Complete cohomogeneity one solitons for G_2 Laplacian flow

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These slides available at
http://people.bath.ac.uk/jlpn20/g2sol4.pdf

Introduction

Context: Riemannian 7-manifolds with holonomy group G_2,
a special kind of Ricci-flat manifolds

Bryant's Laplacian flow: a cousin of Ricci flow for closed G_2-structures

G_2 solitons: self-similar solutions to Laplacian flow

We have found asymptotically conical G_2 solitons of cohomogeneity one on $\Lambda^2_- \mathbb{C}P^2$ and $\Lambda^2_- S^4$, of all three types (shrinker, expander and steady), as well as complete solitons with different end behaviours.

Outline

1. Laplacian flow, solitons, and AC G_2-structures
2. Main existence results
3. Invariant soliton ODE and singular initial value problem
4. Forward-completeness

1. Laplacian flow, solitons, and AC G_2-structures Riemannian holonomy G_2

$G_2 := \mathrm{Aut}\,\mathbb{O}$, $\quad \mathbb{O} =$ octonions, normed division algebra of real dimension 8.

G_2 acts on $\mathrm{Im}\,\mathbb{O} \cong \mathbb{R}^7$, preserving metric, orientation, cross product

G_2 is the stabiliser in $GL(7, \mathbb{R})$ of a stable $\varphi_0 \in \Lambda^3(\mathbb{R}^7)^*$
(*i.e.* the $GL(7, \mathbb{R})$-orbit of φ_0 is open).

$\varphi \in \Omega^3(M^7)$ pointwise equivalent to φ_0 defines a G_2-structure.

Because $G_2 \subset SO(7)$, such a φ induces a metric and orientation.

$Hol(M) \subseteq G_2 \Leftrightarrow$ metric induced by some G_2-structure φ such that $\nabla\varphi = 0$.
Then call φ *torsion-free*. This is equivalent to the first-order non-linear PDE

$$d\varphi = d^*\varphi = 0.$$

Metrics with holonomy G_2 are always Ricci-flat.

All known constructions of examples on closed manifolds (**Joyce 1994**...) solve the elliptic PDE by gluing together pieces with dimensional reduction.

Bryant's Laplacian flow

Solve

$$\frac{d\varphi_t}{dt} = \Delta_{\varphi_t}\varphi_t$$

with initial condition φ_0 satisfying $d\varphi_0 = 0$. (Then $d\varphi_t = 0$ for all t.)

Starting observations

- $d\varphi = dd^*\varphi = 0 \Rightarrow d^*\varphi = 0$, so the stationary points are exactly the torsion-free G_2-structures.
- *Upward* gradient flow for $\mathrm{vol}(\varphi)$ restricted to the cohomology class of φ_0 (the Hitchin functional)
- Induced metric evolves by

$$\frac{dg_t}{dt} = -2\mathrm{Ric}(g_t) + \text{ terms quadratic in the torsion } d^*\varphi_t$$

(while $\mathrm{Ric}(g_t)$ is linear in $d^*\varphi_t$)

What do we know?

Theorem (Bryant-Xu 2011, Lotay-Wei 2017)

- *Short-time existence and uniqueness.*
- *The stationary points are stable:*
  initial conditions close to a torsion-free G_2-structure φ_0 lead to flow defined for all time, with limit isomorphic to φ_0.

Lotay-Wei also show that the Riemann curvature or gradient of torsion must blow up at a finite-time singularity, but little is known about the long-term behaviour in general.

Absent an analogue of Perelman's no-local-collapse theorem for Ricci flow it is not known how to obtain blow-up models for singularities.

Nevertheless, solitons for the flow should play a role in the eventual picture...

G_2 soliton equations

G_2-structure φ, vector field X, dilation constant $\lambda \in \mathbb{R}$ satisfying

$$\begin{cases} d\varphi = & 0, \\ \Delta_\varphi \varphi = & \lambda\varphi + \mathcal{L}_X \varphi. \end{cases}$$

$\Leftrightarrow$ self-similar solution of Laplacian flow

$$\varphi_t = k(t)^3 f^* \varphi, \qquad \frac{df}{dt} = k(t)^{-2} X, \qquad k(t) = \frac{3 + 2\lambda t}{3}$$

$\lambda > 0$: expanders (immortal solutions)
$\lambda = 0$: steady solitons (eternal solutions)
$\lambda < 0$: shrinkers (ancient solutions)

- Non-steady soliton $\Rightarrow \varphi$ exact
- Scaling behaviour: (φ, X) is a λ-soliton $\Leftrightarrow (k^3\varphi, k^{-1}X)$ is a $k^{-2}\lambda$-soliton.

