Topology and Special Holonomy

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These slides available at
`http://people.bath.ac.uk/jlpn20/TopologyAndHolonomy.pdf`

Questions and overview

Focus on G_2 case.

- Which closed 7-manifolds admit metrics with holonomy G_2?
  - Finitely many??
  - Obstructions
  - Where do examples sit in classification of 7-manifolds?
- For a fixed closed M^7, the moduli space $\mathcal{M} = \{\text{holonomy } G_2 \text{ metrics on } M\}/\mathrm{Diff}(M)$ is an orbifold of dimension $b_3(M)$.
  - Global topology of $\mathcal{M}$? Connected?
  - Can the same connected component of $\mathcal{M}$ have boundary points exhibiting different degenerations?

Outline

1. Obstructions
2. Invariants, classification results and applications
3. Constructions

1. Obstructions
G_2-structures and 3-forms

First of two ways we will link G_2-structures to topology.

$G_2 \subset SO(7)$ can be defined as the stabiliser of a definite 3-form

$$\varphi_0 = dx^{123} + dx^{145} + dx^{167} + dx^{246} - dx^{257} - dx^{347} - dx^{356} \in \Lambda^3(\mathbb{R}^7)^*.$$

Therefore G_2-structure on $M^7 \leftrightarrow \varphi \in \Omega^3(M)$ pointwise equivalent to φ_0.
G_2-structure induces a metric. A metric has $\mathrm{Hol} \subseteq G_2$ if and only if it is induced by a G_2-structure that is torsion-free, *ie* satisfies

$$d\varphi = d^*\varphi = 0.$$

In particular, φ represents a de Rham cohomology class

$$[\varphi] \in H^3(M).$$

G_2-structures and spinors

Second link of G_2-structures to topology.

$Spin(7) \to SO(7)$ is a double cover, and $G_2 \hookrightarrow SO(7)$ has a lift $G_2 \hookrightarrow Spin(7)$.
The spin representation Δ of $Spin(7)$ is real of rank 8.
The image of G_2 in $Spin(7)$ is precisely the stabiliser of a non-zero $s_0 \in \Delta$ (unique up to scale).
Therefore a G_2-structure on M^7 is equivalent to

(orientation $+$) spin structure $+$ metric $+$ nowhere vanishing spinor field (up to scale)

Note: because spinor bundle of a spin M^7 has rank 8, nowhere-vanishing sections always exist.

$$M^7 \text{ admits } G_2\text{-structure} \leftrightarrow \ M \text{ is spin}$$

Topological invariants of closed spin 7-manifolds

Can we express obstructions to existence of holonomy G_2 metrics on a closed spin 7-manifold M in terms of established invariants?

Basic invariants:

- Fundamental group $\pi_1(M)$ (and higher homotopy groups)
- Cohomology algebra $H^*(M)$
- First Pontrjagin class $p_1(M) \in H^4(M)$
  (Stiefel–Whitney classes of a closed spin 7-manifold M all vanish)

Later consider more subtle invariants:

- Eells-Kuiper
- Massey triple products

Known obstructions

Let M closed 7-manifold

- M admits a G_2-structure $\Leftrightarrow$ M is orientable and spin
- If a metric has $Hol \subset G_2$, then

$$\mathrm{Hol} = G_2 \Leftrightarrow \pi_1(M) \text{ finite}$$

If φ is a torsion-free G_2-structure then

- φ is harmonic, so $b_3(M) \geq 1$.
- $\int_M p_1(M) \smile [\varphi] < 0$; in particular $p_1(M) \neq 0$.
- $\int_M x^2 \smile [\varphi] < 0$ for any non-zero $x \in H^2(M)$.

(So there is an open halfspace in $H^4(M)$ that contains both $p_1(M)$ and the image of $H^2(M) \setminus \{0\} \to H^4(M)$, $x \to x^2$.)

