Differential Topology
Example Sheet 4

1. Let M^m and N^n be closed manifolds. Meditate on the formulas

$$\chi(M \times N) = \chi(M)\chi(N),$$
$$\chi(M_1 \# M_2) = \chi(M_1) + \chi(M_2) - \begin{cases} 0 \text{ if } n \text{ is odd} \\ 2 \text{ if } n \text{ is even} \end{cases}$$

in light of the Poincaré-Hopf index theorem ($m = n$ for the second formula).

2. Let M^n, N^n smooth closed connected oriented manifolds of equal dimension, and $f : M \to N$ a map of non-zero degree. Does the pull-back $f^* : H^*(N; \mathbb{Z}) \to H^*(M; \mathbb{Z})$ on cohomology with integer coefficients need to be injective?

3. Let $f : B^n \to X$ be a closed (*i.e.* mapping closed subsets to closed subsets) continuous surjection. Let $Y = f(S^{n-1})$, and suppose that the restriction of f to the interior of B^n is a homeomorphism onto $e^n = X \setminus Y$. Show that X is homeomorphic to $Y \cup_\varphi B^n$, where $\varphi = f_{|S^{n-1}} : S^{n-1} \to Y$.

4. Let Y be a topological space and $\varphi_0, \varphi_1 : S^{n-1} \to Y$. Show that if $\varphi_0 \simeq \varphi_1$ then the spaces $Y \cup_{\varphi_i} B^n$ obtained by attaching n-cells to Y by φ_0 and φ_1 are homotopy equivalent.

5. Let $T = B^2/\sim$, where $z_1 \sim z_2$ if $z_i \in S^1$ and $z_1^3 = z_2^3$. Compute the singular homology of T with coefficients in $\mathbb{Z}$, $\mathbb{Z}_3$ and $\mathbb{Z}_2$. Is T homotopy equivalent to a closed manifold?

6. (a) For any path-connected topological space X, prove that $H_1(X; \mathbb{Z}_2)$ is the two-elementary part of $\pi_1(X)$, *i.e.* the quotient by the subgroup generated by all squares.
   (b) Prove that if M is a non-orientable manifold, then $H_1(M; \mathbb{Z}_2)$ is non-trivial.

7. Let $Z \subset \mathbb{R}^3$ be the union of the unit sphere $\{(x, y, z) : x^2 + y^2 + z^2 = 1\}$ and the disc $\{(x, y, z) : x^2 + y^2 \leq 1, z = 0\}$. Identify a cell complex homeomorphic to Z, and compute $H_k(Z; \mathbb{Z})$ for all k. Is Z homotopy equivalent to a closed manifold?

8. (a) Let $0 \to C_n \to \cdots \to C_1 \to 0$ be a chain complex of finitely generated free $\mathbb{Z}$-modules. Let $H_*(C_*)$ be the associated homology groups (which are finitely generated $\mathbb{Z}$-modules), and $\chi(C_*) = \sum(-1)^i \operatorname{rk} H_i(C_*)$. For any field F, $C_* \otimes_{\mathbb{Z}} F$ is a chain complex, and its homology groups are vector spaces over F. Show that

$$\chi(C_*) = \sum(-1)^i \operatorname{rk} C_i = \sum(-1)^i \dim_F H_i(C_* \otimes F).$$

   (b) Now drop the condition that the free $\mathbb{Z}$-module C_* is finitely generated. Show that if $H_*(C_*)$ is finitely generated, then

$$\chi(C_*) = \sum(-1)^i \dim_F H_i(C_* \otimes_{\mathbb{Z}} F).$$

9. Let C_* be a chain complex of free $\mathbb{Z}$-modules with $b_k = \operatorname{rk} H_k(C_*)$ finite. Show that $H_k(C_*; S^1) \cong T^{b_k} \times T(H_{k-1}(C_*))$ (the first term is a torus, the second a torsion subgroup).

10. (*Combinatorial proof of the Brouwer fixed point theorem*)

   (a) Let Δ be an n-simplex $[v_0, \ldots, v_n]$, and give each vertex v_i a different colour c_i. Consider any simplicial subdivision of Δ, and colour each of the vertices in the subdivision subject to the following constraint: if v belongs to the k-dimensional face $[v_{i_0}, \ldots, v_{i_k}]$ of Δ, then v has one of the $k+1$ colours $c_{i_0}, \ldots, c_{i_k}$. Show that there is an n-simplex in the subdivision for which all $n+1$ vertices have different colours.
   (*Hint:* Consider the discriminant polynomial $\prod_{i<j}(c_i - c_j) \in \mathbb{Z}[c_0, \ldots, c_n]$.)

(b) Deduce the Brouwer fixed point theorem: any continuous map $f : \Delta \to \Delta$ has a fixed point.

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