Massey tensors and the rational homotopy type of 7- and 8-manifolds

Johannes Nordström

3 May 2023

Joint work with Diarmuid Crowley and Csaba Nagy
arXiv:1505.04184, arXiv:2105.13660

These slides available at
`http://people.bath.ac.uk/jlpn20/masseyt.pdf`

Overview

Massey products are a basic tool when studying rational homotopy of spaces/quasi-isomorphism of commutative differential graded algebras, but sometimes complicated to use due to dependence on choices.

Multiplying a Massey product by a further element can reduce the dependence on choices, and under Poincaré duality does not lose any information.

For Massey triple and fourfold products we can relate the result to tensors that capture *more* information than the original Massey products, often enough to completely determine rational homotopy class or at least formality.

1. Background
2. Triple products, Bianchi-Massey tensor and simply-connected 7-manifolds
3. Fourfold products, pentagonal Massey tensor and simply-connected 8-manifolds
4. Dependence on choices

1. Background
Rational homotopy

$f : X \to Y$ is a rational homotopy equivalence map if
$f_* : \pi_i(X) \otimes \mathbb{Q} \to \pi_i(Y) \otimes \mathbb{Q}$ are isomorphisms.

X is rationally homotopy equivalent to Y if they can be connected by a zig-zag of rational homotopy equivalence maps.

Rational homotopy equivalence of spaces translates to quasi-isomorphism of commutative differential graded algebras.

Manifold M $\rightsquigarrow$ de Rham complex $\Omega^*(M)$ of diff forms $\rightsquigarrow$ $H^*_{dR}(M)$
Simplicial space X $\rightsquigarrow$ piecewise linear de Rham complex $\rightsquigarrow$ $H^*(X;\mathbb{Q})$.

A CDGA homomorphism $\phi : \mathcal{A} \to \mathcal{B}$ is a quasi-isomorphism if
$\phi_\# : H^*(\mathcal{A}) \to H^*(\mathcal{B})$ is an isomorphism.

Formality

The simplest CDGA with a given cohomology algebra is that algebra itself regarded as a CDGA with trivial differential.

Call a CDGA formal if it is quasi-isomorphic to its cohomology algebra. Closed simply-connected manifolds of dimension ≤ 6 are always formal.

Theorem (Deligne-Griffiths-Morgan-Sullivan)

Any closed Kähler manifold is formal (so "its rational homotopy type is a formal consequence of its cohomology").

What about closed 7-manifolds with holonomy G_2 or 8-manifolds with holonomy $Spin(7)$??

Additional structures on the cohomology can be used to detect non-formality, or more generally to distinguish spaces up to rational homotopy equivalence.

I am especially interested in finding invariants capable of classifying closed simply-connected 7- and 8-manifolds up to rational homotopy equivalence.

Massey triple products

Let $\mathcal{A}$ be a commutative differential graded algebra.

For $x_1, x_2, x_3 \in H^2(\mathcal{A})$ such that $x_1x_2 = x_2x_3 = 0$ we can define a Massey triple product as follows.

Pick representatives $\alpha_i \in \mathcal{A}^2$ of x_i.

By hypothesis, $\alpha_1\alpha_2$ and $\alpha_2\alpha_3$ are exact, say

$$\alpha_1\alpha_2 = d\gamma_{12}, \quad \alpha_2\alpha_3 = d\gamma_{23}$$

for some $\gamma_{ij} \in \mathcal{A}^3$.

Then $\alpha_1\gamma_{23} - \gamma_{12}\alpha_3$ is closed, so defines an element of $H^5(\mathcal{A})$.

Changing choice of γ_{12} can change the result by adding an element of $x_3H^3(\mathcal{A})$. But if we let

$$\langle x_1, x_2, x_3\rangle \subseteq H^5(\mathcal{A})$$

be the set of classes that can be obtained by any choices then that is a well-defined $x_1H^3(\mathcal{A}) + x_3H^3(\mathcal{A})$ coset.

