Differential Topology
Example Sheet 3

1. Prove the following part of the Five lemma: if the diagram below commutes, rows are exact, ψ_2 and ψ_4 are surjective and ψ_5 is injective then ψ_3 is surjective.

$$\begin{array}{ccccccccc} A^1 & \xrightarrow{f} & A^2 & \xrightarrow{f} & A^3 & \xrightarrow{f} & A^4 & \xrightarrow{f} & A^5 \\ \downarrow{\scriptstyle \psi_1} & & \downarrow{\scriptstyle \psi_2} & & \downarrow{\scriptstyle \psi_3} & & \downarrow{\scriptstyle \psi_4} & & \downarrow{\scriptstyle \psi_5} \\ B^1 & \xrightarrow{g} & B^2 & \xrightarrow{g} & B^3 & \xrightarrow{g} & B^4 & \xrightarrow{g} & B^5 \end{array}$$

2. Let T^2 be the 2-torus $\mathbb{R}^2/\mathbb{Z}^2$. If x^1, x^2 coordinates on $\mathbb{R}^2$, then $[dx^1], [dx^2]$ form a basis for $H^1(T^2)$. Identify oriented submanifolds $X_i \subset T^2$ that are Poincaré dual to this basis, i.e. $\int_{X_i} \alpha = \int_{T^2} dx^i \wedge \alpha$ for any $[\alpha] \in H^1(T^2)$. What is the number of intersection points of X_1 and X_2?

3. Let M^{n+1} and N^n be smooth oriented manifolds such that M has boundary but N does not. If $f : M \to N$ is a proper map, show that $\deg f_{|\partial M} = 0$.

4. Let $f : M \to N$ be a smooth map between connected oriented manifolds without boundary, of equal dimension. Suppose M is compact while N is non-compact. Show that $\deg f = 0$.

5. (a) Let f be a degree d polynomial in $\mathbb{C}$. Show that the degree of the smooth map $f : \mathbb{C} \to \mathbb{C}$ equals d. Deduce the fundamental theorem of algebra (provided that you did not assume it in the proof).

   (b) Let f, g be coprime polynomials in $\mathbb{C}$. Show that the degree of the meromorphic map $f/g : \mathbb{C}P^1 \to \mathbb{C}P^1$ is the maximum of the degrees of the polynomials f and g.

6. Prove the "hairy-ball theorem": S^n has a nowhere-vanishing vector field if and only if n is odd.

7. Let M be a smooth manifold with boundary, and $\mathring{M} = M \backslash \partial M$ its interior. Show that the pull-back of the inclusion $i : \mathring{M} \to M$ is an isomorphism $i^* : H^*(M) \to H^*(\mathring{M})$.

8. Let M^n, N^n smooth closed connected oriented manifolds of equal dimension. If $f : M \to N$ has non-zero degree, show that the pull-back on de Rham cohomology $f^* : H^*(N) \to H^*(M)$ is injective.

9. Let M^n be an oriented manifold without boundary, and suppose its cohomology is finite-dimensional. Let $H^k_0(M) \subset H^k(M)$ be the subset of classes that can be represented by forms with compact support (*i.e.* the image of the natural map $H^k_c(M) \to H^k(M)$). Show that there is a non-degenerate pairing $H^k_0(M) \times H^{n-k}_0(M) \to \mathbb{R}$.

Questions and corrections to `j.nordstrom@imperial.ac.uk`.
February 7, 2012