Asymptotically conical solitons

Solitons for the Laplacian flow should play a role in the eventual picture...
...but not compact ones.

- There can be no compact shrinkers, because the Laplacian flow is the upward gradient flow for volume.
- Any compact steady soliton must be static (φ torsion-free, $X = 0$)
- No known examples of compact expanders (or even exact G_2-structures)

A natural class of non-compact solitons that has been studied in Ricci flow and mean curvature flow is that of asymptotically conical ones.

- AC shrinkers provide models for the formation of an isolated conical finite-time singularity
- AC expanders provide models for resolving an isolated conical singularity
- A shrinker and an expander whose asymptotic cones match provide a model for “flowing through” a singularity.

Asymptotic cones of Laplacian solitons

$SU(3)$-structure on Σ^6
$\leftrightarrow$ Hermitian 2-form $\omega \in \Omega^2(\Sigma)$ and (normalised) real part $\alpha \in \Omega^3(\Sigma)$ of complex 3-form with respect to some almost complex structure
$\leftrightarrow$ conical G_2-structure φ_C on $\mathbb{R}_{>0} \times \Sigma$.

$$\varphi_C = r^2 dr \wedge \omega + r^3 \alpha \rightsquigarrow g_C = dr^2 + r^2 g_\Sigma$$

φ_C is torsion-free if the $SU(3)$-structure (ω, α) is "nearly Kähler".

For $X = -\frac{\lambda r}{3} \frac{\partial}{\partial r}$

$$\mathcal{L}_X \varphi_C = -\lambda \varphi_C,$$

so if φ_C is torsion-free then (φ_C, X) is a "Gaussian" λ-soliton.

If merely $d\varphi_C = 0$, then because $\Delta_\varphi \varphi$ has lower order (provided $\lambda \neq 0$), (φ_C, X) is still a sensible asymptotic model for an AC λ-soliton.

But for $\lambda = 0$ the only reasonable asymptotic cones are static, *i.e.* φ_C torsion-free and $X = 0$.

Bryant-Salamon AC G_2 metrics

Where to look for Laplacian solitons (M, φ) that are asymptotically conical, *i.e.* $M \setminus (\text{compact set}) \cong \mathbb{R}_+ \times \Sigma^6$ and, for some "asymptotic rate" $\nu < 0$,

$$\varphi = r^2 dr \wedge \omega + r^3 \alpha + O(r^\nu)?$$

Try spaces with well-known AC torsion-free G_2-structures (ideally exact).

Bryant-Salamon (1987) found the first examples of complete G_2 metrics.

These examples are AC, and moreover they have a cohomogeneity 1 action by a group G, *i.e.* the generic orbit Σ has dimension 6.

M	G	Σ	ν
$\Lambda^2_- S^4$	$Sp(2)$	$\mathbb{C}P^3$	-4
$\Lambda^2_- \mathbb{C}P^2$	$SU(3)$	$SU(3)/T^2$	-4
$S^3 \times \mathbb{R}^4$	$SU(2)^3$	$S^3 \times S^3$	-3

Remark: In the last two cases, Σ has a $\mathbb{Z}_3$ of automorphisms that do not extend to diffeomorphisms of M

$\rightsquigarrow$ G_2 conifold transitions: 3 topologically distinct ways to glue in zero section to resolve conical singularity $\mathbb{R}_{>0} \times \Sigma$.

Invariant G_2-structures on $\Lambda^2_- S^4$ and $\Lambda^2_- \mathbb{C}P^2$

$Sp(2)$-invariant G_2-structures φ on $\Lambda^2_- S^4 \setminus \text{zero section} \cong \mathbb{R}_{>0} \times \mathbb{C}P^3$

$\leftrightsquigarrow f_1, f_2 : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$

$\leftrightarrow$ scale of base and S^2 fibres of $\mathbb{C}P^3 \to S^4$.

$SU(3)$-invariant G_2-structures on $\Lambda^2_- \mathbb{C}P^2 \setminus \text{zero section} \cong \mathbb{R}_{>0} \times SU(3)/T^2$

$\leftrightsquigarrow f_1, f_2, f_3 : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$

$\leftrightarrow$ scale of S^2 fibres of three different fibrations $SU(3)/T^2 \to \mathbb{C}P^2$.

Cones: G_2-structure φ_C defined by $f_i = c_i r$.

Closed cones: $d\varphi_C = 0 \ \Leftrightarrow\ $ scale-normalisation of c

Unique holonomy G_2 cone: $f_1 = f_2 = f_3 = \frac{r}{2}$

The Bryant-Salamon G_2 metrics on $\Lambda^2_- S^4$ and $\Lambda^2_- \mathbb{C}P^2$ are defined by the same f_1 and f_2, with $f_3 = f_2$ in the latter case. Each $\frac{f_i}{r} \to \frac{1}{2}$ as $r \to \infty$.