Constraints on $p_1(M)$ and x^2 for $x \in H^2(M)$

$\Lambda^2(\mathbb{R})^* = \Lambda^2_7(\mathbb{R})^* \oplus \Lambda^2_{14}(\mathbb{R})^*$, where

$$\Lambda^2_7(\mathbb{R})^* = \{v \lrcorner \varphi_0 : v \in \mathbb{R}^7\},$$
$$\Lambda^2_{14}(\mathbb{R})^* = \{\alpha \in \Lambda^2(\mathbb{R})^* : *\varphi \wedge \alpha = 0\}$$

If $\alpha \in \Lambda^2_{14}(\mathbb{R})^*$ then

$$\alpha^2 \wedge \varphi = -\|\alpha\|^2 \,\mathrm{vol}\,.$$

Hodge theory: If M is closed, any $x \in H^2(M)$ is represented by a harmonic $\alpha \in \Omega^2(M)$.
If M has holonomy G_2, then $\alpha \in \Omega^2_{14}(M)$. Hence

$$\int_M x^2[\varphi] = \int_M \alpha^2 \wedge \varphi = -\int_M \|\alpha\|^2 \,\mathrm{vol} < 0.$$

Chern-Weil theory: $p_1(M) = \frac{1}{8\pi^2}[\mathrm{Tr}(R \wedge R)]$, where R is curvature of M.
If M has holonomy G_2, then 2-form part of R takes values in $\Lambda^2_{14} T^*M$ so

$$\int_M \frac{1}{8\pi^2} \mathrm{Tr} R \wedge R \wedge \varphi = -\frac{1}{8\pi^2} \mathrm{Tr} \int_M \|R\|^2 \,\mathrm{vol} < 0$$

Formality and Massey products

Slogan-definition: X is formal if it is the simplest rational homotopy type with a fixed rational cohomology algebra $H^*(X;\mathbb{Q})$.

Formality forces vanishing of Massey triple products.
If $a, b, c \in H^2(M)$ such that $ab = bc = 0 \in H^4(M)$ are represented by closed forms $\alpha, \beta, \gamma \in \Omega^2(M)$, and $\eta, \tau \in \Omega^3(M)$ with

$$d\eta = \alpha \wedge \beta, \qquad d\tau = \beta \wedge \gamma$$

then

$$d(\alpha \wedge \tau + \eta \wedge \gamma) = 0.$$

The Massey triple product $\langle a, b, c \rangle \subset H^5(M)$ is the set of classes obtained for some η and τ.
If M is formal then $0 \in \langle a, b, c \rangle$.

Formality of G_2-manifolds?

Simply-connected closed manifolds of dimension ≤ 6 are always formal, but simply-connected 7-manifolds can be non-formal (provided $b_2 \geq 2$).

Theorem (Deligne-Griffiths-Morgan-Sullivan 1975)

Any closed Kähler manifold X is formal.

Proof relies on Hodge decomposition, but attempts to use Hodge decomposition on G_2-manifolds to prove formality have been unsuccessful.

Formality is largely independent of the known obstructions to G_2 metrics.

Proposition (Cavalcanti 2006, Crowley-N 2019)

If M^7 is closed simply-connected, $b_2(M) \leq 3$ and $\exists[\varphi] \in H^3(M)$ such that $[\varphi]x^2 < 0$ for all non-zero $x \in H^2(M)$ then M is formal.
There exist non-formal simply-connected M satisfying all known necessary conditions for existence of holonomy G_2 metrics with any $b_2(M) \geq 4$.

Non-fibration by 4-folds

Theorem (Baraglia 2010)

If M is closed and satisfies the known necessary condition for admitting a holonomy G_2 metric, then there is no smooth fibration $\pi : M \to B$ with smooth 4-dimensional fibres.

Proof.

Without loss of generality, M and B are simply-connected and the fibres F are connected.