2. The Bianchi-Massey tensor
Tensorifying the triple products

If $x_i \in H^2(\mathcal{A})$ such that $x_1x_2 = x_2x_3 = x_3x_4 = x_4x_1 = 0$ then

$$m(x_1, x_2, x_3, x_4) := \langle x_1, x_2, x_3 \rangle x_4 \in H^7(\mathcal{A})$$

does not depend on the choices.

On the other hand, if $\mathcal{A}$ satisfies 7-dimensional Poincaré duality, then $\langle x_1, x_2, x_3 \rangle$ can be recovered from knowing $m(x_1, x_2, x_3, x_4)$ for all x_4.

If all products of classes in $H^2(\mathcal{A})$ vanish in $H^4(\mathcal{A})$, then we can think of m as defining a linear map

$$H^2(\mathcal{A})^{\otimes 4} \to H^7(\mathcal{A}).$$

But this does not work very well if the product on $H^2(\mathcal{A})$ is non-trivial, because the conditions $x_1x_2 = x_2x_3 = x_3x_4 = x_4x_1 = 0$ do not define a linear subspace of $H^2(\mathcal{A})^{\otimes 4}$.

Better tensor

Notation:

- $P^k V :=$ k^{th} symmetric power of vector space V
- $E(\mathcal{A}) := \ker\left(P^2 H^2(\mathcal{A}) \to H^4(\mathcal{A})\right)$

Think of m as composition of two maps.

- $b : \{x_1x_2 = x_2x_3 = x_3x_4 = x_4x_1 = 0\} \to P^2 E$
  $$(x_1, x_2, x_3, x_4) \mapsto (x_1x_2)(x_3x_4) - (x_2x_3)(x_4x_1)$$
  b takes values in the kernel $K[P^2 E]$ of full symmetrisation $P^2 E \to P^4 H^2(\mathcal{A})$.
- Given "uniform choice of representatives" $\alpha : H^2(\mathcal{A}) \to \mathcal{A}^2$ and $\gamma : E \to \mathcal{A}^3$ such that $\alpha^2_{|E} = \gamma$, consider

  $$P^2 E \to \mathcal{A}^7,\ uv \mapsto \gamma(u)\alpha^2(v).$$

  Differential is α^4, so factors through $P^4 H^2(\mathcal{A})$, and vanishes on $K[P^2 E]$. Thus it induces a linear map

  $$\mathcal{F} : K[P^2 E] \to H^7(\mathcal{A}).$$

  This is independent of choices! We call it the *Bianchi-Massey tensor*.

Features of the Bianchi-Massey tensor

If the image of b spans $K[P^2E]$ then the Bianchi-Massey tensor $\mathcal{F} : K[P^2E] \to H^7(\mathcal{A})$ is completely determined by the Massey triple products.

Under Poincaré duality $\mathcal{F}$ determines all Massey triple products.

Functoriality: For a CDGA homomorphism $\phi : \mathcal{A} \to \mathcal{B}$ and $w \in K[P^2E(\mathcal{A})]$

$$\mathcal{F}_{\mathcal{B}}(\phi_{\#} w) = \phi_{\#} \mathcal{F}_{\mathcal{A}}(w) \in H^7(\mathcal{B}).$$

Obstruction to formality: $\mathcal{A}$ formal $\Rightarrow$ $\mathcal{F} = 0$.

Classification

Theorem

The quasi-isomorphism type of a simply-connected CDGA $\mathcal{A}$ *(i.e.* $H^0(\mathcal{A}) = \mathbb{Q}$, $H^1(\mathcal{A}) = 0$*) with 7-Poincaré duality is determined by isomorphism class of its cohomology algebra and* $\mathcal{F}_{\mathcal{A}}$*.*

Same is true for $(n-1)$-connected m-Poincaré duality CDGAs for $m \leq 5n - 3$; just need to use graded commutative powers instead of P^k.