Anti-self-dual bundle construction

Multiplication by -1 on the fibres of $\Lambda^2_- S^4$ is an isometry of any $Sp(2)$-invariant G_2-structure.

Multiplication by -1 on the fibres of $\Lambda^2_- \mathbb{C}P^2$ is an isometry of the $SU(3)$-invariant G_2-structure defined by $(f_1, f_2, f_3) \Leftrightarrow f_2 = f_3$

The equations encoding the torsion-free condition for $SU(3) \times \mathbb{Z}_2$-invariant and $Sp(2)$-invariant G_2-structures are identical.

Indeed, Bryant-Salamon derived the same equations for certain G_2-structures on $\Lambda^2_- X$ for any positive Einstein self-dual X (but $X = S^4$ or $\mathbb{C}P^2$ are the only complete smooth possibilities).

Similarly, the ODEs for $Sp(2)$-invariant solitons can be regarded as a subsystem of the $SU(3)$-invariant soliton ODEs.

2. Main existence results
Explicit asymptotically conical shrinkers

Shrinkers provide models for formation of singularities, and tend to be rare.

Theorem A
There is a complete $SU(3) \times \mathbb{Z}_2$-invariant shrinking G_2-soliton on $\Lambda^2_- \mathbb{C}P^2$, and an $Sp(2)$-invariant one on $\Lambda^2_- S^4$, defined by $\lambda = -1$ and

$$f_1 = r, \quad f_2 = f_3 = \sqrt{\frac{9}{4} + \frac{r^2}{4}}, \quad X = \left(\frac{t}{3} + \frac{4t}{9 + t^2}\right) \frac{\partial}{\partial r}.$$

This is asymptotically conical because $\dfrac{f_i}{r} \to c_i$ (for $c = (1, \frac{1}{2}, \frac{1}{2})$).

The asymptotic cone is characterised by $\lim\limits_{r \to \infty} \dfrac{f_1}{f_2} = 2$.

The asymptotic rate is $\nu = -2$ because $\dfrac{f_i}{r} = c_i + O(r^{-2})$.

Conjecture: This is the unique complete $Sp(2)$-invariant shrinker.

Asymptotically conical expanders on $\Lambda^2_- S^4$

Expanders provide models for how the flow can smooth out a singularity.

Theorem B

Every complete $Sp(2)$-invariant expanding G_2-soliton on $\Lambda^2_- S^4$ is AC with rate -2.

Up to scale, there is precisely a 1-parameter family of such expanders.

Their asymptotic limits are distinct, bijecting with $(0,1)$ by $\lim_{r\to\infty} \frac{f_1}{f_2}$.

Keeping the scale fixed, the family can be parametrised by $\lambda > 0$.

Limit as $\lambda \to 0$ is the Bryant-Salamon G_2-manifold (which has $\lim_{r\to\infty} \frac{f_1}{f_2} = 1$), considered as a static G_2-soliton.

Remark

The asymptotic cone of the explicit AC shrinker on $\Lambda^2_- S^4$ ($\lim_{r\to\infty} \frac{f_1}{f_2} = 2$) does *not* match the cone of any AC $Sp(2)$-invariant expander ($\lim_{r\to\infty} \frac{f_1}{f_2} < 1$).

Asymptotically conical expanders on $\Lambda^2_- \mathbb{C}P^2$

Theorem B also yields a corresponding 1-parameter family of $SU(3) \times \mathbb{Z}_2$-invariant expanders on $\Lambda^2_- \mathbb{C}P^2$.

AC expander ends are stable, making it possible to perturb this 1-parameter family of $SU(3) \times \mathbb{Z}_2$-invariant solutions to obtain

Theorem C

Up to scale, $\Lambda^2_- \mathbb{C}P^2$ admits a 2-parameter family of $SU(3)$-invariant expanding G_2-solitons that are AC with rate -2.

We do not expect *every* complete $SU(3)$-invariant expander to be AC.

$$f_1 = \frac{3}{\lambda r}, \; f_2^2 = f_3^2 = r \, e^{\frac{\lambda r^2}{6}}$$

solves the soliton ODEs to leading order, and can be corrected to forward-complete solutions with doubly exponential volume growth.

Conjecture: The boundary of the 2-parameter family of $SU(3)$-invariant AC expanders corresponds to complete expanders with such ends.

Flowing through conical singularity?

If a singularity forms modelled on the explicit AC shrinker on $\Lambda^2_- S^4$, then no $Sp(2)$-invariant expander provides a model for how to smooth it out again.

