore

$$\mathrm{Hol}^0(\Sigma) = \begin{cases} 1 & \text{if } \Sigma \text{ is flat} \\ SO(2) & \text{otherwise.} \end{cases}$$

This is a simple example of the close relationship between holonomy and curvature.

The holonomy group controls the existence of parallel tensor fields.

Proposition 6. *Let M be a connected Riemannian manifold, $E = TM^a \otimes T^*M^b$, $p \in M$, $s_p \in E_p$. There exists a parallel section s of E with $s(p) = s_p$ if and only if s_p is invariant under* $\mathrm{Hol}_p(M)$.

Proof. If s exists, then for any loop γ based at p the parallel transport of s_p is simply the restriction of s to γ, so $P_\gamma s_p = s_p$. Conversely, if s_p is invariant under $\mathrm{Hol}_p(M)$ then setting $s(q) = P_\gamma s_p$ for any path γ from p to q gives a well-defined parallel section s. □

Example 7.

- M^n oriented $\Leftrightarrow \exists$ parallel n-form $\Leftrightarrow \mathrm{Hol}(M) \subseteq SO(n)$
- M^{2m} Kähler manifold $\Leftrightarrow \exists$ parallel complex structure $\Leftrightarrow \mathrm{Hol}(M) \subseteq U(m)$

Now let $\alpha_0 \in \Lambda^k(\mathbb{R}^n)^*$ be a calibration on $\mathbb{R}^n$, with stabiliser $G \subseteq O(n)$. If $\mathrm{Hol}(M) \subseteq G$, then there is a calibration $\alpha \in \Omega^k(M)$ pointwise equivalent to α_0 ($\nabla\alpha = 0 \Rightarrow d\alpha = 0$). If G is not too small then G-invariance of the set of calibrated k-planes in $\mathbb{R}^n$ implies that there are enough calibrated planes at each $p \in M$ for a rich geometry.

Lecture two

Berger's list. The possible holonomy groups of Riemannian manifolds were classified in 1955 by Berger [1]. Each case corresponds to a certain kind of special geometry.

Theorem 8. *Let (M, g) be a simply-connected, complete, irreducible, non-symmetric Riemannian manifold. Then $(\dim M, \mathrm{Hol}(M))$ is one of the following:*

- $(m, SO(m))$ — *Generic*
- $(2m, U(m))$ — *Kähler*
- $(2m, SU(m))$ — *Calabi-Yau (Ricci-flat Kähler)*
- $(4m, Sp(m))$ — *Hyper-Kähler (Ricci-flat, several different complex structures)*
- $(4m, Sp(m)Sp(1))$ — *Quaternionic Kähler (Einstein, not Kähler)*
- $(7, G_2)$ — *Exceptional (Ricci-flat)*
- $(8, Spin(7)$ — *Exceptional (Ricci-flat)*

On any manifold M^n, a generic metric has holonomy $SO(n)$ or $O(n)$ (depending on orientability). The other groups on the list are the *special holonomy groups*. They come in 4 infinite families, plus two exceptional cases. G_2 is the automorphism group of the octonions $\mathbb{O}$, an 8-dimensional normed algebra, and has a natural 7-dimensional representation on the imaginary part of $\mathbb{O}$ (more on this later). $Spin(7)$ is the double cover of $SO(7)$, and has a unique real 8-dimensional spin representation.

Explanation of hypotheses. The hypotheses in Berger's theorem exclude various trivial possibilities from the list.

- If $\pi_1(M) \neq 1$, then the universal cover $\tilde{M}$ has $\mathrm{Hol}(\tilde{M}) = \mathrm{Hol}^0(M) = id$ component of $\mathrm{Hol}(M)$. So $\mathrm{Hol}(M)$ is just a finite extension of a group on the list.

- If M is a product $M_1 \times M_2$, then $\mathrm{Hol}(M) = \mathrm{Hol}(M_1) \times \mathrm{Hol}(M_2)$. Conversely, it is a theorem of de Rham that if M is complete and simply-connected and the representation of $\mathrm{Hol}(M)$ on $\mathbb{R}^n$ is reducible, then M is a product manifold.
- M is symmetric if for each $p \in M$, there is an isometric involution $\sigma_p \colon M \to M$ with p as an isolated fixed point. Then the group G generated by products $\sigma_p \sigma_q$ acts transitively on M, so $M \cong G/H$, where $H \subset G$ is the stabiliser of a point. Symmetric spaces were classified by Cartan. If M is a symmetric space G/H, then $\mathrm{Hol}(M) \cong H$ (with adjoint representation on $\mathfrak{g}/\mathfrak{h}$).