By Leray-Serre, $\pi^* : H^3(B) \to H^3(M)$ is surjective, so an isomorphism since $b_3(M) \geq 1$.
Then also $H^2(M) \cong H^2(F)$. The condition that

$$H^2(M) \times H^2(M) \to \mathbb{R}, (x, y) \mapsto \int_M xy[\varphi]$$

is definite forces that the intersection form of F is definite.
By Donaldson's diagonalisation theorem, the intersection form is therefore odd.
But it is also even because F is spin, so actually $H^2(F)$ is trivial.
In particular the signature of F is trivial, and hence $p_1(F) = 0$ by the signature theorem.
As $H^4(M) \cong H^4(F)$, that contradicts $p_1(M) \neq 0$. □

Other fibrations

Have not ruled out the existence of fibrations of a G_2-manifold M by 4-manifolds if some of the fibres are allowed to be singular.

Indeed, expect many examples of closed G_2-manifolds with such fibrations by coassociative submanifolds.

Have also not ruled out smooth fibrations by 3-folds.
Indeed, consider the unit sphere bundle M_k in the total space of rank 4 vector bundle $V \to S^4$ with Euler class $e(V) = 0$ and Pontrjagin class $p_1(V) = 4k$ times a generator of $H^4(S^4)$.

These M_k are arguably the simplest 7-manifolds satisfying all the known necessary conditions for admitting a holonomy G_2 metric.

Since $b_3(M_k) = 1$, a holonomy G_2 metric on M_k would be rigid up to scale.
Therefore the existing arguments for constructing G_2 metrics cannot possibly apply.

However, *eg* $M_4 \# (S^3 \times S^4)^{\# 2n}$ does admit holonomy G_2 metrics for all $30 \leq n \leq 73$ (Corti-Haskins-N-Pacini 2014).

2. Invariants and classification
Homeomorphism classification

Let M closed 7-manifold. Focus on:

- M 2-connected, *ie* $\pi_1(M) = \pi_2(M) = 0$, because that is so far only context where we have complete classification results so far
- $H^4(M)$ torsion-free, to simplify statements

Then $p_1(M) = dx$ for some primitive $x \in H^4(M)$ and $d(M)$ divisible by 4.
(Set $d(M) := 0$ if $p_1(M) = 0$.)

Theorem (Wilkens 1972)

Closed 2-connected M are classified up to homeomorphism by the pair $(b_3(M), d(M))$.
A pair (b_3, d) is realised if and only if d is divisible by 4 (and $d = 0$ if $b_3 = 0$)

Indeed by $M_d\#(S^3 \times S^4)^{\#b_3-1}$ where M_d in the total space of the rank 4 vector bundle $V \to S^4$ with Euler class $e(V) = 0$ and Pontrjagin class $p_1(V) = 4d$ times generator of $H^4(S^4)$.

Applications to G_2-manifolds

Classification theorems for diffeomorphism or G_2-structures require further invariants. With those, we can exhibit the following phenomena.

Example 1 (Crowley-N 2018)

There are closed G_2-manifolds with $b_3 = 89$ and $d = 16$ that are homeomorphic but not diffeomorphic.

Example 2 (Crowley-Goette-N 2018, Wallis 2019)

There is a closed 7-manifold with $b_3 = 71$ and $d = 12$ that admits at least 3 different holonomy G_2 metrics, such that no two of the associated G_2-structures are related by diffeomorphism and homotopy of G_2-structures (*ie* deformation through a pth of G_2-structures). In particular, the moduli space of holonomy G_2 metrics on this manifold has at least 3 connected components.

Example 3 (Crowley-Goette-N 2018)

There is a closed 7-manifold with $b_3 = 109$ and $d = 4$ that admits two G_2-metrics whose associated G_2-structures are homotopic, but the metrics are in different components of the moduli space.