Harder to control which closed $SU(3)$-invariant cones over $SU(3)/T^2$ appear as asymptotic limits of complete $SU(3)$-invariant expanders on $\Lambda^2_- \mathbb{C}P^2$, but numerics suggest:

the asymptotic cone of the shrinker on $\Lambda^2_- \mathbb{C}P^2$

does match

the asymptotic cone of some $SU(3)$-invariant expander on $\Lambda^2_- \mathbb{C}P^2$

after applying an order 3 automorphism to $SU(3)/T^2$ that does not extend to $\Lambda^2_- \mathbb{C}P^2$ (instead permuting 3 different S^2-fibrations $SU(3)/T^2 \to \mathbb{C}P^2$)

$\rightsquigarrow$ potential model for "flowing through" the singularity, crushing a $\mathbb{C}P^2$ and inflating it again in one of two topologically different ways.

This would realise a "G_2 conifold transition" (**Atiyah-Witten (2001)**).

Complete steady solitons

All known complete examples of steady Ricci solitons have sub-Euclidean volume growth. In contrast

Theorem D
There is precisely a 1-parameter family of $SU(3)$-invariant AC steady G_2 solitons on $\Lambda^2_-\mathbb{C}P^2$, all asymptotic with rate -1 to the unique $SU(3)$-invariant torsion-free cone ($f_i = \frac{r}{2} + O(1)$).

One limit is again the static soliton on the Bryant-Salamon AC G_2-manifold.

The other limit is an explicit complete steady G_2-soliton:

$$f_1 = \sqrt{1+e^{-r}},\ f_2 = \sqrt{1+e^{r}},\ f_3 = 2\sinh\frac{r}{2}, \quad X = \tanh\frac{r}{2}\frac{\partial}{\partial r}$$

Asymptotic geometry:

In one *other* fibration $SU(3)/T^2 \to \mathbb{C}P^2$, the S^2 fibres have constant size, and the base is the sinh cone over $\mathbb{C}P^2$, i.e. the negative Einstein metric

$$dr^2 + (\sinh r)^2 g_{\mathbb{C}P^2} \quad \text{on} \quad \mathbb{R}_+ \times \mathbb{C}P^2.$$

3. Initial value problem for invariant solitons
Invariant G_2-structures

$SU(3)$-invariant G_2-structures φ on $\mathbb{R}_+ \times SU(3)/T^2$ such that

- $\|\frac{\partial}{\partial r}\| = 1$, and
- restriction to each slice $SU(3)/T^2$ is closed

are parametrised by triples of functions $f_1, f_2, f_3 : \mathbb{R}_+ \to \mathbb{R}_+$.

$$\varphi = dr \wedge (f_1^2\omega_1 + f_2^2\omega_2 + f_3^2\omega_3) + f_1 f_2 f_3 \alpha$$

for $\omega_i \in \Omega^2(SU(3)/T^2)$ and $\alpha \in \Omega^3(SU(3)/T^2)$ $SU(3)$-invariant.

f_i = scale of S^2 fibres of one of three possible fibrations $SU(3)/T^2 \to \mathbb{C}P^2$

$\mathrm{Vol}(SU(3)/T^2)$ proportional to $(f_1 f_2 f_3)^2$

Ones with $f_2 = f_3$ have extra $\mathbb{Z}_2$ symmetry, and also define analogous $Sp(2)$-invariant G_2-structures on $\mathbb{R}_+ \times \mathbb{C}P^3$.
(Then f_1 = scale of S^2 fibres of $\mathbb{C}P^3 \to S^4$, and $f_2 = f_3$ a scale of base.)

Closure and soliton ODE

$$d\varphi = 0 \;\Leftrightarrow\; 2\frac{d}{dr}(f_1 f_2 f_3) = f_1^2 + f_2^2 + f_3^2$$

Cones $\leftrightarrow$ $f_i = c_i r$ linear

Then $d\varphi = 0 \;\Leftrightarrow\; 6c_1c_2c_3 = c_1^2 + c_2^2 + c_3^2$ is a scale-normalisation:

Unique choice of "cone angle" to make a closed cone for each homothety class on $SU(3)/T^2$
$\rightsquigarrow$ 2-parameter family of closed cones

The soliton condition for $\varphi = f_1^2\omega_1 + f_2^2\omega_2 + f_3^2\omega_3 + f_1 f_2 f_3 \alpha$ and $X = u\frac{\partial}{\partial r}$ is naively a 2nd-order ODE system for (f_1,f_2,f_3,u) (with some constraints).