Selling points for the Bianchi-Massey tensor

- Completely independent of choices
- Computable
- Symmetries transparent, so clear precisely how many components need to be computed (*i.e.* $\dim K[P^2E]$)

3. The pentagonal Massey tensor
Massey fourfold products

Fourfold Massey product of $x_1, x_2, x_3, x_4 \in H^2(\mathcal{A})$ defined if $x_1 x_2 = x_2 x_3 = x_3 x_4 = 0 \in H^4(\mathcal{A})$ *and* one can choose $\alpha_i \in \mathcal{A}^2$, $\gamma_{ij} \in \mathcal{A}^3$ so that the resulting representatives of $\langle x_1, x_2, x_3 \rangle$ and $\langle x_2, x_3, x_4 \rangle$ both vanish. Then for $\sigma_{123}, \sigma_{234} \in \mathcal{A}^4$ such that

$$\alpha_1 \gamma_{23} - \gamma_{12} \alpha_3 = d\sigma_{123}, \quad \alpha_2 \gamma_{34} - \gamma_{23} \alpha_4 = d\sigma_{234}$$

we get a representative

$$-\alpha_1 \sigma_{234} + \gamma_{12} \gamma_{34} - \sigma_{123} \alpha_4$$

of the fourfold product $\langle x_1, x_2, x_3, x_4 \rangle \subseteq H^6(\mathcal{A})$.

Dependence on choices more complicated now, but if we fix the choices of α_i and γ_{ij} (or if $H^3(\mathcal{A}) = 0$) then the remaining ambiguity is a coset for $x_1 H^4(\mathcal{A}) + x_4 H^4(\mathcal{A})$.

Tensor

If $x_i \in H^2(\mathcal{A})$ with $x_1x_2 = x_2x_3 = x_3x_4 = x_4x_5 = x_5x_1 = 0$ then choosing representatives $\alpha_i \in \mathcal{A}^2$ and $\gamma_{ij} \in \mathcal{A}^3$ as above, the product of the representative of $\langle x_1, x_2, x_3, x_4, x_5 \rangle$ in $H^6(\mathcal{A})$ with x_5 is

$$\begin{aligned}
[-\alpha_1\sigma_{234} + \gamma_{12}\gamma_{34} - \sigma_{123}\alpha_4][\alpha_5] &= [-(d\gamma_{51})\sigma_{234} + \gamma_{12}\gamma_{34}\alpha_5 - \sigma_{123}d\gamma_{45}] \\
&= [-\gamma_{51}d\sigma_{234} + \gamma_{12}\gamma_{34}\alpha_5 + (d\sigma_{123})\gamma_{45}] \\
&= \left[\sum_{\text{cyc}} \alpha_1\gamma_{23}\gamma_{45}\right] \in H^8(\mathcal{A})
\end{aligned}$$

Think of the representative on the right hand side as composition of

- $\star : \{x_1x_2 = x_2x_3 = x_3x_4 = x_4x_5 = x_5x_1 = 0\} \to H^2(\mathcal{A}) \otimes \Lambda^2 E$
$$(x_1, x_2, x_3, x_4, x_5) \mapsto \sum_{\text{cyc}} x_1(x_2x_3) \wedge (x_4x_5)$$
- Given $\alpha : H^2(\mathcal{A}) \to \mathcal{A}^2$ and $\gamma : E \to \mathcal{A}^3$ as in definition of $\mathcal{F}$, consider

$$H^2(\mathcal{A}) \otimes \Lambda^2 E \to \mathcal{A}^8, \; x(u \wedge v) \mapsto \alpha(x)\gamma(u)\gamma(v).$$

Pentagonal symmetries

If the product $P^2H^2(\mathcal{A}) \to H^4(\mathcal{A})$ is trivial, so that $E = P^2H^2(\mathcal{A})$, then ☆ induces a linear map

$$☆ : H^2(\mathcal{A})^{\otimes 5} \to H^2(\mathcal{A}) \otimes \Lambda^2 P^2 H^2(\mathcal{A}).$$

Its image equals the kernel of

$$\begin{aligned} p : H^2(\mathcal{A}) \otimes \Lambda^2 P^2 H^2(\mathcal{A}) &\to P^3 H^2(\mathcal{A}) \otimes P^2 H^2(\mathcal{A}) \\ q \otimes (xy \wedge zw) &\mapsto qxy \otimes zw - qzw \otimes xy \end{aligned}$$

This is an irreducible representation of $GL(H^2(\mathcal{A}))$ of dimension $6\binom{r+2}{5}$, corresponding to the partition (3,1,1).