Existence. All the groups on the list do appear as holonomy groups of complete Riemannian manifolds, but this was not proved by Berger, he only excluded the groups not on the list (except $(16, Spin(9))$, which Alekseevski later showed can occur as the holonomy only of spaces locally isometric to the octonionic projective plane, which is symmetric). For the exceptional cases, even existence of local metrics was not proved until 1985 by Bryant [3]. Compact examples are yet harder to find. Significantly, Yau proved a general existence result for compact holonomy $SU(n)$ metrics in 1977 [12] (see theorem 9). The first compact manifolds with exceptional holonomy were constructed by Joyce in 1995 [6, 7].

Interesting calibrated submanifolds.
- Special Lagrangian m-folds in Calabi-Yau manifolds of (complex) dimension m.
- Associative 3-folds and coassociative 4-folds in G_2-manifolds.
- Cayley 4-folds in $Spin(7)$-manifolds.

We will discuss the calibrations on Calabi-Yau and G_2-manifolds in more detail.

Idea of Berger's proof. As explained, the hypotheses that M is simply-connected, complete and irreducible imply that G is connected and acts irreducibly on $\mathbb{R}^n$.

If $\mathrm{Hol}(M) \subseteq G$ then the Riemannian curvature tensor R takes values in $S^2\mathfrak{g} \subseteq S^2\mathfrak{so}(n) \cong S^2(\Lambda^2 T^*M)$. In fact, R must then lie in a further subspace $\mathcal{R}_G \subseteq S^2\mathfrak{g}$ of tensors satisfying the Bianchi identity. In some cases $\mathcal{R}_G$ is trivial; then any metric with holonomy contained in G is actually flat, so has trivial holonomy, and G therefore cannot be a holonomy group. More generally, $\mathfrak{g}$ is in a certain sense generated by the values of R, so to exclude G as a possible holonomy group it is enough that $\mathcal{R}_G$ is small.

For groups that do pass this test, one can then consider ∇R. For all groups except those on the list, the second Bianchi identity now forces $\nabla R = 0$, which implies that M is locally isometric to a symmetric space.

Calabi-Yau manifolds. Let M be a Riemannian manifold of dimension $2n$. $\mathrm{Hol}(M) \subseteq SU(n)$ implies:

M is Kähler, since $SU(n) \subset U(n)$.

The complex determinant $\Omega_0 = dz^1 \wedge \cdots \wedge dz^n$ on $\mathbb{C}^n$ is invariant under $SU(n)$. Therefore M has a parallel complex n-form Ω.

$\nabla\Omega = 0$ implies that Ω is non-vanishing, and that Ω is holomorphic (because M is Kähler, the Chern connection coincides with the Levi-Civita connection); we say that "Ω is a holomorphic volume form".

In particular, the canonical bundle $K_M = \Lambda^n T^*_{\mathbb{C}} M$ is holomorphically trivial. By definition, the fact that it is trivial as a complex line bundle implies that the first Chern class $c_1(M) \in H^2(M; \mathbb{Z})$ vanishes.

Also, the connection on K_M is flat. There is a linear relation between the Ricci curvature of a Kähler metric and the curvature of the induced connection on K_M; therefore M is Ricci-flat.

Conversely, if M is Ricci-flat Kähler, then K_M flat implies $\mathrm{Hol}^0(M) \subseteq SU(n)$, and that $c_1(M) = 0 \in H^2(M; \mathbb{R})$. If M is simply-connected then $\mathrm{Hol}(M) \subseteq SU(n)$ and K_M is trivial, but otherwise this does not have to be the case.

Yau's proof of the Calabi conjecture makes it possible to construct millions of manifolds with holonomy $SU(n)$ using algebraic geometry.