Useful to rewrite it as a 1st-order ODE in $(f_1, f_2, f_3, \tau_1, \tau_2, \tau_3)$, where τ_i capture the torsion by $d^*\varphi = \tau_1\omega_1 + \tau_2\omega_2 + \tau_3\omega_3$.

This is a system in 5 variables after taking into account that

$$d\varphi = 0 \;\Rightarrow\; \varphi \wedge d^*\varphi = 0 \;\Rightarrow\; \frac{\tau_1}{f_1^2} + \frac{\tau_2}{f_2^2} + \frac{\tau_3}{f_3^2} = 0.$$

Smooth extension problem

Suppose that $H \subset G$, that H acts on a vector space V, and that the action is transitive on the unit sphere in V, with stabiliser $K \subset H$.

Then think of the vector bundle $G \times_K V := (G \times V)/K \to G/K$ as

$$\text{zero section } G/K \quad \sqcup \quad \mathbb{R}_+ \times G/H.$$

To find complete structures on $G \times_K V$, the first step is to ask:

Which solutions on $(0, \epsilon) \times G/H$ extend smoothly over G/K at $r = 0$?

Applying methods **Eschenburg-Wang (2000)**

- Identify conditions on $f_i : [0, \epsilon) \to \mathbb{R}_+$ to ensure smooth extension of $\varphi = f_1^2\omega_1 + f_2^2\omega_2 + f_3^2\omega_3 + f_1 f_2 f_3 \alpha$ from $(0, \epsilon) \times SU(3)/T^2$ to $\mathbb{C}P^2$:
  - f_1 odd with $f_1'(0) = 1$ (so that S^2 fibres shrink to zero at right rate)
  - f_2 and f_3 even with $f_2(0) = f_3(0) = b = \sqrt[4]{\mathrm{Vol}(\mathbb{C}P^2)} > 0$
- Then solve by power series.

Solutions to the soliton initial value problem

Proposition

For each $\lambda \in \mathbb{R}$, there is a 2-parameter family $\varphi_{b,c}$ of solutions to the G_2-soliton equation with dilation constant λ defined for small r that extend smoothly to (a neighbourhood of zero section in) $\Lambda^2_- \mathbb{C}P^2$.

Two scale-invariant parameters: λb^2 and $\frac{c}{b}$.

$\rightsquigarrow$ up to scale there are 2-parameter families of local expanders and shrinkers on $\Lambda^2_- \mathbb{C}P^2$, and 1-parameter family of local steady solitons.

The parameter b is $\sqrt[4]{\mathrm{Vol}(\mathbb{C}P^2)}$, while c controls the leading term in $f_2 - f_3$.

The subfamily $\varphi_{b,0}$ has $f_2 = f_3$, so

- has extra $\mathbb{Z}_2$-symmetry (multiplication by -1 on fibres of $\Lambda^2_- \mathbb{C}P^2$)
- also defines solution near zero section of $\Lambda^2_- S^4$.

$\rightsquigarrow$ up to scale there are 1-parameter families of local expanders and shrinkers on $\Lambda^2_- S^4$, and a unique local steady soliton.

The latter defines the static soliton on the Bryant-Salamon G_2-manifold.

Hence there are no non-trivial $Sp(2)$-invariant steady solitons on $\Lambda^2_- S^4$.

4. Forward-completeness Scale decoupling and AC ends for steady solitons

The steady case $\lambda = 0$ has a very different character because the scale

$$g := \sqrt[3]{f_1 f_2 f_3} = \sqrt[6]{\mathrm{vol}(\Sigma)}$$

essentially decouples from the homothety class

$$\left(\frac{f_1}{g}, \frac{f_2}{g}, \frac{f_3}{g}\right)$$

The latter evolves in a surface under a 2nd order autonomous ODE
$\Leftrightarrow$ 1st order ODE in 4 parameters

Torsion-free cone $c_1 = c_2 = c_3 = \frac{1}{2}$ is unique fixed point, and stable

$\Rightarrow$ Solutions with $\frac{f_i}{g}$ bounded are asymptotic to the torsion-free cone.
Eigenvalues of linearisation at fixed point give rate -1.
Since $\varphi_{b,0}$ is AC (static Bryant-Salamon), $\varphi_{b,c}$ is AC too for c near 0.