When the product $P^2H^2(\mathcal{A}) \to H^4(\mathcal{A})$ is not trivial, we instead focus on

$$\mathcal{D} := (\ker p) \cap H^2(\mathcal{A}) \otimes \Lambda^2 E.$$

The pentagonal Massey tensor

Given a "uniform choice of representatives" $\alpha : H^2(\mathcal{A}) \to \mathcal{A}^2$ and $\gamma : E \to \mathcal{A}^3$ such that $\alpha^2_{|E} = d\gamma$, the restriction of

$$H^2(\mathcal{A}) \otimes \Lambda^2 E \to \mathcal{A}^8,\ x(u \wedge v) \mapsto \alpha(x)\gamma(u)\gamma(v).$$

to $\mathcal{D} := \ker\left(p : H^2(\mathcal{A}) \otimes \Lambda^2 E \to P^3 H^2(\mathcal{A}) \otimes P^2 H^2(\mathcal{A})\right)$ takes closed values, and defines

$$\mathcal{P} : \mathcal{D} \to H^8(\mathcal{A}).$$

In general $\mathcal{P}$ still depends on the choices of α and γ, but

- $\mathcal{P}$ is independent of choices if $H^3(\mathcal{A}) = 0$
- Poincaré duality $+$ $\mathcal{F} = 0$ $\Rightarrow$ canonical class of choices yielding unique value of $\mathcal{P}$.

Formality obstruction:

- On a formal CDGA there must exist choices of α and γ that make $\mathcal{P} = 0$.
- For a formal CDGA with Poincaré duality $\mathcal{F} = \mathcal{P} = 0$.

Classification?

Theorem (Nagy-N)

For $m \leq 5n-2$, any $(n-1)$-connected m-Poincaré duality CDGA with $\mathcal{F} = \mathcal{P} = 0$ is formal.

Conjecture

For $m \leq 5n-2$, $(n-1)$-connected m-Poincaré duality CDGA with $\mathcal{F} = 0$ are classified up to quasi-isomorphism by the cohomology algebra and $\mathcal{P}$.

(Statement without $\mathcal{F} = 0$ requires explaining functoriality of $\mathcal{P}$, and first clarifying the dependence on choices.)

Theorem (Nagy)

Simply-connected 8-manifolds with isomorphism $H_2 \cong \mathbb{Z}^r$ and $H_3 = H_4 = 0$ form a group under parametric connected sum, isomorphic to $\mathbb{Z}^a \times (\mathbb{Z}/2)^b$ for

$$a = 6\binom{r+2}{5}, \quad b = 2\binom{r+3}{4} - \binom{r-1}{2} + 2.$$

$\mathcal{P}$ "detects" the free part of this group.

4. Dependence on choices
Classes of choices

A complete picture of the pentagonal Massey tensor requires details of the dependence on the choice of a pair $c = (\alpha, \gamma)$ such that

- $\alpha : H^2(\mathcal{A}) \to \mathcal{A}^2$ picks out a representative of each cohomology class
- $\gamma : E \to \mathcal{A}^3$ satisfies $d\gamma = \alpha^2_{|E}$

where E is the kernel of the algebra product $P^2 H^2(\mathcal{A}) \to H^4(\mathcal{A})$.

Given two choices c, c' we can define a "difference" $\delta : E \to H^3(\mathcal{A})$:
Pick $\beta : H^2(\mathcal{A}) \to \mathcal{A}^1$ such that $d\beta = \alpha' - \alpha$.
Then $\gamma' - \gamma - \beta(\alpha + \frac{1}{2} d\beta)$ maps E to closed elements of $\mathcal{A}^3$, so defines a map $\delta : E \to H^3(\mathcal{A})$.

This δ itself depends on choice of β, but we call c and c' equivalent if there is some choice of β such that $\delta = 0$.
If c and c' are equivalent, then all invariants we define in terms of those choices will agree.

Uniform triple products

The dependence of the pentagonal Massey tensor on choices is expressed in terms of some "triple product like" information that is not quite captured by the Bianchi-Massey tensor, but by a "uniform" version of the triple product defined using similar ideas.