Theorem 9. *Let M be a compact Kähler manifold with $c_1(M) = 0 \in H^2(M;\mathbb{R})$. Then every Kähler class on M (*i.e.* a class in $H^2(M;\mathbb{R})$ that can be represented by the Kähler form of some Kähler metric) contains a unique Ricci-flat Kähler metric.*

Example 10. Let M be a smooth quintic 3-fold (so real dimension 6) in $\mathbb{C}P^4$, *e.g.*

$$\{X_0^5 + X_1^5 + X_2^5 + X_3^5 + X_4^5 = 0\} \subset \mathbb{C}P^4.$$

By the adjunction formula, M has trivial canonical bundle.

The special Lagrangian calibration.

Linear version. Let $z^k = x^k + iy^k$ be complex coordinates on $\mathbb{C}^n$. $SU(n)$ preserves

- the Kähler form $\omega_0 = dx^1 \wedge dy^1 + \cdots + dx^n \wedge dy^n \in \Lambda^2_{\mathbb{R}}\mathbb{C}^n$, and
- the complex determinant $\Omega_0 = dz^1 \wedge \cdots dz^n \in \Lambda^n_{\mathbb{C}}\mathbb{C}^n$.

Definition 11. The *special Lagrangian calibration* on $\mathbb{C}^n$ is $\operatorname{Re}\Omega_0 \in \Lambda^n_{\mathbb{R}}\mathbb{C}^n$.

By definition, $\omega_0^n \neq 0$ means that ω_0 is a symplectic form. Let $V \subset \mathbb{C}^n$ be a subspace of real dimension n. V is Lagrangian if $\omega_{0|V} = 0$. If V is oriented, then the metric defines a volume form $\mathrm{vol}_V \in \Lambda^n V^*$, and $\Omega_{0|V} = z\,\mathrm{vol}_V$ for some $z \in \mathbb{C}$.

Fact: $|z| \leq 1$, with equality if and only if V is Lagrangian. Then we can write $z = e^{i\theta}$, and θ is called the *Lagrangian angle* of V (defined modulo $2\pi\mathbb{Z}$; reversing the orientation of V shifts θ by π). So

$$V \text{ SLag} \Leftrightarrow V \text{ Lagrangian and } \theta = 0 \Leftrightarrow \omega_{0|V} = \operatorname{Im}\Omega_{0|V} = 0 \text{ (and } V \text{ correctly oriented).} \quad (12)$$

An alternative interpretation of the SLag condition is to consider $\mathbb{R}^n = \{y^1 = \cdots = y^n = 0\} \subset \mathbb{C}^n$. It is easy to check that this is SLag. The Lagrangian n-planes in $\mathbb{C}^n$ are precisely the $U(n)$-orbit of $\mathbb{R}^n$, while the SLag n-planes are the $SU(n)$-orbit.

Lecture three

Special Lagrangian submanifolds of Calabi-Yau manifolds. Let M^{2n} be a Riemannian manifold with $\mathrm{Hol}(M) \subseteq SU(n)$. Then M is a Kähler manifold. Let ω be the Kähler form, and choose a parallel complex n-form Ω. We normalise it so that $(T_pM, \omega, \Omega) \cong (\mathbb{C}^n, \omega_0, \Omega_0)$ for each $p \in M$ (this fixes Ω up to phase, *i.e.* multiplication by $e^{i\theta}$).

Definition 13. The SLag calibration on M is $\operatorname{Re}\Omega \in \Omega^n(M)$.

$\omega^n \neq 0$ and $d\omega = 0$ means that ω is a symplectic form. Let $L \subset M$ be a submanifold of dimension n. Then from (12),

$$L \text{ SLag (for some choice of orientation)} \Leftrightarrow \omega_{|L} = \operatorname{Im}\Omega_{|L} = 0. \quad (14)$$

If L is Lagrangian (*i.e.* $\omega_{|L} = 0$) then there is a Lagrangian angle function $\theta\colon L \to \mathbb{R}/2\pi\mathbb{Z}$. $d\theta \in \Omega^1(L)$ is well-defined and, up to choice of phase of Ω,

$$L \text{ Slag} \Leftrightarrow d\theta = 0.$$

We know that calibrated implies volume-minimising. In particular, Slag implies minimal, *i.e.* mean curvature $H = 0$.