Coboundary defect invariants

Consider invariants of a class of compact manifolds with boundary that are additive under gluing boundaries. E.g. for oriented 8-manifolds W whose boundary M has $p_1(M) = 0$

- signature $\sigma(W)$ of intersection form on $H^4(W, M)$
- $p_1(W)^2 \in \mathbb{Z}$
  ($p_1(M) = 0 \Rightarrow p_1(W)$ has a preimage in $H^4(W, M)$, whose square is independent of choice)

Linear combinations that vanish for closed manifolds are then invariants of the boundary M.
Eg Hirzebruch signature theorem gives

$$45\sigma(X) + p_1(X)^2 = 7p_2(X)$$

for any closed oriented 8-manifold X, so that

$$3\sigma(X) + p_1(X)^2 \equiv 0 \mod 7.$$

Therefore

$$3\sigma(W) + p_1(W)^2 \in \mathbb{Z}/7$$

depends only on the smooth manifold M, and not on W. This invariant of M was used by Milnor (1956) to detect non-standard smooth structures on the 7-sphere.

The Eells-Kuiper invariant

For a closed spin 8-manifold X, the Atiyah singer index theorem for the index of the Dirac operator $\not{D}_X$

$$\operatorname{ind} \not{D}_X = \frac{7p_1^2 - 4p_2}{45 \cdot 2^7}$$

combined with the Hirzebruch signature theorem gives

$$\frac{p_1(X)^2 - 4\sigma(X)}{32} = 28 \operatorname{ind} \not{D}_X.$$

For a closed spin 7-manifold M with $p_1(M) = 0$ and spin coboundary W

$$\mu(M) = \frac{p_1(W)^2 - 4\sigma(W)}{32} \in \mathbb{Z}/28$$

is thus a well-defined diffeomorphism invariant.
It distinguishes all 28 classes of smooth structures on S^7.

Generalised Eells-Kuiper invariant

If $p_1(M) \neq 0$ then we cannot interpret $p_1(W)^2$ as a well-defined element of $\mathbb{Z}$.
But if $H^4(M)$ is torsion-free and $p_1(M)$ is divisible by d, then $p_1(W)^2 \in \mathbb{Z}/8\tilde{d}$ is well-defined, where $\tilde{d} := \mathrm{lcm}(8, d)$. Therefore

$$\mu(M) := \frac{p_1(W)^2 - 4\sigma(W)}{32} \in \mathbb{Z}/\gcd\left(28, \tfrac{\tilde{d}}{8}\right)$$

is a well-defined diffeomorphism invariant of M.

Theorem (Crowley-N 2019)

Closed 2-connected M with $H^4(M)$ torsion-free are classified up to diffeomorphism by $(b_3(M), d(M), \mu(M))$.

To find "exotic" G_2-manifolds as in Example 1:
generate many examples of 2-connected G_2-manifolds with torsion-free $H^4(M)$, compute invariants, and look for pairs where values b_3 and d agree while μ do not.

Invariants of G_2-structures

On a spin 8-manifold X, the spinor bundle S_X is real of rank 8.
If X is closed and $s_+ \in \Gamma(S_X)$ has transverse zeros, then $\# s_+^{-1}(0)$ (counted with signs) does not depend on s_+. It equals the Euler class $e(S_X)$, related to Euler characteristic $\chi(X)$ by

$$-3\sigma(X) + \chi(X) - 2\,\# s_+^{-1}(0) = -48 \operatorname{ind} D_X,$$

$$\frac{3p_1(X)^2 - 180\sigma(X)}{8} + 7\chi(X) - 14\,\# s_+^{-1}(0) = 0.$$

For W compact spin 8-manifold with boundary M, $\# s_+^{-1}(0)$ of $s_+ \in \Gamma(S_W)$ depends only on W and $s := s_{+|M} \in \Gamma(S_M)$. Therefore

$$\nu(M,s) := 3\sigma(X) + \chi(X) - 2\,\# s_+^{-1}(0) \in \mathbb{Z}/48,$$

$$\xi(M,s) := \frac{3p_1(W)^2 - 180\sigma(W)}{8} + 7\chi(W) - 14\,\# s_+^{-1}(0) \in \mathbb{Z}/\tfrac{3}{2}\tilde{d}$$

are well-defined diffeomorphism invariants of (M,s), *ie* of M equipped with a G_2-structure.
Also clear that ν and ξ are invariant under continuous deformation of a G_2-structure.