Let $K[H^2(\mathcal{A}) \otimes E]$ be the kernel of symmetrisation $H^2(\mathcal{A}) \otimes E \to P^3 H^2(\mathcal{A})$. Given $c = (\alpha, \gamma)$, the restriction of $H^2(\mathcal{A}) \otimes E \to \mathcal{A}^5,\ xu \mapsto \alpha(x)\gamma(u)$ to $K[H^2(\mathcal{A}) \otimes E]$ takes closed values, so induces a map

$$\mathcal{T}_c : K[H^2(\mathcal{A}) \otimes E] \to H^5(\mathcal{A}).$$

If the "difference" between c and c' is $\delta : E \to H^3(\mathcal{A})$ then $\mathcal{T}_{c'} - \mathcal{T}_c$ is the restriction of the product of $\mathrm{Id} : H^2(\mathcal{A}) \to H^2(\mathcal{A})$ and δ.

Under Poincaré duality, the set of values of $\mathcal{T}_c$ is determined by $\mathcal{F}$ (in particular $\mathcal{T}_c = 0$ for some c iff $\mathcal{F} = 0$), but the transformation rule for $\mathcal{P}$ depends on the actual map $c \mapsto \mathcal{T}_c$ and not just its image.

Transformation rule for $\mathcal{P}$

Recall that $\mathcal{P}$ is defined on $\mathcal{D} := \ker\left(H^2(\mathcal{A}) \otimes \Lambda^2 E \to P^3 H^2(\mathcal{A}) \otimes E\right)$.
The natural map $H^2(\mathcal{A}) \otimes \Lambda^2 E \to H^2(\mathcal{A}) \otimes E \otimes E$ maps $\mathcal{D}$ to a subspace of $K[H^2(\mathcal{A}) \otimes E] \otimes E$.
For $T : K[H^2(\mathcal{A}) \otimes E] \to H^5(\mathcal{A})$ and $\delta : E \to H^3(\mathcal{A})$, let

$$T \owedge \delta : \mathcal{D} \to H^8(\mathcal{A})$$

be the restriction of the product $T\delta : K[H^2(\mathcal{A}) \otimes E] \otimes E \to H^8(\mathcal{A})$ to $\mathcal{D}$.

Proposition

The difference between the pentagonal Massey tensors defined with two choices c and c' differing by $\delta : K[H^2(\mathcal{A}) \otimes E] \to H^3(\mathcal{A})$ is

$$\mathcal{P}_{c'} - \mathcal{P}_c = \mathcal{T}_c \owedge \delta + \mathrm{Id}\delta^2.$$

- If $\mathcal{T}_c = 0$ for some c, then $\mathcal{P}_c$ takes the same value for all such c.
- Formality $\Rightarrow$ $\mathcal{T}_c = \mathcal{P}_c = 0$ for some c.
- $\mathcal{P}_{c+\delta+\epsilon} - \mathcal{P}_{c+\delta} - \mathcal{P}_{c+\epsilon} + \mathcal{P}_c = \mathrm{Id}\,\delta\wedge\epsilon$ is bilinear in δ and ϵ and independent of c: "affine quadratic" dependence.

Formal domination

Functoriality:

For a homomorphism $\phi : \mathcal{A} \to \mathcal{B}$ and choices $c_\mathcal{A}$ and $c_\mathcal{B}$, there is an analogous comparison $\delta : E_\mathcal{A} \to H^3(\mathcal{B})$. Then

$$\mathcal{P}_{c_\mathcal{B}} \circ \phi_\# - \phi_\# \circ \mathcal{P}_{c_\mathcal{A}} = (\phi_\# \circ \mathcal{T}_{c_\mathcal{A}}) \owedge \delta + (\phi_\#)\delta^2.$$

Corollary

Let $f : M \to N$ be a map of non-zero degree between closed manifolds. If $\mathcal{F}_M = \mathcal{P}_M = 0$ then $\mathcal{F}_N = \mathcal{P}_N = 0$.

In particular, if N is $(n-1)$-connected of dimension $\leq 5n-2$ and M is formal then N is formal too.