Trichotomy for $SU(3)$-invariant steady ends

Proposition

Any initial condition for an $SU(3)$-invariant steady soliton on $\mathbb{R} \times SU(3)/T^2$ leads to one of the following behaviours forward in time (up to permuting f_i)

- **(i)** *AC with rate -1 to torsion-free cone*
- **(ii)** *Complete with exponential volume growth: $f_1 \to \frac{1}{k}$, while $f_2 \sim f_3 \sim e^{kr}$.*
- **(iii)** $f_1 = O(\sqrt{r_* - r})$, $f_2, f_3 = O((r_* - r)^{-1/4})$ *near finite extinction time r_*.*

We can decide the type of each smoothly closing local solution $\varphi_{b,c}$ thanks to spotting an explicit solution of type (ii) corresponding to $\varphi_{\sqrt{2},3}$

$$f_1 = \sqrt{1 + e^{-r}},\ f_2 = \sqrt{1 + e^{r}},\ f_3 = 2\sinh\tfrac{r}{2}, \quad u = \tanh\tfrac{r}{2}.$$

Theorem D

For $\lambda = 0$, the local solution $\varphi_{b,c}$ is

- **(i)** *AC for $\frac{c^2}{b^2} < \frac{9}{2}$.*
- **(ii)** *Exponentially growing for $\frac{c^2}{b^2} = \frac{9}{2}$.*
- **(iii)** *Incomplete for $\frac{c^2}{b^2} > \frac{9}{2}$.*

Non-steady AC ends

The scale does not decouple for $\lambda \neq 0$. On the contrary, scaling up any point in the phase space makes λ terms more dominant, causing

Proposition

Any $SU(3)$-invariant non-steady soliton with all ratios $\frac{f_i}{f_j}$ bounded in forward time is AC with rate -2.

Schematically, because λ has dimensions of length^{-2}, the other factor S of those terms has dimensions of length^2, and satisfies an equation

$$\frac{dS}{dt} = -\lambda \alpha S + \beta$$

where α and β have dimension of length, and $\alpha > 0$ involves only f_i (not $\frac{df_i}{dr}$)

If $\frac{\beta}{\alpha} \to m$ as $r \to \infty$ then $S \to \frac{m}{\lambda}$.

S converging despite having dimension length^2 $\rightsquigarrow$ AC rate is -2

This behaviour is **stable** for $\lambda > 0$
unstable for $\lambda < 0$

AC end solutions

Proposition

For each $\lambda \neq 0$ and (c_1, c_2, c_3) such that $c_1^2 + c_2^2 + c_3^2 = 6c_1c_2c_3$ there exists

- *a unique AC end solution if $\lambda < 0$*
- *a 2-parameter family of AC end solutions if $\lambda > 0$*

asymptotic to the corresponding closed cone (i.e. $\frac{f_i}{r} \to c_i$).

Letting $u = r^{-2}$, the sign of λ becomes significant in an ODE analogous to

$$\frac{dx}{du} = \lambda \frac{x}{u^2} + a\frac{x}{u} + b \qquad \text{with } x \to 0 \text{ as } u \to 0.$$

Setting $a = b = 0$, the general solution for $\lambda \neq 0$ is $\exp\left(\frac{-\lambda}{u}\right)$.

But $\exp\left(\frac{-\lambda}{u}\right) \to 0$ as $u \to 0$ only for $\lambda > 0$.

Haskins-Khan-Payne (2022) prove rigidity of AC ends for gradient shrinking G_2 solitons without cohomogeneity one assumption

AC shrinkers

Heuristic for $\lambda < 0$:

Invariant shrinkers on $\mathbb{R}_+ \times SU(3)/T^2$ are flow lines in 5-dim phase space.
In 4-dimensional space of flow lines

- 2-dimensional submanifold extends across zero section $\mathbb{C}P^2 \subset \Lambda^2_- \mathbb{C}P^2$
- 2-dimensional submanifold has AC behaviour

Expect isolated intersections $\rightsquigarrow$ *finitely* many AC shrinkers on $\Lambda^2_- \mathbb{C}P^2$.

Theorem A

For $\lambda = -1$, $\varphi_{\frac{3}{2},0}$ is the explicit solution

$$f_1 = r, \quad f_2^2 = f_3^2 = \frac{9}{4} + \frac{r^2}{4}, \quad u = \frac{r}{3} + \frac{4r}{9 + r^2}.$$

$\rightsquigarrow$ *$SU(3) \times \mathbb{Z}_2$-invariant shrinker on $\Lambda^2_- \mathbb{C}P^2$ and $Sp(2)$-invariant shrinker on $\Lambda^2_- S^4$, AC with rate -2 to cone with $c_1 = 1$, $c_2 = c_3 = \frac{1}{2}$.*

Conjecture: Unique complete shrinker with this symmetry.

Trichotomy for $Sp(2)$-invariant shrinker ends

In the $Sp(2)$-invariant case, the AC end solutions are a 1-dimensional submanifold of a 2-dimensional space of end solutions.

We can characterise the other possible end behaviours.