Classification of G_2-structures

Theorem (Crowley-N 2015)

Let M_i be closed 2-connected 7-manifolds with torsion-free $H^4(M_i)$, and G_2-structures φ_i. Then there is a diffeomorphism $f : M_1 \to M_2$ such that $f^*\varphi_2$ is homotopic to φ_1 if and only if b_3, d, ν and ξ agree.

To detect components of G_2 moduli space by homotopy of G_2-structures as in Example 2:

Generate many 2-connected G_2-manifolds with H^4 torsion-free, and compute invariants.
Look for examples where b_3 and d (and μ if not vacuous) agree, so that the G_2 metrics are on the same smooth manifold, but where different values of ν or ξ distinguish the G_2-structures.

To detect components of G_2 moduli space within the same homotopy class of G_2-structures as in Example 3:

Look for G_2-manifolds where b_3, d, ν and ξ all agree, so that by Theorem 3 we get two homotopic torsion-free G_2-structures on the same smooth manifold.
Use an analytic refinement $\hat{\nu} \in \mathbb{Z}$ of ν to show that they cannot be connected by a path of torsion-free G_2-structures.

Formality of 7-manifolds as a coboundary defect

If X is a closed oriented 8-manifold and $a, b, c, d \in H^2(X)$ then

$$(ac)(bd) - (ad)(bc) = 0.$$

If we interpret cup product as a map $\operatorname{Sym}^2 \operatorname{Sym}^2 H^2(X) \to \mathbb{R}$, that vanishes when restricted to

$$\mathcal{B} := \ker\left(\operatorname{Sym}^2 \operatorname{Sym}^2 H^2(X) \to \operatorname{Sym}^4 H^2(X)\right).$$

For a compact W^8 with $\partial W = M^7$, let $\widetilde{E} \subset \operatorname{Sym}^2 H^2(W)$ be the pre-image of

$$E := \ker(\operatorname{Sym}^2 H^2(M) \to H^4(M)).$$

Cup product and intersection form of W defines $\operatorname{Sym}^2 \widetilde{E} \to \mathbb{Q}$. Restriction to $\mathcal{B} \cap \operatorname{Sym}^2 \widetilde{E}$ factors through a

$$\mathcal{F} : \mathcal{B} \cap \operatorname{Sym}^2 E \to \mathbb{Q}$$

which is a rational homotopy invariant of M.

Theorem (Crowley-N)

A simply-connected closed 7-manifold M is formal if and only if $\mathcal{F} = 0$.

Defect invariants and the h-cobordism theorem

Strategy for finding diffeomorphism between two closed simply-connected manifolds M_1 and M_2 of dimension ≥ 5 (**Browder**, **Novikov**, **Sullivan**, **Wall**, ..., **Kreck**):

First check whether there is a cobordism, *ie* a compact W such that $\partial W = M_1 \sqcup -M_2$.

Try to use surgery to improve W to an h-cobordism, *ie* $M_i \hookrightarrow W$ homotopy equivalences.

Smale (2012): Then W is a product cylinder, so $M_1 \cong M_2$.

W has characteristic numbers such as $\sigma(W)$, $p_1(W)^2, \ldots$, unchanged by surgery.

If W has appropriate structure, the characteristic numbers are the only obstruction to improving to h-cobordism by surgery.

All defect invariants of M_1 and M_2 agree
$\Leftrightarrow$ characteristic numbers of W equal those of a closed manifold X
$\Leftrightarrow$ $W \# -X$ is a cobordism with vanishing characteristic numbers.

