Differential Topology
Example Sheet 5

1. Let $j : \mathbb{R}P^{n-1} \to \mathbb{R}P^n$ be the inclusion of a hyperplane. Use the Mayer-Vietoris sequence for $H_*(-;\mathbb{Z}_2)$ to show $j_* : H_k(\mathbb{R}P^{n-1};\mathbb{Z}_2) \to H_k(\mathbb{R}P^n;\mathbb{Z}_2)$ is an isomorphism for $0 \le k \le n-1$.

2. Let G be a group order m acting freely on M, and let $\pi : M \to M/G$ be the quotient map. For any abelian group A, show that the kernel of $\pi^* : H^k(M/G;A) \to H^k(M;A)$ is contained in $T_m(H^k(M/G;A))$ and that the image contains $mH^k(M;A)^G$ (where $-^G$ denotes the G-invariant part).

3. Let X be a cell complex with a single 0-cell, 2-cell and 4-cell, such that the attaching map for the 4-cell is the composition of a degree 2 map $S^3 \to S^3$ with the Hopf map $S^3 \to S^2$. Is X homotopy equivalent to a closed manifold?

4. (a) Let M be a compact manifold and $X \subset M$ a closed submanifold. Let $U = M \setminus X$, and let $i\colon U \to M$ and $j\colon X \to M$ denote the inclusion maps. Use the existence of a tubular neighbourhood of X to show that $j^* : \Omega^*(M) \to \Omega^*(X)$ is surjective. Let $\Omega^*(M,X)$ be the kernel of j^*, and $H^*(M,X)$ the cohomology of this cochain complex. One can prove that the cochain map $i_* : \Omega^*_c(U) \to \Omega^*(M,X)$ induces an isomorphism $H^*_c(U) \cong H^*(M,X)$. Deduce that there is a long exact sequence

$$0 \to H^0_c(U) \xrightarrow{i_*} H^0(M) \xrightarrow{j^*} H^0(X) \xrightarrow{\delta} H^1_c(U) \to \cdots,$$

and describe δ.

   (b) If M is compact and oriented, show that the sequence obtained by setting $X = \partial M$ in (a) is Poincaré dual to itself. (The Poincaré dual of a map between two cohomology groups means the composition of the dual map with the two Poincaré duality isomorphisms.)

   (c) Let M compact oriented of dimension $n = 2k+1$, and L the image of $H^k(M) \to H^k(\partial M)$. Show that $L = L^\perp$, where $L^\perp = \{[\alpha] \in H^{2k+1}(\partial M) : \int_{\partial M}[\alpha] \wedge [\beta] = 0 \text{ for any } [\beta] \in L\}$ (so L is an isotropic subspace of dimension exactly half the dimension of $H^k(\partial M)$). Deduce that if X^{4k} is the boundary of a compact manifold then the signature of X is 0. (*Hint:* Recall that if $f : V \to W$ is a linear map, then the kernel of the dual map $f^\vee : W^\vee \to V^\vee$ is the annihilator of the image of f.)

5. Let $Q \subset \mathbb{C}P^4$ be the hypersurface $\{(z_0 : \cdots : z_4) : \sum_{j=0}^4 z_j^2 = 0\}$. Show that $\mathbb{C}P^4 \setminus Q \simeq \mathbb{R}P^4$, and deduce that Q has the same Betti numbers as $\mathbb{C}P^3$. Is $Q \simeq \mathbb{C}P^3$?
(*Hint*: Consider $\mathbb{C}P^4 \setminus Q$ as the image of $\{(z_0,\dots,z_4) \in \mathbb{C}^5 : \sum_{j=0}^4 z_j^2 = 1\}$.)

6. (a) Show that any antipode-preserving map $f : S^n \to S^n$ (*i.e.* $f(-x) = -f(x)$ for all $x \in S^n$) has odd degree.

   (b) Deduce the Borsuk-Ulam theorem: for any smooth $f : S^n \to \mathbb{R}^n$ there is some $x \in S^n$ such that $f(x) = f(-x)$.

7. (a) Let $p, q : \mathbb{R}P^{n-1} \times \mathbb{R}P^{n-1} \to \mathbb{R}P^{n-1}$ be the projection maps, and $c \in H^1(\mathbb{R}P^{n-1};\mathbb{Z}_2)$ the generator. Show that $(p^*c + q^*c)^n = 0 \in H^n(\mathbb{R}P^{n-1} \times \mathbb{R}P^{n-1};\mathbb{Z}_2)$ if and only if n is a power of 2.

   (b) Deduce the Hopf theorem: the dimension of any division algebra over $\mathbb{R}$ (*i.e.* an algebra where every non-zero element has a multiplicative inverse; the multiplication need not be associative or commutative) is a power of 2.
(*Remark:* In fact the only possibilities are 1, 2, 4 and 8.)