Fact: If L is Lagrangian (in a Calabi-Yau manifold), then $H = J\nabla\theta$ (where $J \in End(TM)$ is the complex structure on TM). Thus (modulo phase)

$$\text{SLag} \Leftrightarrow \text{minimal Lagrangian.}$$

Lagrangian mean curvature flow. If a submanifold N of a Riemannian manifold M is not minimal, its volume is decreased fastest by pushing it along its mean curvature vector field H. So one can attempt attempt to find minimal submanifolds as limits of "mean curvature flow". The problem is that singularities can form during the flow. This is an area of current research.

MCF on a symplectic manifold preserves the Lagrangian condition. Therefore, if MCF of a Lagrangian L in a Calabi-Yau manifold converges then the limit is SLag. The Lagrangian angle evolves by $\frac{d\theta}{dt} = \Delta\theta$.

Deformations of SLag submanifolds. Let L be a (smooth) closed SLag submanifold of a Calabi-Yau manifold M. The moduli space of L is the space $\mathcal{M}_L$ of (smooth) deformations of L that are SLag.

Theorem 15 (McLean [10]). *$\mathcal{M}_L$ is a smooth manifold of dimension $b^1(L)$.*

For any Lagrangian submanifold L in a symplectic manifold, the Lagrangian neighbourhood theorem implies that Lagrangian deformations correspond to closed 1-forms on L. In order to prove theorem 15 we show that the SLag deformations essentially correspond to the subspace of 1-forms that are also coclosed, *i.e.* the harmonic ones.

Proof. It suffices to find a chart for a neighbourhood of L in $\mathcal{M}_L$.

$$\{\text{small deformations of } L\} \leftrightarrow \{\text{normal vector fields to } L \text{ with small norm}\}$$

via the exponential map (L itself corresponds to the zero normal field). For any Lagrangian L,

$$N_L \to T^*L$$
$$u \mapsto u \lrcorner \omega$$

is an isomorphism (where the contraction is defined by $(u \lrcorner \omega)(v) = \omega(u, v)$). Let $\alpha \mapsto u_\alpha$ be the inverse. So for any $\alpha \in \Omega^1(L)$ there is a corresponding deformation L_α, which is the image of an embedding $f_\alpha = \exp u_\alpha \colon L \to M$. By (14), the failure of L_α to be Slag is measured by

$$F(\alpha) = (f_\alpha^*\omega, f_\alpha^* \operatorname{Im} \Omega).$$

So the problem is to identify the zeros of

$$F : \Omega^1(L) \to \Omega^2(L) \times \Omega^n(L)$$

in a neighbourhood of $0 \in \Omega^1(L)$ (F is only defined near 0). Because f_α is homotopic to the inclusion $L \hookrightarrow M$, F actually takes values in the subspace of exact forms, *i.e.*

$$F : \Omega^1(L) \to d\Omega^1(L) \times d\Omega^{n-1}(L).$$

To apply the Implicit Function Theorem, we need the derivative at the origin

$$D_0F : \Omega^1(L) \to d\Omega^1(L) \times d\Omega^{n-1}(L)$$

to be surjective. (D_0F is the "deformation operator", and its cokernel is the "obstruction space"; we want to show that the latter is trivial, so that the deformation problem is unobstructed.) Differentiating exp gives a Lie derivative, so

$$D_0F(\beta) = (\mathcal{L}_{u_\beta}\omega_{|L},\ \mathcal{L}_{u_\beta} \operatorname{Im} \Omega_{|L})$$

Using Cartan's formula for the Lie derivative of a form gives

$$\mathcal{L}_{u_\beta}\omega = u_\beta \lrcorner d\omega + d(u_\beta \lrcorner \omega) = d\beta,$$

since $d\omega = 0$ and $u_\beta \lrcorner \omega = \beta$ by definition. Similarly

$$\mathcal{L}_{u_\beta} \operatorname{Im} \Omega = u_\beta \lrcorner (d \operatorname{Im} \Omega) + d(u_\beta \lrcorner \operatorname{Im} \Omega) = d{*_L}\beta,$$

where $*_L \colon \omega^1(L) \to \Omega^{n-1}(L)$ is the Hodge star map on L; the definition of the Hodge star involves the volume form, and we use that $\mathrm{vol}_L = \operatorname{Re} \Omega_{|L}$ by hypothesis.

