Imperial College London

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2012

This paper is also taken for the relevant examination for the Associateship.

M4P54/M5P54

Differential Topology

Date: 23rd May 2012 Time: 2pm – 4pm

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

Throughout, 'manifold' means smooth manifold and 'closed' means compact without boundary. $H^*(-)$ denotes de Rham cohomology, while $H^*(-;G)$ denotes singular cohomology with coefficients in an abelian group G.

1. (i) Let M^n be an oriented manifold without boundary. Show that if $[\alpha] \in H^r(M)$, $[\beta] \in H^s_c(M)$ then the wedge product $[\alpha] \wedge [\beta] = [\alpha \wedge \beta] \in H^{r+s}_c(M)$ is well-defined. Define the Poincaré pairing and state the Poincaré duality theorem for de Rham cohomology on M.

   (ii) Let $j : X \hookrightarrow M$ be the embedding of a closed oriented submanifold, $\alpha \in \Omega^r(X)$ a closed form, and $[\alpha] \in H^r(X)$ its de Rham cohomology class. Show that $\alpha = j^*\beta$ for some closed $\beta \in \Omega^r(M)$ if and only if $[\alpha]$ lies in the image of $j^* : H^r(M) \to H^r(X)$.
   (You may assume that X has a tubular neighbourhood.)

   (iii) Let k be the dimension of the closed submanifold X. Define the Poincaré dual $PD(X) \in H^{n-k}(M)$. For $[\beta] \in H^r(M)$, show that $PD(X) \wedge [\beta] \in H^{n-k+r}(M)$ depends only on $j^*[\beta] \in H^r(X)$.

2. Let M^m and N^n be closed manifolds. Let $p : M \times N \to M$ and $q : M \times N \to N$ denote the projection maps.

   (i) Prove the Künneth formula: the linear map $\bigoplus_{i+j=k} H^i(M) \otimes H^j(N) \to H^k(M \times N)$ induced by the bilinear maps $H^i(M) \times H^j(N) \to H^{i+j}(M \times N),\ ([\alpha], [\beta]) \mapsto [p^*\alpha \wedge q^*\beta]$ is an isomorphism. You may assume the Mayer-Vietoris theorem, the Poincaré lemma and the existence of a good cover of M. You may also assume the Five lemma, and you do not need to prove commutativity of the diagram that you apply it to.

   (ii) State a formula for the Betti numbers of a connected sum of closed oriented manifolds. Calculate the Betti numbers of the manifold $X = (S^3 \times S^3) \# (S^3 \times S^3)$ and show that it is not diffeomorphic to a product of closed manifolds of lower dimension.

3. (i) State the singular cohomology with integer coefficients of $S^2 \times S^2$ and $\mathbb{C}P^2$. Pick generators for $H^2(S^2 \times S^2; \mathbb{Z})$ and $H^2(\mathbb{C}P^2; \mathbb{Z})$ and describe (with brief justification) the cup products of pairs of generators (including the product of each generator with itself).

   (ii) Let $f : M \to N$ be a smooth map between closed oriented manifolds of the same dimension n. Describe the degree of f in terms of generators $u \in H^n(M; \mathbb{Z})$, $v \in H^n(N; \mathbb{Z})$ compatible with the orientations. Show that

      1. any smooth map $f : \mathbb{C}P^2 \to \mathbb{C}P^2$ has non-negative degree;
      2. any smooth map $f : \mathbb{C}P^2 \to S^2 \times S^2$ has degree zero;
      3. any smooth map $f : S^2 \times S^2 \to \mathbb{C}P^2$ has even degree.

4. (i) Identify $\mathbb{R}P^2$ with a finite cell complex and compute $H_*(\mathbb{R}P^2;\mathbb{Z}_2)$, the singular homology with coefficients in the group of order 2.

(ii) Let $S^3 = \{(x_1,x_2,x_3,x_4) \in \mathbb{R}^4 : x_1^2+x_2^2+x_3^2+x_4^2 = 1\}$ and let $f : S^3 \to S^3$ be the orientation-reversing diffeomorphism $(x_1,x_2,x_3,x_4) \mapsto (x_1,-x_2,-x_3,-x_4)$. Let $\sim$ be the equivalence relation on Z defined by $x \sim y$ if and only if $x = y$ or $x = f(y)$, and Z the quotient topological space $S^3/{\sim}$. Compute $H_*(Z;\mathbb{Z}_2)$.
(*Hint:* Apply the Mayer-Vietoris sequence for $H_*(-;\mathbb{Z}_2)$.)

(iii) Is Z homotopy equivalent to any closed manifold?