Imperial College London

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2013

This paper is also taken for the relevant examination for the Associateship.

M4P54/M5P54

Differential Topology

Date: 29th May 2013 Time: 10am – 12am

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

Throughout, 'manifold' means smooth manifold, 'closed' means compact without boundary, and $H^*(-)$ denotes de Rham cohomology.

1. Let M be an n-dimensional oriented manifold without boundary.

   (i) Define the Poincaré pairing $H^k(M) \times H_c^{n-k}(M) \to \mathbb{R}$, and state the Poincaré duality theorem for de Rham cohomology. Explain what is meant by a good cover of M, and prove the Poincaré duality theorem for M when M has a finite good cover. You may assume appropriate forms of the Poincaré lemma. You may also assume the Five lemma, and you do not need to prove commutativity of the diagram that you want to apply it to.

   (ii) Define the Euler characteristic $\chi(M)$. If M is closed show that

   (a) if n is odd then $\chi(M) = 0$.

   (b) if $n = 4k + 2$ then $\chi(M)$ is even.

   (c) if $n = 4k$ and $\chi(M)$ is odd then there is no orientation-reversing diffeomorphism $f : M \to M$.

   (You may state without proof any results you require about non-degenerate bilinear forms on vector spaces.)

2. (i) Let M be a manifold, and $V_1, V_2 \subset M$ open subsets such that $M = V_1 \cup V_2$. Let $i_r : V_r \hookrightarrow M$, $j_r : V_1 \cap V_2 \hookrightarrow V_r$ $(r = 1, 2)$ denote the inclusion maps. If $V_1 \cap V_2$ is connected, show that M is orientable if and only if both V_1 and V_2 are. State the Mayer-Vietoris theorem for de Rham cohomology.

   (ii) Let M be a closed connected orientable manifold of dimension n. Let $p \in M$, and $\dot{M} = M \backslash \{p\}$. Show that $H^k(M) \cong H^k(\dot{M})$ for $0 < k < n$. (*Hint:* Use Poincaré duality to show that $H^n(\dot{M}) = 0$.)

   (iii) Let M_1 and M_2 be closed connected orientable manifolds of equal dimension n. Pick points $p_r \in M_r$ and coordinate charts $f_r : U_r \to \mathbb{R}^n$ such that $f_r(p_r) = 0$. Let $\dot{M}_r = M_r \backslash \{p_r\}$, and let $M_1 \# M_2$ be the quotient of the disjoint union $\dot{M}_1 \sqcup \dot{M}_2$ by the equivalence relation $x_1 \sim x_2$ if $x_i \in U_i$ and $f_1(x_1) = \frac{f_2(x_2)}{|f_2(x_2)|^2}$. Show that

   $$b_k(M_1 \# M_2) = b_k(M_1) + b_k(M_2)$$

   for $0 < k < n$.

3. (i) Let X^n and Y^n be closed oriented manifolds of equal dimension n with Y connected, and $f : X \to Y$ a smooth map. Define the degree of f in terms of integration, and prove that it equals the number of pre-images of any regular value x of f counted with signs. You should explain how to define the signs, but may assume that x has a neighbourhood $U \subset Y$ such that the restriction of f to each component of $f^{-1}(U)$ is a diffeomorphism.

(ii) Let S^1 denote the unit circle $\{x_1^2 + x_2^2 = 1\}$, and consider the following functions $S^1 \to \mathbb{R}^3$.

* $f^{(1)} : (x_1, x_2) \mapsto (x_1, x_2, 0)$
* $f^{(2)} : (x_1, x_2) \mapsto (x_1, -x_2, 0)$
* $f^{(3)} : (x_1, x_2) \mapsto (x_1 + 2, x_2, 0)$
* $g \quad : (x_1, x_2) \mapsto (x_1 - 1, 0, x_2)$

For a pair of distinct points $p, q \in \mathbb{R}^3$, let $\pi(p, q) = \frac{p-q}{|p-q|} \in S^2$. For $i = 1, 2, 3$ we define the smooth map $R^{(i)} : S^1 \times S^1 \to S^2, (r, s) \mapsto \pi(f^{(i)}(r), g(s))$. Show that

(a) $\deg R^{(1)} = 1$.

(b) $\deg R^{(2)} = -1$.

(c) $\deg R^{(3)} = 0$.

You do not need to explain the choice of orientations on $S^1 \times S^1$ and S^2 in detail, but should use the same choice in (a), (b) and (c). (*Hint:* Consider preimages of $(0, 0, 1)$.)

(iii) Do there exist homotopies $F_t, G_t : S^1 \times S^1 \times [0, 1] \to S^2$ such that the images under F_t and G_t of $S^1 \times S^1$ are disjoint for each $t \in [0, 1]$ and

(a) $F_0 = f^{(1)}$, $F_1 = f^{(2)}$ and $G_0 = G_1 = g$?

(b) $F_0 = f^{(1)}$, $F_1 = f^{(3)}$ and $G_0 = G_1 = g$?

4. (i) Let M and N be closed manifolds, and let $p : M \times N \to M$ and $q : M \times N \to N$ denote the projection maps. Show that the bilinear map $H^i(M) \times H^j(N) \to H^{i+j}(M \times N)$, $([\alpha], [\beta]) \mapsto [p^*\alpha \wedge q^*\beta]$ is well-defined. State the Künneth theorem for the de Rham cohomology of $M \times N$.

(ii) Let $M = N = \mathbb{Z}$, considered as a dimension 0 manifold. Show that the Künneth formula does not hold for the de Rham cohomology of $M \times N$.

(iii) State the Betti numbers of $\mathbb{C}P^n$, and describe the algebra structure on $H^*(\mathbb{C}P^n)$. Suppose M and N are manifolds of dimension r and s respectively, and that $M \times N$ is diffeomorphic to $\mathbb{C}P^n$ (so $r + s = 2n$). Show that M and N are both connected and orientable, and prove that in fact one of them is a point. (*Hint:* What are $b_2(M)$ and $b_2(N)$?)