Singular cohomology with compact supports

It is possible to define singular cohomology of M^n with compact supports. It is the cohomology $H^*_c(M)$ of the complex $C^*_c(M)$, where $C^k_c(M) \subseteq C^k(M)$ is the subgroup of cochains α such that there is a compact $K \subset M$ such that $\alpha(\sigma) = 0$ for any chain σ contained in $M \backslash K$.

If M is non-compact (and connected) then $H_n(M) \cong 0$, so the ad hoc definition of orientability in terms of existence of a fundamental class is no longer suitable. An equally ad hoc definition that does work regardless of whether M is compact or not (as long as M is connected and $\partial M = \emptyset$) is that M is R-orientable if $H^n_c(M; R) \cong R$.

A proper map between topological spaces induces pull-backs on the compactly supported cohomology. For a proper map $f : M_1^n \to M_2^n$ between connected manifolds without boundary of equal dimension, we can define a mod 2 degree in terms of $f^* : H^n_c(M_2; \mathbb{Z}_2) \to H^n_c(M_1; \mathbb{Z}_2)$. If M_i and f are smooth we can prove that this coincides with the degree mod 2 defined in terms of counting pre-images. The argument is similar to the one we used in the oriented case to show that the signed count of pre-images equals the degree defined in terms of de Rham cohomology (using that $H^n_c(U; \mathbb{Z}_2) \cong H^n_c(M_i; \mathbb{Z}_2)$ for any open ball $U \subset M_i$). Similarly, if M_i are oriented with corresponding generators $u_i \in H^n_c(M_i; \mathbb{Z})$, then defining $\deg f$ by $f^* u_2 = (\deg f) u_1$ is equivalent to our previous definitions of degree.

For non-compact manifolds, compactly supported cohomology can be used to state Poincaré duality. Note that $H^k(M; \mathbb{R}) \cong H_k(M; \mathbb{R})^\vee$, so under the de Rham isomorphism we recover the de Rham cohomology version of Poincaré duality.

Theorem ([2, Theorem 3.35]). *If M^n is R-orientable and $\partial M = \emptyset$ then $H_k(M; R) \cong H^{n-k}_c(M; R)$.*

If M is closed and X is a closed submanifold, then $H^k_c(M \setminus X; G)$ is isomorphic to the 'relative' cohomology $H^k(M, X; G)$ of the pair (M, X). The latter is defined for any pair of topological spaces $M \supset X$, and fits into a long exact sequence analogous to the one we saw on the coursework:

$$\cdots \to H^{k-1}(X; G) \to H^k(M, X; G) \to H^k(M; G) \to H^k(X; G) \to \cdots$$

Cup products and transverse intersections

The cup product on singular cohomology is harder to work out from first principles than the wedge product on de Rham cohomology. But on manifolds one can interpret the cup product as dual to transverse intersection under Poincaré duality,

The homology class represented by a closed R-oriented submanifold $Y^{n-j} \subset M^n$ is the image $[Y] \in H_{n-j}(M; R)$ of the fundamental class of Y under the push-forward of the inclusion. If M too is closed R-oriented, let $PD_M(Y) \in H^j(M; R)$ denote the image of $[Y]$ under the Poincaré isomorphism.

Now suppose that M and Y is smooth, and that X is another smooth manifold. A smooth map $f : X \to M$ is called transverse to Y if $Df_x(T_x X) + T_y Y = T_y M$ for each $y \in Y$ and $x \in f^{-1}(y)$. Then $f^{-1}(Y) \subset X$ is a smooth submanifold of codimension i. If X too is R-oriented, then $f^{-1}(Y)$ has a natural R-orientation as well.

Proposition (*cf.* [1, p.69]). *$f^* PD_M(Y) = PD_X(f^{-1}(Y)) \in H^j(N; R)$.*

Proof idea. Let N_Y be the normal bundle of Y in M, and identify it with a tubular neighbourhood of Y. Like in Q1 of the coursework, we can identify $PD_M(Y)$ with a push-forward of $PD_{N_Y}(Y) \in H^j_c(N_Y; R)$. The Poincaré dual of the zero section on a closed manifold is equal to the 'Thom class' $\Phi(N_Y)$, which is naturally associated to any oriented vector bundle.

Note that the normal bundle of $f^{-1}(Y)$ in X is $f^* N_Y$. The naturality of the Thom class implies that $\Phi(f^* N_Y) = f^* \Phi(N_Y)$. Now by the same argument that is the Poincaré dual of the zero section, and its push-forward in $H^j(X; R)$ is $PD_X(f^{-1}(Y))$. □

If $n \geq 10$ then not every homology class in M^n need be represented even by a continuous image of a manifold (this problem was studied by Thom [4]). Therefore the results below cannot be regarded as a complete characterisation of the cup product even on smooth manifolds, but it is still useful.