3. Constructions
Sources of closed G_2-manifolds

- **Joyce (1995)**
  Orbifold construction
  Resolve singularities of T^7/Γ using QALE Calabi-Yau spaces
  - **Joyce-Karigiannis (2018)**
    Resolve singularities of $(CY^3 \times S^1)/\mathbb{Z}_2$
- **Kovalev (2003)**, **Corti-Haskins-N-Pacini (2014)**
  Twisted connected sums
  Glue asymptotically cylindrical Calabi-Yaus $\times S^1$
  - **Crowley-Goette-N (2018)**
    Extra-twisted connected sums
- **Foscolo-Haskins-N (202X)**
  Collapse to orientifolds
  Bulk: Circle bundle over $CY^3/\mathbb{Z}_2$
  Degenerations of bundle modelled on fibrations by Taub-NUT or Atiyah-Hitchin spaces.

Twisted connected sums

Ingredients:

- Closed simply-connected Kähler 3-folds Z_+, Z_-
- $\Sigma_\pm \subset Z_\pm$ anticanonical K3 divisors ($[\Sigma_\pm] = c_1(Z_\pm)$) with trivial normal bundle
- $\mathfrak{r} : \Sigma_+ \to \Sigma_-$ diffeomorphism

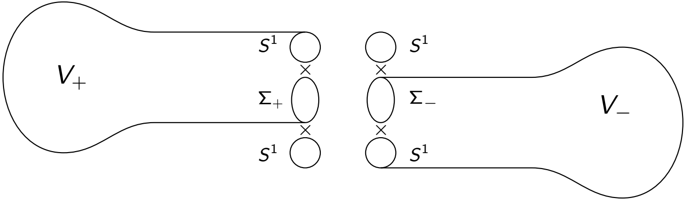
Let $V_\pm := Z_\pm \setminus$ tubular neighbourhood $\Sigma_\pm \times \Delta$; so $\partial V_\pm = \Sigma_\pm \times S^1$.
Form simply-connected M^7 by gluing boundaries of $V_+ \times S^1$ to $V_- \times S^1$ by

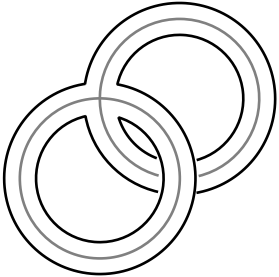
$$\Sigma_+ \times S^1 \times S^1 \to \Sigma_- \times S^1 \times S^1,$$
$$(x, u, v) \mapsto (\mathfrak{r}(x), v, u)$$

Tian-Yau, Haskins-Hein-N: $V_\pm$ admits asymptotically cylindrical Calabi-Yau metrics.
$\rightsquigarrow$ metric on $V_\pm \times S^1$ with holonomy $SU(3) \subset G_2$.
For carefully chosen $\mathfrak{r}$, these metrics glue to a holonomy G_2 metric on M.

Coboundary of twisted connected sum

$V_\pm := Z_\pm \setminus \Sigma_\pm \times \Delta$. Glue the $\partial(V_+ \times S^1) = \Sigma_+ \times S^1 \times S^1$ by $(x, u, v) \mapsto (\mathrm{r}(x), v, u)$.





Form an 8-manifold W by gluing $Z_+ \times \Delta$ to $Z_- \times \Delta$ along open subsets

$$\Sigma_+ \times \Delta \times \Delta \to \Sigma_- \times \Delta \times \Delta,$$
$$(x, z, w) \mapsto (\mathrm{r}(x), w, z)$$

Then ∂W is the twisted connected sum M.

Invariants of twisted connected sums

Twisted connected sum M is simply-connected.

$H^*(M)$ can be computed in terms of $H^*(Z_\pm)$, $c_2(Z_\pm)$ and $\mathfrak{r}^* : H^2(\Sigma_-) \to H^2(\Sigma_+)$.

While W is not spin but only spin^c, it can still be used to compute μ, from the same data.

