The induction step in the proof of Poincaré duality

Consider the following diagram, where the top row is the Mayer-Vietoris sequence for H^* and the bottom row is the dual of the Mayer-Vietoris sequence for H_c^*.

$$\begin{array}{ccccccccc}
\cdots\ H^{k-1}(V_1 \cap V_2) & \xrightarrow{\delta} & H^k(V_1 \cup V_2) & \xrightarrow{I} & H^k(V_1) \oplus H^k(V_2) & \xrightarrow{J} & H^k(V_1 \cap V_2)\ \cdots \\
\downarrow{\scriptstyle P_{V_1 \cap V_2}} & & \downarrow{\scriptstyle P_{V_1 \cup V_2}} & & \downarrow{\scriptstyle P_{V_1} + P_{V_2}} & & \downarrow{\scriptstyle P_{V_1 \cap V_2}} \\
\cdots H_c^{n-k-1}(V_1 \cap V_2)^\vee & \xrightarrow{\delta_c^\vee} & H_c^{n-k}(V_1 \cup V_2)^\vee & \xrightarrow{I_c^\vee} & H_c^{n-k}(V_1)^\vee \oplus H_c^{n-k}(V_2)^\vee & \xrightarrow{J_c^\vee} & H_c^{n-k}(V_1 \cap V_2)^\vee \cdots
\end{array}$$

If we prove that the diagram commutes, then the result follows from the Five lemma. In fact, some squares only commute up to sign, but that's not a problem since changing the signs of some maps doesn't affect the exactness of the rows and the application of the Five lemma.

Commutativity of the squares involving I and J is straight-forward and boils down to the fact that if $i : U \to M$ is inclusion of an open set then this square commutes:

$$\begin{array}{ccc}
H^k(M) & \xrightarrow{i^*} & H^k(U) \\
\downarrow{\scriptstyle P_M} & & \downarrow{\scriptstyle P_U} \\
H^{n-k}(M)^\vee & \xrightarrow{i_*^\vee} & H^{n-k}(U)^\vee
\end{array}$$

This means that if $[\alpha] \in H^k(M)$ and $[\beta] \in H^{n-k}(U)$ then $P_U(i^*[\alpha])[\beta] = i_*^\vee(P_M([\alpha]))[\beta]$. The LHS is $\int_U \alpha \wedge \beta$, while by definition of the dual of a map the RHS is $P_M([\alpha])(i_*[\beta]) = \int_M \alpha \wedge \beta$, so they are equal.

Finally we prove that

$$(-1)^k P_{V_1 \cup V_2} \circ \delta \;=\; \delta_c^\vee \circ P_{V_1 \cap V_2} \;:\; H^k(V_1 \cap V_2) \to H_c^{n-k-1}(V_1 \cup V_2)^\vee.$$

For $[\alpha] \in H^k(V_1 \cap V_2)$ and $[\beta] \in H_c^{n-k-1}(V_1 \cup V_2)$ we need to compare

$$P_{V_1 \cup V_2}(\delta[\alpha])[\beta] = \int_{V_1 \cup V_2} \delta[\alpha] \wedge [\beta]$$

with

$$\delta_c^\vee(P_{V_1 \cap V_2}[\alpha])[\beta] = P_{V_1 \cap V_2}[\alpha](\delta_c[\beta]) = \int_{V_1 \cap V_2} [\alpha] \wedge \delta_c[\beta].$$

But $\delta[\alpha] = [d\rho_1 \wedge \alpha] \in H^{k+1}(M)$ and $\delta_c[\beta] = [d\rho_1 \wedge \beta] \in H_c^{n-k}(M)$. Thus

$$\int_{V_1 \cup V_2} \delta[\alpha] \wedge [\beta] = \int_{V_1 \cup V_2} d\rho_1 \wedge \alpha \wedge \beta = (-1)^k \int_{V_1 \cup V_2} \alpha \wedge d\rho_1 \wedge \beta = (-1)^k \int_{V_1 \cap V_2} [\alpha] \wedge \delta_c[\beta].$$

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