Theorem

Any initial condition for an $Sp(2)$-invariant shrinker on $\mathbb{R} \times \mathbb{C}P^3$ leads to one of the following behaviours forward in time

- **(i)** *AC with rate -2 to a closed cone*
- **(ii)** *Complete, exponential growth: $f_1 \sim 4e^{\mu r}$ and $f_2^2 \sim \frac{1}{\mu} e^{\mu r}$ for $\mu = \sqrt{\frac{-\lambda}{18}}$.*
- **(iii)** $f_1 = O(\sqrt{r_* - r})$, $f_2 = O((r_* - r)^{-1/4})$ *near finite extinction time r_*.*

The end in (ii) is modelled on an explicit shrinking soliton on $\mathbb{R} \times$ Iwasawa manifold, found by **Fowdar (2022)**.

There is an essentially unique such end solution, appearing as a boundary point for the curve of AC ends solutions.

The incomplete end behaviour (iii) is the only stable type.

Trichotomy for $Sp(2)$-invariant expander ends

In the expander case,

$$f_1 = \frac{3}{\lambda r}, \; f_2 = A\sqrt{r}\, e^{\frac{\lambda r^2}{12}}$$

can be uniquely corrected to a forward-complete end solution for each $A > 0$.

This 1-parameter family of end solutions with doubly exponential growth forms a "wall" between two different stable types of forward-evolution.

Theorem
Any initial condition for an $Sp(2)$-invariant expander on $\mathbb{R} \times \mathbb{C}P^3$ leads to one of the following behaviours forward in time

(i) *AC with rate -2 to a closed cone*

(ii) *Complete with doubly exponential growth: $f_1 \sim \frac{3}{\lambda r}$, while $f_2 \sim \sqrt{r} e^{\frac{\lambda r^2}{12}}$.*

(iii) $f_1 = O(\sqrt{r_* - r})$, $f_2 = O((r_* - r)^{-1/4})$ *near finite extinction time r_*.*

All smoothly-closing $Sp(2)$-invariant solutions $\varphi_{b,0}$ on $\Lambda^2_- S^4$ fall in case (i), but we expect an analogous transition to be relevant for smoothly-closing $SU(3)$-invariant expanders $\varphi_{b,c}$ on $\Lambda^2_- \mathbb{C}P^2$.

$Sp(2)$-invariant AC expanders on $\Lambda^2_- S^4$

Theorem B

(i) *Every $Sp(2)$-invariant local expander $\varphi_{b,0}$ defined near the zero section of $\Lambda^2_- S^4$ is AC.*

(ii) *Moreover $\mathcal{L} : b \mapsto \lim_{r\to\infty} \frac{f_1}{f_2}$ is a continuous bijection $(0, \infty) \to (0, 1)$ (so each closed $Sp(2)$-invariant cone $f_i = c_i t$ such that $c_1 < c_2$ is the asymptotic cone of unique complete expander).*

Elementary techniques suffice for (i) and non-decreasingness of $\mathcal{L}$, checking that certain inequalities are preserved.

$\mathcal{L}$ *strictly* increasing builds on detailed understanding of AC ends with given asymptotic cone.

$b \to 0$ limit equivalent up to scale to keeping b fixed and letting $\lambda \to 0$.

This limit is the Bryant-Salamon manifold (with $c_1 = c_2 = \frac{1}{2}$), so $\mathcal{L}(b) \to 1$

5. Twistor bundles and rescaling limits of ODEs Rescalings of the $Sp(2)$-invariant soliton ODE

Recall variables in $Sp(2)$-invariant soliton ODEs:

$f_2 \leftrightarrow$ scale of the base of $\Lambda^2_- S^4 \to S^4$

$f_1 \leftrightarrow$ scale of S^2 fibres when foliating by sphere bundles

Changing variable f_2 to $\epsilon f_2 \Leftrightarrow$ rescaling "reference metric" on base of $\Lambda^2_- S^4$

Taking $\epsilon \to 0$

$\rightsquigarrow$ ODE for solitons on $\Lambda^2_- \mathbb{R}^4$ invariant under $\mathrm{Isom}(\mathbb{R}^4) = SO(4) \ltimes \mathbb{R}^4$.

This limit ODE is easier to analyse due to additional scale invariance and a conserved quantity.

The expander system has an explicit solution $f_1 = \dfrac{3}{\lambda r}$, $f_2^2 = r\, e^{\frac{\lambda r^2}{6}}$, incomplete at $r = 0$ but helps understand transition in original system.

Limit of $\varphi_{b,0}$ as $b \to \infty$ $\rightsquigarrow$ AC solution in limit system $\rightsquigarrow$ $\mathcal{L}(b) \to 0$

Taking $\epsilon = \sqrt{-1}$

$\rightsquigarrow$ ODE for solitons on $\Lambda^2_- H^4$ invariant under $\mathrm{Isom}(H^4) = SO(4,1)$.

