Differential Topology
Example Sheet 2

1. Let $0 \to A^* \xrightarrow{f} B^* \xrightarrow{g} C^* \to 0$ be a short exact sequence of cochain complexes. Show that the sequence

$$\cdots \to H^k(B^*) \xrightarrow{g} H^k(C^*) \xrightarrow{\delta} H^{k+1}(A^*) \to \cdots$$

in the Snake lemma is exact at $H^k(C^*)$.

2. The Euler characteristic of a manifold M^n is the alternating sum of its Betti numbers (if finite): $\chi(M) = \sum_{i=0}^{n}(-1)^i b_i(M)$.

   (a) Let U and V be open subsets of a manifold M. Show that (if the terms are well-defined)

$$\chi(U) + \chi(V) = \chi(U \cup V) + \chi(U \cap V).$$

   (b) Let M_1^n, M_2^n be manifolds. Show that

$$\chi(M_1 \# M_2) = \chi(M_1) + \chi(M_2) - \begin{cases} 0 \text{ if } n \text{ is odd} \\ 2 \text{ if } n \text{ is even} \end{cases}$$

   (c) Let M^m, N^n be manifolds. Show that $\chi(M \times N) = \chi(M)\chi(N)$.

3. Let U, V open connected in $\mathbb{R}^n$ such that $\mathbb{R}^n = U \cup V$. Show that $U \cap V$ is connected.

4. Show that $b_k(T^n) = \binom{n}{k}$ for all $0 \le k \le n$.

5. (a) Show that $S^2 \times S^4$ is not homotopy equivalent to $\mathbb{C}P^3$.

   (b) Are the punctured manifolds $(S^2 \times S^4) \setminus \{pt\}$ and $\mathbb{C}P^3 \setminus \{pt\}$ homotopy equivalent to any closed manifolds?

6. Let M_1^n, M_2^n be closed connected oriented manifolds of equal dimension. Find smooth maps $\phi_i : M_1 \# M_2 \to M_i$ such that $\phi_i^* : H^*(M_i) \to H^*(M_1 \# M_2)$ are injective, $H^k(M_1 \# M_2) = \phi_1^*(H^k(M_1)) \oplus \phi_2^*(H^k(M_2))$ for $0 < k < n$, and $\phi_1^*[\alpha] \wedge \phi_2^*[\beta] = 0$ for any $[\alpha] \in H^a(M_1)$, $[\beta] \in H^b(M_2)$, $a, b > 0$.
(*Remark:* Since $H^0(M_i)$ are both naturally isomorphic to $\mathbb{R}$, there is a well-defined notion for a pair $(h_1, h_2) \in H^*(M_1) \oplus H^*(M_2)$ to have equal degree 0 components. Let $A \subset H^*(M_1) \oplus H^*(M_2)$ be the subset of such pairs. $H^*(M_1) \oplus H^*(M_2)$ is not an algebra because it has no multiplicative identity, but A is. The question proves that $H^*(M_1 \# M_2)$ is isomorphic as a graded algebra to the quotient A/K, where $K = \left\{([\omega_1], [\omega_2]) \in H^n(M_1) \oplus H^n(M_2) : \int_{M_1} \omega_1 = \int_{M_2} \omega_2 \right\}$.)

7. Show that there is an orientation-reversing diffeomorphism $\mathbb{C}P^n \to \mathbb{C}P^n$ if and only if n is odd.

8. Find a pair of connected oriented manifolds M, N that do not have orientation-reversing self-diffeomorphisms such that $M \times N$ and $M \# N$ do.

9. Let G^n be a compact Lie group, i.e. a smooth compact manifold of dimension n with a group structure such that the multiplication and inverse maps are smooth. For $g \in G$, let $L_g : G \to G$ be the left multiplication by g, i.e. $L_g : h \mapsto gh$.

   (a) Show that there is a left-invariant orientation form on G, i.e. a non-zero $\omega \in \Omega^n(G)$ such that $L_g^*\omega = \omega$ for any $g \in G$. Show that for any $f : G \to \mathbb{R}$ (or $f : G \to V$ for some vector space V)

$$\int_G f\omega = \int_G (f \circ L_g)\omega.$$

(b) Let M^m be a smooth manifold with a smooth G-action, i.e. a smooth map $G \times M \to M$, $(g, x) \mapsto gh$ such that $g(hx) = (gh)x$. Let $\Omega^k_G(M) = \{\alpha \in \Omega^k(M) : g^*\alpha = \alpha \text{ for all } g \in G\}$. If G is connected, show that there is a projection $A : \Omega^k(M) \to \Omega^k_G(M)$ such that $[A(\alpha)] = [\alpha] \in H^k(M)$ for any closed $\alpha \in \Omega^k(M)$.

(c) $U(n) \subset GL(n, \mathbb{C})$ is the subgroup that preserves the hermitian inner product $h_0 = \sum dz^j \otimes d\bar{z}^j$. Alternatively it can be characterised as preserving the real part g_0 of the hermitian product (which is just the Euclidean inner product), or the "Kähler form" $\omega_0 = g_0(i\cdot, \cdot) = \frac{i}{2} \sum dz^j \wedge d\bar{z}^j \in \Lambda^2_{\mathbb{R}}(\mathbb{C}^n)^*$. Show that the natural action of $U(n+1)$ on $\mathbb{C}P^n$ is transitive, with stabiliser isomorphic to $U(1) \times U(n)$. Describe the action of the stabiliser of a point on the tangent space at that point, and show that $\Omega^2_{U(n+1)}(\mathbb{C}P^n)$ is 1-dimensional (a certain normalised non-zero element ω is called the *Fubini-Study form*). Deduce that a generator $c \in H^2(\mathbb{C}P^n)$ satisifies $c^k \neq 0 \in H^{2k}(\mathbb{C}P^n)$ for $0 \leq k \leq n$. (*Hint*: Prove or assume that up to scaling ω_0 is the only real 2-form on $\mathbb{C}^n$ invariant under $U(n)$.)

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