Proposition. *Let M^n be a smooth closed R-oriented manifold, and X^{n-i}, Y^{n-j} smooth closed R-oriented submanifolds that intersect transversely,* i.e. $T_xX + T_xY = T_xM$ *for every* $x \in X \cap Y$ *(then $X \cap Y$ is a smooth submanifold of codimension $i+j$ in M, and it has a natural orientation).*

(i) $PD_X(X \cap Y) = PD_M(Y)_{|X} \in H^j(X; R)$

(ii) $PD_M(X) \cup PD_M(Y) = PD_M(X \cap Y) \in H^{i+j}(M; R)$

Proof. (i) is obtained by applying the previous proposition to the inclusion map $f : X \hookrightarrow M$. (ii) follows using standard relations for the cap product: $([M] \cap [\alpha]) \cap [\beta] = [M] \cap ([\alpha] \cup [\beta])$ for $[\alpha], [\beta] \in H^*(M; R)$, and $i_*([X] \cap i^*[\alpha]) = i_*[X] \cap [\alpha]$ for $i : X \to M$. See Hutchings [3, §1 and §4] for more detail. $\square$

In particular, for transverse oriented submanifolds whose codimensions add up to n, the cup product of the Poincaré duals can be interpreted as the intersection number. One application is to give a geometric derivation of the cohomology rings of projective spaces.

Any hyperplane $P \subset \mathbb{C}P^n$ represents the same class $[P] \in H_{2n-2}(\mathbb{C}P^n; \mathbb{Z})$. $PD_{\mathbb{C}P^n}(P)^n \in H^{2n}(\mathbb{C}P^n; \mathbb{Z}) \cong \mathbb{Z}$ corresponds to the number of intersection points of n transverse hyperplanes, which is 1. Therefore $PD_{\mathbb{C}P^n}(P)^k$ is a generator for $H^{2k}(\mathbb{C}P^n; \mathbb{Z})$ for $0 \leq k \leq n$, and $H^*(\mathbb{C}P^n; \mathbb{Z})$ is isomorphic as a ring to the truncated polynomial ring $\mathbb{Z}[x]/(x^{n+1})$.

Similarly if $[\alpha] \in H^1(\mathbb{R}P^n; \mathbb{Z}_2)$ is the dual of hyperplane in $\mathbb{R}P^n$ then $[\alpha]^n \in H^n(\mathbb{R}P^n; \mathbb{Z}_2) \cong \mathbb{Z}_2$ corresponds to the mod 2 intersection number of n transverse hyperplanes. This is again 1, so $[\alpha]^k \in H^k(\mathbb{R}P^n; \mathbb{Z}_2)$ is a generator for each $0 \leq k \leq n$, and $H^*(\mathbb{R}P^n; \mathbb{Z}_2)$ is isomorphic as a ring to the truncated polynomial ring $\mathbb{Z}_2[x]/(x^{n+1})$.

Recognising closed manifolds

Poincaré duality is a special property of closed manifolds; it can fail if there is even a single singular point (see Q4 on the sample exam). We can therefore use it (together with results about orientability) to show that certain topological spaces cannot be homotopy equivalent to closed manifolds. (Being homotopy equivalent to a non-closed manifold is not much of a restriction: any reasonably nice subset of $\mathbb{R}^n$ is homotopy equivalent to some sort of tubular neighbourhood.)

Let X be a connected topological space. If we suppose that X is homotopy equivalent to an *orientable* closed manifold then we can detect its dimension as the highest degree in which the Betti number is non-zero. When we know the dimension n we can check whether $b_n(X) = 1$ and various consequences of Poincaré duality are satisfied, most easily $b_i(X) = b_{n-i}(X)$ and that $b_{2k+1}(X)$ is even if $n = 4k + 2$. If any of these properties fails, then X cannot be homotopy equivalent to an orientable closed manifold.

One can sometimes rule out the possibility that X is homotopy equivalent to a nonorientable manifold by using that then $\pi_1(X)$ must contain an index 2 subgroup (in particular a nonorientable manifold is never simply-connected). Equivalently $H_1(X; \mathbb{Z}_2)$ must be non-trivial (see Q3 on example sheet 4). In general the easiest way is perhaps to work out the homology with $\mathbb{Z}_2$ coefficients. If X is homotopy equivalent to a closed manifold (whether orientable or not) then the dimension must be the maximal n such that $H_n(X; \mathbb{Z}_2)$ is non-trivial. Poincaré duality then imposes $\dim H_i(X; \mathbb{Z}_2) = \dim H_{n-i}(X; \mathbb{Z}_2)$.

References

[1] Raoul Bott and Loring Tu, *Differential Forms in Algebraic Topology*, Springer, 1982

[2] Allen Hatcher, *Algebraic Topology*, Cambridge University Press, 2002

[3] Michael Hutchings, `http://math.berkeley.edu/~hutching/teach/215b-2011/cup.pdf`

[4] René Thom, *Quelques propriétés globales des variétés differentiables.* Comm. Math. Helv. **28** (1954) 17–86

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