Twistor space interpretation

For X^4 oriented, consider on the sphere bundle in $\Lambda^2_- X$ (twistor space)

- volume form ω_1 on fibres
- tautological 2-form ω_2

For X self-dual Einstein, the conditions for the Isom(X)-invariant

$$\varphi = dr \wedge (f_1^2\omega_1 + f_2^2\omega_2) + f_1 f_2^2 d\omega_1 \in \Omega^3(\Lambda^2_- X)$$

to be a soliton depend only on the scalar curvature κ. So

$\kappa = 1 \rightsquigarrow Sp(2)$-invariant ODE on $\Lambda^2_- S^4$;
in general get the $\epsilon = \sqrt{\kappa}$ rescaling.

In particular, the $\epsilon = 0$ limit can be interpreted as ODE for warped product solitons on $\mathbb{R} \times S^2 \times Y^4$, with Y hyper-Kähler.

The steady case was considered by **Ball (2022)**.

Rescalings of the $SU(3)$-invariant soliton ODE

In a similar way, rescaling f_2 and f_3 in the $SU(3)$-invariant soliton ODEs is related to considering $\Lambda^2_- X$ for self-dual *Kähler*-Einstein X^4 for different κ.

(In particular, one could consider $SU(2,1)$-invariant solitons of $\Lambda^2_- \mathbb{C}H^2$.)

$f_2 = f_3 \Leftrightarrow$ invariance under under multiplying fibres of $\Lambda^2_- X$ by -1, recovering ODE from the case not requiring Kähler.

The limit of rescalings of the explicit exponentially growing solution $f_1 = \sqrt{1 + e^{-r}},\ f_2 = \sqrt{1 + e^r},\ f_3 = 2\sinh\frac{r}{2}$ give an explicit solution

$$f_1 = 1,\ f_2 = f_3 = e^r$$

on $\mathbb{R}_{>0} \times S^2 \times Y$ for Y hyperKähler

$\rightsquigarrow$ complete steady G_2 soliton that is a metric product of S^2 and $dr^2 + e^r g_Y$

Another rescaling of $Sp(2)$-invariant soliton ODE

Changing variables $f_1 \to \epsilon^2 f_1$, $f_2 \to \epsilon f_2$ and taking $\epsilon \to 0$

$\Leftrightarrow$ rescaling both fibres and base of S^2-fibration $\mathbb{C}P^3 \to S^4$ of link of $\Lambda^2_- S^4$

$\rightsquigarrow$ soliton ODE on $\mathbb{R}\times\mathbb{C}\times\mathbb{C}^2$, invariant under complex Heisenberg group $\mathcal{H}_3$.

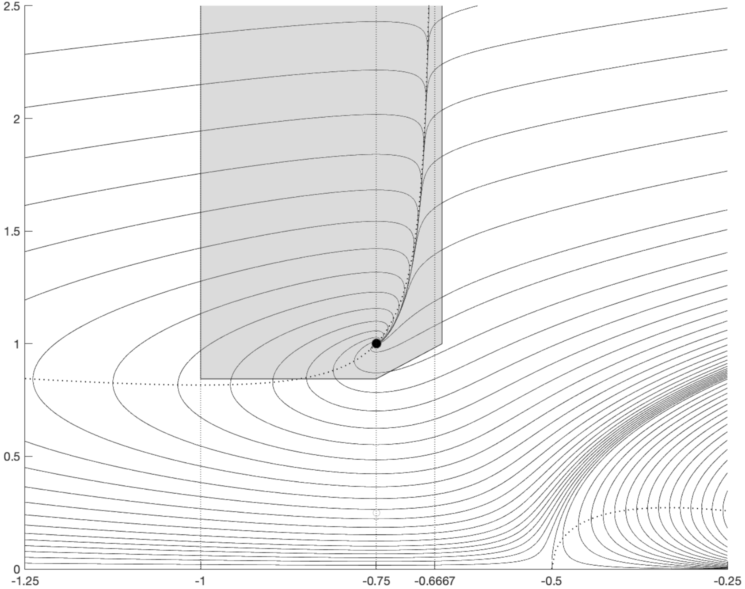
Same as soliton ODE for certain G_2-structures on $\mathbb{R} \times$ suitable T^2-bundle over hyper-Kähler Y^4.

This system can be reduced to one in two variables.

Fowdar's explicit shrinker corresponds to a fixed point.

We find another complete shrinker with one AC end and one cusp end modelled on Fowdar's.

The proof of the trichotomy for shrinkers relies on analysing this limit system.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