By Hodge theory,

$$\Omega^1(L) = \mathcal{H}^1(L) \oplus d\Omega^0(L) \oplus d^*\Omega^2(L),$$

and the formula for D_0F shows that it vanishes on the first term, maps the second isomorphically to $d\Omega^{n-1}(L)$, and the third isomorphically to $d\Omega^1(L)$. By IFT, a neighbourhood of 0 in $F^{-1}(0)$ is a graph over $\mathcal{H}^1(L)$, which has dimension $b^1(L)$.

Technical issue: to apply IFT, we would really require that the vector spaces $\Omega^1(L)$ etc are complete, which they are not. To get around this problem, complete them with respect to suitable Sobolev or Hölder norms. IFT now shows that $F^{-1}(0)$ in the completion of $\Omega^1(L)$ is a smooth manifold. The elements of this manifold are a priori just Sobolev/Hölder regular forms, but elliptic regularity shows that they are actually smooth. □

Saying that $\mathcal{M}_L$ is a manifold gives no information about its global properties, whereas if one knew that it were compact (possibly with some boundary) one would at least have some control. The reason one should not expect $\mathcal{M}_L$ to be compact is that one could deform L continuously to a limit that is still SLag, but with singularities (a basic analogy would be how $\{xy = t\} \subset \mathbb{R}^2$ is smooth for $t \neq 0$, but becomes singular at $t = 0$). It is an important problem to understand how one could define a compactification $\overline{\mathcal{M}_L}$ by adding boundary points to $\mathcal{M}_L$ corresponding to singular SLag submanifolds.

Special cases.

- $b^1(L) = 0$
  Then $\dim \mathcal{M}_L = 0$, so it is a discrete set of points, finite if it is compact. For any small deformation of the Calabi-Yau structure (ω, Ω) on M, there is a unique deformation of L that keeps it SLag, provided the obvious necessary topological condition that $[\omega_{|L}] \in H^2(L; \mathbb{R})$ and $[\operatorname{Im} \Omega_{|L}] \in H^n(L; \mathbb{R})$ vanish; this is proved by an argument similar to theorem 15. Nevertheless, the number of elements of $\mathcal{M}_L$ need not stay constant, because of the possibility of singular degenerations.
- $L \cong T^n$
  Then $\dim \mathcal{M}_L = n$. If L is close to flat, then harmonic 1-forms on L have no zeros, so the small deformations of L do not intersect. Thus M is locally fibred by deformations of L. Global fibrations of Calabi-Yau manifolds by SLag tori would be of relevance for mirror symmetry (SYZ conjecture), but to construct them one would need to be able to handle singular fibres.

Lecture four

G_2-manifolds.

The group G_2. The *octonions* $\mathbb{O}$ are the unique real normed division algebra of dimension 8. The multiplication has no zero divisors, but is not commutative or even associative. $\|a\|^2 = a\bar{a}$ for a conjugation map $^-\colon \mathbb{O} \to \mathbb{O}$, splitting $\mathbb{O} = \mathbb{R} \oplus \operatorname{Im} \mathbb{O}$.

Any $a \in \mathbb{O} \setminus \mathbb{R}$ generates a subalgebra $\cong \mathbb{C}$. Any $a, b \in \mathbb{O}$ that are linearly independent (also from 1) generate a subalgebra $\cong \mathbb{H}$ (the quaternions).

Setting $a \times b = \operatorname{Im}(ab)$ defines a *cross product* on $\operatorname{Im} \mathbb{O}$: it is anti-symmetric and satisfies the relation $\|a \times b\|^2 + \|a \wedge b\|^2 = \|a\|^2 \|b\|^2$.

