Differential Topology
Assessed Coursework

Submit solutions (with cover sheet) by 4pm on Friday 24 February.

1. Let M^n be an oriented manifold without boundary. Let $X^{n-k} \subset M$ be an oriented closed submanifold (i.e. X is compact without boundary), and $Y^k \subset M$ an oriented submanifold without boundary that is closed as a subspace of M (but not necessarily compact).

   (a) Show that there is a unique element $PD(Y) \in H^{n-k}(M)$ such that $\int_Y \alpha = \int_M \alpha \wedge PD(Y)$ for any $\alpha \in \Omega^k_c(M)$ such that $d\alpha = 0$.

   (b) Show that there is a unique element $PD_c(X) \in H^k_c(M)$ such that $\int_X \alpha = \int_M \alpha \wedge PD_c(X)$ for any $\alpha \in \Omega^{n-k}(M)$ such that $d\alpha = 0$ (you may assume that the de Rham cohomology of M is finite-dimensional). Show that the image of $PD_c(X)$ under the natural map $H^k_c(M) \to H^k(M)$ is $PD(X)$.

   Recall that a real rank k *vector bundle* over X with total space E and projection map $\pi : E \to X$ is defined by *local trivialisations*: every $x \in X$ has a neighbourhood $U \subset X$ with a diffeomorphism $\psi_U : \pi^{-1}(U) \to U \times \mathbb{R}^k$ such that $p_1 \circ \psi_U = \pi$ (where $p_1 : U \times \mathbb{R}^k$ is projection to the first factor), and the *transition functions* $\psi_V \circ \psi_U^{-1} : (U \cap V) \times \mathbb{R}^k \to (U \cap V) \times \mathbb{R}^k$ have the form $(y, v) \mapsto (y, g_{UV}(y)v)$ for some smooth map $g_{UV} : U \cap V \to GL(k, \mathbb{R})$.

   Recall that X has a tubular neighbourhood T, i.e. a neighbourhood with a diffeomorphism $j : N_X \to T$ from the total space of the normal bundle of X in M, identifying $X \subset T$ with the zero section in N_X.

   (c) Let $\tau \in \Omega^k_c(N_X)$ be a representative of the Poincaré dual of X *in* N_X. Let $F = \pi^{-1}(x)$ be a fibre in N_X (note that F has a natural orientation, as it is the quotient T_xM/T_xX of oriented vector spaces). Show that $\int_F \tau = 1$, and that $j_*\tau = PD_c(X)$.
   (*Hint:* Consider $\int_{\pi^{-1}(U)} \pi^*\rho \wedge \tau$, for $\rho \in \Omega^{n-k}(X)$ supported in a trivialising neighbourhood U for N_X near x.)

   X and Y are said to intersect transversely if $T_xM = T_xX \oplus T_xY$ for every $x \in X \cap Y$. Let $\epsilon(x) = 1$ if the orientation of T_xM agrees with that on the right hand side coming from the orientations of Y and X. The intersection number of X with Y is $\sum_{x \in X \cap Y} \epsilon(x)$, which is finite since $X \cap Y$ is a discrete compact subset of X. (If $k(n-k)$ is odd then $\epsilon(x)$, and hence the intersection number, depends on the ordering of X and Y.)

   (d) If X and Y intersect transversely, show that the intersection number of X with Y equals $\int_M PD_c(X) \wedge PD(Y)$.
   (*Hint:* If the tubular neighbourhood T of X is thin enough, U is small trivialising neighbourhood for N_X near $x \in X \cap Y$, then the image of $T \cap Y$ in $U \times \mathbb{R}^k$ is a graph over $\mathbb{R}^k$.)

   [**7** marks]

2. (a) Identify sets of submanifolds of $\mathbb{R}^2 \backslash \{0\}$ whose Poincaré duals forms bases for $H^1(\mathbb{R}^2 \backslash \{0\})$ and $H^1_c(\mathbb{R}^2 \backslash \{0\})$ respectively. Is $\mathbb{R}^2 \backslash \{0\}$ diffeomorphic to the interior of a compact manifold with boundary?

   (b) Compute $H^1(\mathbb{R}^2 \backslash \mathbb{Z})$ and $H^1_c(\mathbb{R}^2 \backslash \mathbb{Z})$ (where $\mathbb{Z}$ is identified with $\{(n, 0) \in \mathbb{R}^2 : n \in \mathbb{Z}\}$). Is $\mathbb{R}^2 \backslash \mathbb{Z}$ diffeomorphic to the interior of a compact manifold with boundary?
   (*Hint:* Consider $U_n = \{(x, y) \in \mathbb{R}^2 \backslash \mathbb{Z} : \ |x - n| < \frac{3}{4}\}$.)

   [**6** marks]

3. Let M be a compact manifold and $X \subset M$ a closed submanifold. Let $U = M \setminus X$, and let $i \colon U \to M$ and $j \colon X \to M$ denote the inclusion maps.

   (a) Show that $j^* : \Omega^*(M) \to \Omega^*(X)$ is surjective. Let $\Omega^*(M, X)$ be its kernel, and $H^*(M, X)$ the cohomology of this cochain complex. Show that for any $\beta \in \Omega^k(M, X)$ such that $d\beta = 0$ on some tubular neighbourhood T of X, there is a $\gamma \in \Omega_c^{k-1}(T)$ such that the support of $\beta + d\gamma$ is contained in U. Deduce that the map $H_c^*(U) \to H^*(M, X)$ induced by the chain map $\Omega_c^*(U) \hookrightarrow \Omega^*(M, X)$ is injective. Prove that it is also surjective, and deduce that there is a long exact sequence

$$0 \to H_c^0(U) \xrightarrow{i_*} H^0(M) \xrightarrow{j^*} H^0(X) \to H_c^1(U) \to \cdots$$

   (b) Recall that a plane cubic curve $E \subset \mathbb{C}P^2$ is diffeomorphic to T^2. Compute the Betti numbers of $U = \mathbb{C}P^2 \setminus E$. What is the image of $H_c^2(U) \to H^2(U)$? (*Hint*: The Fubini-Study form $\omega \in \Omega^2(\mathbb{C}P^2)$ restricts to an orientation form on any complex curve.)

[**7** marks]

Questions and corrections to `j.nordstrom@imperial.ac.uk`.
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