Example 1 (Crowley-N 2018) There are 2-connected twisted connected sums with $H^4(M)$ torsion-free, $b_3(M) = 89$ and $d(M) = 16$, and $\mu = 0, 1 \in \mathbb{Z}/2$.

We cannot use ν or its analytic refinement to distinguish components of the G_2 moduli space reached by twisted connected sums.

Theorem (Crowley-N 2015, Crowley-Goette-N 2018)

Any twisted connected sum G_2-manifold has $\nu = 24$, and $\bar{\nu} = 0$.

But W can also be used to compute ξ.

Example 2a (Wallis 2019) There are 2-connected twisted connected sums with $H^4(M)$ torsion-free, $b_3(M) = 71$ and $d(M) = 12$, and $\xi = 0, 12 \in \mathbb{Z}/36$.

Extra-twisted connected sums

Extra-twisted connected sums: gluing of finite quotients of $V_+ \times S^1$ and $V_- \times S^1$.

We do not know coboundaries, but can compute by eta invariants instead.

Example 2b (Crowley-Goette-N 2018)

There is a 2-connected extra-twisted connected sum with torsion-free $H^4(M)$, $b_3(M) = 71$ and $d = 12$, and $\nu = 36$.

Thus M is diffeomorphic to manifold from Example 2a, but the torsion-free G_2-structure is distinguished from both the previous G_2-structures, which have $\nu = 0$.

Example 3 (Crowley-Goette-N 2018)

There is both a twisted connected sum and an extra-twisted connected sum with that are 2-connected with torsion-free $H^4(M)$, $b_3(M) = 109$ and $d = 4$, and the extra-twisted connected sum has $\bar{\nu} = 48$.

Because both torsion-free G_2-structures have $\nu = 24$ (and ξ is vacuous when $d = 4$) the manifolds are diffeomorphic, and moreover the diffeomorphism can be chosen so that the torsion-free G_2-structures are homotopic.

Need for further classification results

Twisted connected sums generate many 2-connected examples, but also many with $b_2 \geq 1$.

Examples from Joyce's orbifold construction (nearly) always have $b_2 \geq 1$, as do the tentative collapsing examples.

Existing complete classification results for 7-manifolds that are not 2-connected impose $b_3 = 0$, so cannot apply to G_2-manifolds.

Good news: only finitely many invariants missing.

Theorem (Crowley-N 2019)

Closed simply-connected 7-manifolds M are classified up to finitely many diffeomorphism types by $H^*(M)$, $p_1(M)$ and the rational homotopy invariant $\mathcal{F}$.

~~Bad~~ Interesting news: While some of the remaining finite ambiguity is accounted for by friendly primary invariants like torsion linking form, there will also be subtle secondary invariants.

Further invariants when $b_2 \geq 1$

Beyond 2-connected 7-manifolds, existing classification results require $\pi_2(M)$ torsion-free and $H^4(M)$ finite (+ simplifying assumptions)

Kreck-Stolz (1991)

For $\pi_2 M \cong \mathbb{Z}$ and $b_4(M) = 0$, the secondary data needed is the Eells-Kuiper invariant μ and an additional invariant taking values in a $\mathbb{Z}_{12} \times \mathbb{Z}_2$ coset.

Motivated by metrics of positive sectional curvature on Aloff-Wallach spaces $SU(3)/U(1)$.

Hepworth (2005)

For $\pi_2 M \cong \mathbb{Z}^r$ and $b_4(M) = 0$, the secondary data needed is $\mathcal{F}$, μ and an invariant taking values in a $\mathbb{Z}_{12}^r \times \mathbb{Z}_6^{\frac{r(r-1)}{2}} \times \mathbb{Z}_2^{\frac{r^3-6r^2+11r}{6}}$-coset.

Motivated by 3-Sasakian manifolds.

Wallis (2019): Analysis of which Hepworth-Kreck-Stolz invariants survive when $H^4(M)$ infinite.