The induction step in the proof of the Künneth formula

Let $V_1, V_2 \subset M$ be open subsets such that $M = V_1 \cup V_2$. Suppose that

$$\psi_{V_1} : \bigoplus_{i+j=k} H^i(V_1) \otimes H^j(N) \to H^k(V_1 \times N)$$

$$\psi_{V_2} : \bigoplus_{i+j=k} H^i(V_2) \otimes H^j(N) \to H^k(V_2 \times N)$$

$$\psi_{V_1 \cap V_2} : \bigoplus_{i+j=k} H^i(V_1 \cap V_2) \otimes H^j(N) \to H^k((V_1 \cap V_2) \times N)$$

are isomorphisms for all k. Then

$$\psi_M : \bigoplus_{i+j=k} H^i(M) \otimes H^j(N) \to H^k(M \times N)$$

is also an isomorphism.

Proof. (Based on Bott and Tu, pp. 49) Consider the diagram

$$\begin{array}{ccccccc}
\cdots \displaystyle\bigoplus_{i+j=k} (H^i(V_1) \oplus H^i(V_2)) \otimes H^j(N) & \longrightarrow & \displaystyle\bigoplus_{i+j=k} H^i(V_1 \cap V_2) \otimes H^j(N) & \xrightarrow{\delta} & \displaystyle\bigoplus_{i+j=k} H^{i+1}(M) \otimes H^j(N) \cdots \\
\Big\downarrow \psi_{V_1} + \psi_{V_2} & & \Big\downarrow \psi_{V_1 \cap V_2} & & \Big\downarrow \psi_M \\
\cdots \quad H^k(V_1 \times N) \oplus H^k(V_2 \times N) & \longrightarrow & H^k((V_1 \cap V_2) \times N) & \xrightarrow{\delta} & H^{k+1}(M \times N) \quad \cdots
\end{array}$$

Here the top row is obtained from the Mayer-Vietoris sequence for $M = V_1 \cup V_2$ by first taking the tensor product with $H^j(N)$ for $j = 0, 1, \ldots$ to get exact sequences

$$\cdots \to (H^i(V_1) \oplus H^i(V_2)) \otimes H^j(N) \to H^i(V_1 \cap V_2) \otimes H^j(N) \xrightarrow{\delta} H^{i+1}(M) \otimes H^j(N) \to \cdots,$$

and then adding them all up, with the index i shifted by j so that $i + j$ is the same for all the terms in each entry in the sequence. (Each entry has finite number of non-trivial terms; with the dummy variables as written, the sum in the top right corner includes a possibly non-trivial contribution from $i = -1, j = k$.) The bottom row is just the Mayer-Vietoris sequence for $M \times N = (V_1 \times N) \cup (V_2 \times N)$.

The diagram is commutative. The only square for which the commutativity is not straightforward is the one on the right, *i.e.* that $\psi_M \circ \delta = \delta \circ \psi_{V_1 \cap V_2}$. By definition of the maps ψ, this means that for any $[\alpha] \in H^i(V_1 \cap V_2)$, $[\beta] \in H^j(N)$,

$$p^*(\delta[\alpha]) \wedge q^*[\beta] = \delta(p^*[\alpha] \wedge q^*[\beta]) \in H^{i+j+1}(M \times N),$$

where $p : M \times N \to M$ and $q : M \times N \to N$ denote the projection maps (and their restriction to $(V_1 \cap V_2) \times N$). Let $\{\rho_1, \rho_2\}$ be a partition of unity on M relative to $\{V_1, V_2\}$, *i.e.* $\operatorname{spt} \rho_r \subset V_r$ and $\rho_1 + \rho_2 \equiv 1$. From the details of the proof of the Mayer-Vietoris theorem and the Snake lemma, $\delta[\alpha] \in H^{i+1}(M)$ can be represented by an $(i+1)$-form on M whose restriction to V_{3-r} is $(-1)^r d(\rho_r \alpha)$. Since $d\rho_1 = -d\rho_2$ and has support in $V_1 \cap V_2$, this means that $\delta[\alpha] = [d\rho_1 \wedge \alpha]$. So $p^*(\delta[\alpha]) \wedge q^*[\beta]$ is represented by

$$p^*(d\rho_1 \wedge \alpha) \wedge q^*\beta = d(\rho_r \circ p) \wedge (p^*\alpha \wedge q^*\beta).$$

As $\{\rho_1 \circ p, \rho_2 \circ p\}$ is a partition of unity on $M \times N$ relative to $\{V_1 \times N, V_2 \times N\}$, the right hand side represents $\delta(p^*[\alpha] \wedge q^*[\beta])$.

Because the diagram commutes and the rows are exact, and $\psi_{V_1} + \psi_{V_2}$ and $\psi_{V_1 \cap V_2}$ are isomorphisms by hypothesis, the Five lemma implies that ψ_M is also an isomorphism. $\square$

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