$G_2 = Aut(\mathbb{O})$ is an exceptional case in the classification of simple Lie groups. G_2 is 2-connected and has dimension 14. The natural action of G_2 on $\operatorname{Im} \mathbb{O}$ identifies it with a subgroup of $SO(7)$. It preserves the cross product, and hence also the "multiplication table"

$$\varphi_0(a, b, c) = \langle a \times b, c \rangle.$$

φ_0 is anti-symmetric in all three arguments, and can be written in coordinates as

$$\varphi_0 = dx^{123} + dx^{145} + dx^{167} + dx^{246} - dx^{257} - dx^{356} - dx^{347} \in \Lambda^3(\mathbb{R}^7)^*.$$

Fact: G_2 is *exactly* the stabiliser of φ_0 in $GL(7, \mathbb{R})$.

$$\dim GL(7, \mathbb{R})\varphi_0 = 49 - 14 = 35 = \dim \Lambda^3(\mathbb{R}^7)^*,$$

so the orbit of φ_0 is open in $\Lambda^3(\mathbb{R}^7)^*$ ("φ_0 is stable"). Call elements of this orbit *positive 3-forms.*

G_2-structures. Let M^7 be a smooth manifold. Call $\varphi \in \Omega^3(M)$ a *G_2-structure* if it is positive at each $p \in M$, in other words $(T_pM, \varphi) \cong (\mathbb{R}^7, \varphi_0)$. Because φ_0 is stable, any small perturbation of the 3-form φ is also a G_2-structure.

Because $G_2 \subset SO(7)$, a G_2-structure induces a Riemannian metric. Say that φ is *torsion-free* if $\nabla \varphi = 0$; here ∇ is the Levi-Civita connection of the metric induced by φ, so the condition is non-linear. By proposition 6

$$Hol(M, g) \subseteq G_2 \Leftrightarrow g \text{ induced by some torsion-free } G_2\text{-structure} \varphi.$$

Call (M, φ) where φ is torsion-free a *G_2-manifold.*

$SU(3) \subset G_2$ (it is the stabiliser of a non-zero vector). Therefore, if X^6 is a Calabi-Yau manifold (of complex dimension 3), then $X \times S^1$ is a (reducible) G_2-manifold. Its G_2-structure can be written as

$$\varphi = \operatorname{Re}\Omega + d\theta \wedge \omega. \tag{16}$$

But we are most interested in irreducible G_2-manifolds, *i.e.* ones that have full holonomy G_2. In the compact case, there is a simple topological criterion for irreducibility.

Proposition 17. *For M a closed G_2-manifold,*

$$\operatorname{Hol}(M) = G_2 \Leftrightarrow \pi_1(M) \text{ is finite.}$$

Constructions. $\nabla\varphi = 0$ is equivalent to $d\varphi = d^*\varphi = 0$ (again, ∇ and d^* involve the metric induced by φ). To construct G_2-manifolds, solve this non-linear PDE.

- Bryant [3]: first local examples using exterior differential systems (1985)
- Bryant-Salamon [4]: first complete examples (1989)

These complete examples have cohomogeneity 1, *i.e.* there is an isometry group with orbits of codimension 1. This can be used to reduce the PDE for G_2 holonomy to an ODE, and numerous complete G_2 metrics with large isometry groups have been found since.

Compact (irreducible) G_2-manifolds cannot have continuous isometries; on a Ricci-flat manifold any Killing vector field is parallel, so if one exists then the holonomy cannot be irreducible. Instead, the known constructions of compact examples solve the PDE by gluing together reducible pieces to a smooth closed simply-connected manifold, and deduce that it is irreducible from proposition 17.

- Joyce [7]: First compact examples (1995). Resolve a flat orbifold T^7/Γ to a simply-connected smooth closed M^7, in simplest case by replacing singularities of the form $T^3 \times \mathbb{C}^2/\pm 1$ by $T^3 \times Y$. Here Y is the Eguchi-Hansen space, a Ricci-flat Kähler metric on $T^*\mathbb{C}P^2$, asymptotic to $\mathbb{C}^2/\pm 1$. As $\operatorname{Hol}(Y) = SU(2)$, $Y^3 \times SU(2)$ has a torsion-free G_2-structure. Define a G_2-structure φ on M by interpolating between this and the flat G_2-structure on T^7/Γ. With some hard work, one can make sure that the torsion $\nabla\varphi$ is small in a quantifiable sense, and prove that φ can be perturbed to a torsion-free G_2-structure.
- Kovalev [9]: More compact examples (2000). First construct Calabi-Yau 3-folds $V_\pm$ with cylindrical end of the form $\mathbb{R} \times S^1 \times K3$. Then form a closed smooth M^7 from $S^1 \times V_+$ and $S^1 \times V_-$, by truncating the cylindrical ends and gluing the resulting $S^1 \times S^1 \times K3$ boundaries. One takes the gluing map to swap the two circle factors in order to ensure that $\pi_1(M) = 1$; we call M a "twisted connected sum". Finding a gluing map that matches up the torsion-free G_2-structures on $S^1 \times V_+$ and $S^1 \times V_-$ takes some work, but once that is done it is comparatively straight-forward to glue to define a G_2-structure with small torsion and perturb to a torsion-free one.

