Differential Topology
Example Sheet 1

1. Let S^2 be the unit sphere $\{x^2+y^2+z^2=1\} \subset \mathbb{R}^3$.
   (a) Show that $X_1 = S^2 \cup \{x=y=0,\ 0 \le z \le 1\}$ is homotopy equivalent to S^2.
   (b) Show that $X_2 = S^2 \cup \{x=y=0,\ -1 \le z \le 1\}$ is not homotopy equivalent to S^2.

2. Let $A \in O(n+1)$, so that $A_{|S^n}$ maps S^n to itself. Show that $A_{|S^n}$ is orientation-preserving if and only if A is.

3. (a) Let M_1^n, M_2^n be smooth manifolds of equal dimension, $U_i, V_i \subseteq M_i$ open subsets such that $M_i = U_i \cup V_i$, and $g : U_1 \to U_2$ a diffeomorphism. Suppose that $g(U_1 \cap V_1)$ is disjoint from V_2. Let $M_1 \cup_g M_2$ be the quotient topological space $M_1 \sqcup M_2/\sim$, where $x_1 \sim x_2$ if $x_i \in U_i$ and $g(x_1) = x_2$. Show that $M_1 \cup_g M_2$, is Hausdorff, and that it has a smooth structure such that the natural inclusions $j_i : M_i \hookrightarrow M_1 \cup_g M_2$ are smooth.
   (b) Deduce that the connected sum of two smooth M_1, M_2 of equal dimension n is a smooth manifold, where the connected sum is defined as follows. Pick points $p_i \in M_i$, and coordinate charts $f_i : U_i \xrightarrow{\sim} \mathbb{R}^n$ such that $f_i(p_i) = 0$. The connected sum is
   $$M_1 \# M_2 = M_1 \backslash \{p_1\} \cup_g M_2 \backslash \{p_2\},$$
   where g is the composition of f_2^{-1}, $x \mapsto \frac{x}{|x|^2}$ on $\mathbb{R}^n \backslash \{0\} \to \mathbb{R}^n \backslash \{0\}$, and f_1.
   (*Remark:* Up to diffeomorphism, the connected sum is independent of the choice of points p_i and charts f_i, unless M_i are both orientable. In that case one can remove the ambiguity by specifying orientations of both M_i and requiring the charts f_i to be orientation-preserving. Later examples will show that when M_1 and M_2 are oriented, reversing the orientation of one of them really can change the diffeomorphism type of the connected sum.)

4. (a) Let M be a manifold, $U, V \subseteq M$ open subsets such that $U \cap V$ is connected and $M = U \cup V$. Show that M is orientable if and only if U and V are both orientable.
   (b) Show that $M_1 \# M_2$ is orientable if and only if M_1 and M_2 are both orientable.

5. Given an orientation form $\omega \in \Lambda^n V \setminus \{0\}$ on an n-dimensional vector space V, let $\langle \omega \rangle$ denote its image in the (two-element) set of orientations $(\Lambda^n V \setminus \{0\})/\mathbb{R}_{>0}$. The *oriented double cover* of a manifold M is $\hat{M} = \{(x, \langle \omega \rangle) : x \in M,\ \omega \in \Lambda^n T_x M \setminus \{0\}\}$. Let $p : \hat{M} \to M$ denote the projection map $(x, \langle \omega \rangle) \mapsto x$. For any chart $f : U_i \to \mathbb{R}^n$ of M, let $\hat{f} = f \circ p : \hat{U} \to \mathbb{R}^n$, where $\hat{U} = \{(x, \langle f_x^* \mathrm{vol}_{\mathbb{R}^n} \rangle) : x \in U\} \subset \hat{M}$. Show that
   (a) These $\hat{f}$ form an atlas of charts giving $\hat{M}$ the structure of a smooth manifold;
   (b) $\hat{M}$ has a natural orientation;
   (c) The involution $\tau : \hat{M} \to \hat{M},\ (x, \langle \omega \rangle) \mapsto (x, \langle -\omega \rangle)$ is orientation-reversing;
   (d) $p : \hat{M} \to M$ is a 2-to-1 covering map;
   (e) If M is connected, then $\hat{M}$ is connected if and only if M is not orientable.