8. Let M^n be a closed oriented manifold with fundamental class $[M] \in H_n(M;\mathbb{Z})$. Let $[\alpha] \in H^a(M;\mathbb{Z})$ and $[\beta] \in H^{n-a+1}(M;\mathbb{Z})$ be torsion classes, *i.e.* $k[\alpha] = 0$, $m[\beta] = 0$ for some $k, m \in \mathbb{N}$. Then we can write $k\alpha = d\gamma$ for some $\gamma \in C^{a-1}(M;\mathbb{Z})$. Let

$$[\alpha] \vee [\beta] = \frac{1}{k}(\gamma \cup \beta)[M] \in \mathbb{Q}/\mathbb{Z}.$$

   (a) Show that $[\alpha] \vee [\beta]$ is well-defined (independent of choice of representatives $\alpha \in C^a(M;\mathbb{Z})$ and $\beta \in C^{n-a-1}(M;\mathbb{Z})$, and of the choice of γ), and that $[\beta] \vee [\alpha] = (-1)^{a(n-a+1)}[\alpha] \vee [\beta]$.

   (b) Show that the 'torsion linking pairing'

$$T(H^a(M;\mathbb{Z})) \times T(H^{n-a+1}(M;\mathbb{Z})) \to \mathbb{Q}/\mathbb{Z},\ ([\alpha],[\beta]) \mapsto [\alpha] \vee [\beta]$$

   is non-degenerate.

9. (a) Let M be a closed oriented manifold of dimension $4k-1$. Let $\ell(M)$ denote the image of the quadratic form $T(H^{2k}(M;\mathbb{Z})) \to \mathbb{Q}/\mathbb{Z}$, $[\alpha] \mapsto [\alpha] \vee [\alpha]$. If N is a closed oriented manifold of the same dimension and $f : M \to N$, show that $(\deg f)\ell(N) \subseteq \ell(M)$.

   (b) Let M be a closed orientable manifold of dimension 3. Show that if $\pi_1(M) \cong \mathbb{Z}_p$ for a prime p such that -1 is not a quadratic residue mod p, then there is no orientation reversing diffeomorphism $f : M \to M$.

10. Let $\lambda_0 < \cdots < \lambda_n$.

   (a) Let $f : \mathbb{C}P^n \to \mathbb{R}$, $(z_0 : \cdots : z_n) \mapsto \sum \lambda_i |z_i|^2 / \sum |z_i|^2$. Show that f is a Morse function, identify the critical points, and compute $H^*(\mathbb{C}P^n;\mathbb{Z})$.

   (b) Let $f : \mathbb{R}P^n \to \mathbb{R}$, $(x_0 : \cdots : x_n) \mapsto \sum \lambda_i x_i^2 / \sum x_i^2$. Show that f is a Morse function and identify the critical points. Identify the gradient flow (with respect to the standard round metric on $\mathbb{R}P^n$) lines between critical points of adjacent index, and compute $H^*(\mathbb{R}P^n;\mathbb{Z}_2)$. (*Hint:* Lift f to a function on S^n, and show that if $f < \lambda_k$ then $(\operatorname{grad} f).e_k$ has the same sign as x_k.)

11. (a) Let M_1, M_2 be smooth manifolds with boundary, and $f : \partial M_1 \to \partial M_2$ a diffeomorphism. Given collar neighbourhoods of M_i, define a smooth structure on $M_1 \cup_f M_2 = M_1 \sqcup M_2/\sim$, where $x_1 \sim x_2$ if $x_i \in \partial M_i$ and $f(x_1) = x_2$. Show that different choices of collar neighbourhoods give rise to equivalent smooth structures.

   (b) Let $f : S^{n-1} \to S^{n-1}$ be a diffeomorphism. Show that $B^n \cup_f B^n$ is homeomorphic to S^n ($B^n \subset \mathbb{R}^n$ is the closed unit ball, and S^{n-1} is its boundary). We call this a *twisted sphere*. Show that if it is not diffeomorphic to S^n, then f is not smoothly isotopic to $id_{S^{n-1}}$ (*i.e.* there is no smooth homotopy $F_t, t \in [0,1]$ with $F_0 = id_{S^{n-1}}$ and $F_1 = f$, such that F_t is a diffeomorphism for each fixed t).

   (c) Persuade yourself that the (oriented) diffeomorphism classes of smooth n-dimensional oriented manifolds homeomorphic to S^n form an Abelian monoid under connected sums, *i.e.* the binary operation is associative, commutative and has an identity element. (Take for granted that the connect-sum operation is well-defined on connected oriented manifolds.) Show that the twisted spheres form a subgroup, *i.e.* the set is closed under connected sums and inverses exist. How does the operation of reversing orientation act on this subgroup? (*Remark:* For $n \geq 5$, any smooth closed n-manifold homotopy equivalent to S^n is a twisted sphere.)

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