Notice the difference between these constructions and the Calabi-Yau theorem 9: for complex manifolds there is an obvious necessary *topological* condition for admitting a holonomy $SU(n)$ metric that can also be proved to be sufficient, while these constructions rely on first finding a metric that is "almost" a G_2 metric (in the sense that it is induced by a G_2-structure with small torsion). While a few necessary topological conditions are known for a closed manifold to admit a holonomy G_2 metric, there is currently no clear conjecture for what a necessary and sufficient condition might be.

Calibrations on G_2-manifolds. $\varphi_0 \in \Lambda^3(\mathbb{R}^7)^*$ and $*\varphi_0 \in \Lambda^4(\mathbb{R}^7)^*$ are calibrations on $\mathbb{R}^7$. The calibrated planes are called *associative* 3-planes and *coassociative* 4-planes.

$$V^3 \subset \mathbb{R}^7 \text{ associative} \Leftrightarrow V \lrcorner (*\varphi_0) = 0 \Leftrightarrow \mathbb{R} \oplus V \subset \mathbb{O} \text{ is a subalgebra} \cong \mathbb{H}$$

$$V^4 \subset \mathbb{R}^7 \text{ coassociative} \Leftrightarrow \varphi_{0|V} = 0$$

If (M, φ) is a G_2-manifold, then φ and $*\varphi$ are calibrations on M. A smooth coassociative submanifold $C^4 \subset M$ has unobstructed deformations, similar to SLag submanifolds. Since C has dimension 4, the degree 2 cohomology can be split into self-dual and anti-self-dual parts, whose ranks are denoted by $b^2_\pm(C)$.

Theorem 18 (McLean [10])**.** *The moduli space $\mathcal{M}_C$ is a smooth manifold of dimension $b^2_+(C)$.*

T^4 and K3 have $b^2_+ = 3$, so could be fibres in coassociative fibrations. (16) implies that if $C \subset V$ is a complex surface in a Calabi-Yau 3-fold V, then $\{\theta\} \times C \subset S^1 \times V$ is coassociative. It therefore seems feasible to construct twisted connected sum G_2-manifolds that are fibred by coassociative K3s, because often the asymptotically cylindrical Calabi-Yau 3-folds $V_\pm$ in the construction have holomorphic K3 fibrations where the singular fibres are well-behaved (only ordinary double point singularities).

The deformation theory of an associative $A^3 \subset M$ is more complicated. McLean shows that the deformation operator for this problem is a twisted Dirac operator $\not{D}_A$ on A. $\not{D}_A$ is self-adjoint, so the infinitesimal deformation space $\ker \not{D}_A$ is isomorphic to the obstruction space $\operatorname{coker} \not{D}_A$, but neither is controlled by the topology of A (unlike SLag and coassociative cases, where the deformation operator can be written in terms of exterior derivatives). So one can only use the Implicit Function Theorem to prove smoothness of the space of deformations of A when A is in fact rigid.

The expected dimension of the moduli space is 0, so one could try to count its elements. If A is rigid, then it persists under sufficiently small deformations of the G_2-structure on M. But for larger deformations, singularities may develop, or A may become non-rigid and then disappear, so one should not expect a crude count of associatives to stay constant under deformation.

In recent work with Corti, Haskins and Pacini, we show that there are many examples of twisted connected sum G_2-manifolds containing rigid associatives diffeomorphic to $S^1 \times S^2$. The idea is to first find asymptotically cylindrical Calabi-Yau 3-folds V_+ containing rigid complex curves $\mathbb{C}P^1 \cong S^2$. Then (16) shows that $S^1 \times S^2$ is associative in $S^1 \times V_+$, and one can check that the rigidity of the complex curve translates into rigidity of the associative. Therefore it persists under the perturbation of the G_2-structure involved in the gluing construction.

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