6. Let M^n be a connected manifold. Given a closed path $\gamma : S^1 \to M$, write S^1 as a finite union of intervals $[x_0, x_1]$, $[x_1, x_2], \ldots, [x_k, x_0]$ such that the image $\gamma([x_i, x_{i+1}]) \subset M$ is contained in a coordinate chart $f_i : U_i \to \mathbb{R}^n$ and the transition functions $f_{i+1} \circ (f_i)^{-1}$ are orientation-preserving for $i = 0, \ldots, k-1$. Let $h(\gamma) = 1$ if $f_0 \circ f_k^{-1}$ is orientation-preserving, and $h(\gamma) = -1$ otherwise.
   (a) Show that if $h(\gamma) = 1$ for all γ then M is orientable.
   (b) Show that otherwise M is not orientable, and $h(\gamma) = 1$ if and only if $[\gamma] \in \pi_1(M)$ is in the image $p_* \pi_1(\hat{M})$ of the fundamental group of the oriented double cover $\hat{M}$ under the push-forward of the covering map $p : \hat{M} \to M$.

(c) (*If you are familiar with vector bundles*) Recall that up to isomorphism, there are only two real line bundles on $S^1 \cong \mathbb{R}P^1$: the trivial one, and the tautological bundle on $\mathbb{R}P^1$ ("the Möbius line bundle"). Show that $h(\gamma) = 1$ if and only if the pull-back line bundle $\gamma^* \Lambda^n T^* M$ over S^1 is trivial.

7. Let M^n be a compact oriented manifold with boundary ∂M, $\omega \in \Omega^{k-1}(M)$ and $\tau \in \Omega^{n-k}(M)$. Show that
$$\int_M d\omega \wedge \tau = \int_{\partial M} \omega \wedge \tau + (-1)^k \int_M \omega \wedge d\tau.$$

8. Let $r_1 > r_2$, and let $T \subset \mathbb{R}^3$ be the surface $(\sqrt{x^2+y^2} - r_1)^2 + z^2 = r_2^2$. Evaluate
$$\int_T \frac{x^2 dz \wedge (xdy - ydx)}{x^2+y^2}.$$

9. Prove from the definition that there is a unique smooth structure on $\mathbb{R}$, i.e. that any smooth manifold $\hat{\mathbb{R}}$ homeomorphic to $\mathbb{R}$ is also diffeomorphic to $\mathbb{R}$. (*Hint:* $\hat{\mathbb{R}}$ is orientable.)

10. Let T^2 be the 2-dimensional torus $\mathbb{R}^2/\mathbb{Z}^2$, and let $C_1, C_2 \subset T^2$ be the images of $\mathbb{R} \times \{0\}$ and $\{0\} \times \mathbb{R}$. Show that
$$\Omega^1(T^2) \to \mathbb{R}^2, \; \alpha \mapsto \left(\int_{C_1} \alpha, \int_{C_2} \alpha \right)$$
induces an isomorphism $H^1(T^2) \to \mathbb{R}^2$.

11. Let $p : \widetilde{M} \to M$ be a finite cover, and G the group of deck transformations. Show that $p^* : H^k(M) \to H^k(\widetilde{M})$ maps $H^k(M)$ isomorphically to $H^k_G(\widetilde{M}) = \{[\alpha] \in H^k(\widetilde{M}) : g^*[\alpha] = [\alpha] \text{ for all } g \in G\}$ (i.e. the G-invariant part of $H^k(\widetilde{M})$).

12. Show that if M^n is a connected closed smooth manifold that is not orientable then $H^n(M) = 0$.

13. Show that there is no retraction $B^n \to S^{n-1}$ (i.e. a smooth map $B^n \to S^{n-1}$ whose restriction to S^{n-1} is the identity). Prove the Brouwer fixed point theorem: any smooth map $f : B^n \to B^n$ has a fixed point.

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