Asymptotically conical G_2-solitons

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These slides available at
`http://people.bath.ac.uk/jlpn20/g2sol.pdf`

Introduction

Context: Riemannian 7-manifolds with holonomy group G_2,
a special kind of Ricci-flat manifolds

Bryant's Laplacian flow: a G_2 cousin of Ricci flow

G_2 solitons: self-similar solutions to Laplacian flow.

We found asymptotically conical G_2 solitons with cohomogeneity one, $SU(3)$-invariant ones on $\Lambda^2_- \mathbb{C}P^2$ and $Sp(2)$-invariant ones on $\Lambda^2_- S^4$.

Shrinkers provide models for formation of singularities, and tend to be rare.

Theorem
There exists a shrinking G_2-soliton on each of $\Lambda^2_- \mathbb{C}P^2$ and $\Lambda^2_- S^4$.

Expanders provide models for how the flow can smooth out a singularity.

Theorem
There exist families of expanding G_2-solitons on both $\Lambda^2_- \mathbb{C}P^2$ and $\Lambda^2_- S^4$.

These can be viewed as deformations of Bryant-Salamon AC G_2-manifolds.

Introduction

Theorem

There is a 1-parameter family of steady G_2-solitons on $\Lambda^2_-\mathbb{C}P^2$.

This is in contrast to Ricci flow, where all known complete examples of steady solitons have sub-Euclidean volume growth.

Moreover, a steady G_2-soliton with exponential volume growth appears as one limit of the family. (The other limit is the static soliton from the Bryant-Salamon AC G_2-manifold.)

Outline

1. Holonomy G_2
2. G_2 solitons
3. Main results
4. ODE for invariant solitons
5. Complete solutions: initial value problem and infinite lifetime
6. Stability and rigidity for the end problem

1. Holonomy G_2
The group G_2

$G_2 := \mathrm{Aut}\ \mathbb{O}$, $\quad \mathbb{O} =$ octonions, normed division algebra of real dimension 8.

G_2 acts on $\mathrm{Im}\,\mathbb{O} \cong \mathbb{R}^7$, preserving metric, orientation, cross product

$$a \times b := \mathrm{Im}(ab), \text{ and}$$
$$\varphi_0(a, b, c) := \langle a \times b, c \rangle.$$

In terms of basis $e^1, \dots, e^7 \in (\mathbb{R}^7)^*$

$$\varphi_0 = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356} \in \Lambda^3(\mathbb{R}^7)^*.$$

Peculiar algebra facts:

- G_2 is not just contained in stabiliser of φ_0 in $GL(7, \mathbb{R})$, but equality holds.
- The $GL(7, \mathbb{R})$-orbit of φ_0 is open in $\Lambda^3(\mathbb{R}^7)^*$.

Holonomy and G_2-structures

For a Riemannian manifold M (and base point p)

$$Hol(M) = \{\text{parallel transport around loops } \gamma \text{ based at } p\} \subseteq O(T_pM).$$

Parallel tensor fields on Riemannian manifold $M \leftrightarrow$ invariants of $Hol(M)$.

G_2 is an exceptional case in Berger's list of Riemannian holonomy groups.

A metric with holonomy G_2 is always Ricci-flat.

A 3-form $\varphi \in \Omega^3(M^7)$ such that $(T_xM, \varphi) \cong (\mathbb{R}^7, \varphi_0)$ for all $x \in M$ defines a G_2-structure. (This is an *open* condition on φ)

Because $G_2 \subset SO(7)$, this induces a metric and orientation.

$Hol(M) \subseteq G_2 \Leftrightarrow$ metric induced by some G_2-structure φ such that $\nabla\varphi = 0$. Then call φ *torsion-free*. This is equivalent to the first-order non-linear PDE

$$d\varphi = d^*\varphi = 0,$$

elliptic transverse to diffeomorphisms.

Examples of holonomy G_2 metrics

Local examples:

- Bryant (1985) by understanding local generality of solutions

Complete examples:

- Bryant-Salamon (1987) AC examples on $\Lambda^2_- \mathbb{C}P^2$, $\Lambda^2_- S^4$ and $S^3 \times \mathbb{R}^4$.

Examples on closed manifolds:

- Joyce (1994) Resolutions of flat orbifolds
- Kovalev (2003), Corti-Haskins-N-Pacini (2014) Twisted connected sums

All known constructions of examples on closed manifolds solve the elliptic PDE by gluing together pieces with dimensional reduction.

Can sometimes determine topology completely, but little clue as to how to characterise which differentiable manifolds admit holonomy G_2 metrics. Can flow approach do any better?

2. G_2 solitons
Bryant's Laplacian flow

Solve
$$\frac{d\varphi_t}{dt} = \Delta_{\varphi_t}\varphi_t$$
with initial condition φ_0 satisfying $d\varphi_0 = 0$. (Then $d\varphi_t = 0$ for all t.)

- Stationary points are exactly torsion-free G_2-structures.
- Gradient flow for $\mathrm{vol}(\varphi)$ restricted to cohomology class of φ_0.
- Induced metric evolves by

$$\frac{dg_t}{dt} = -2\mathrm{Ric}(g_t) + \text{ terms quadratic in torsion of } \varphi_t$$

Theorem (Bryant-Xu 2011, Lotay-Wei 2017)

Short-time existence and uniqueness.
Torsion-free G_2-structures are stable.

What is long term behaviour?? Expect singularities to form in finite time.
By analogy with other flows, expect solitons as models.

G_2 soliton equations

G_2-structure φ, vector field X, dilation constant $\lambda \in \mathbb{R}$ satisfying

$$\begin{cases} d\varphi &= \quad 0, \\ \Delta_\varphi \varphi = & \lambda\varphi + \mathcal{L}_X \varphi. \end{cases}$$

$\Leftrightarrow$ self-similar solution of Laplacian flow

$$\varphi_t = k(t)^3 f^* \varphi, \qquad \frac{df}{dt} = k(t)^{-2} X, \qquad k(t) = \frac{3 + 2\lambda t}{3}$$

$\lambda > 0$: expanders (immortal solutions)
$\lambda = 0$: steady solitons (eternal solutions)
$\lambda < 0$: shrinkers (ancient solutions)

- Non-steady soliton $\Rightarrow \varphi$ exact
- Solitons on a compact manifold are stationary or expanders
- Scaling behaviour: (φ, X) is a λ-soliton $\Leftrightarrow (k^3\varphi, k^{-1}X)$ is a $k^{-2}\lambda$-soliton.

Asymptotically conical behaviour

A positive 3-form φ on M is asymptotically conical with rate ν if there is a compact set K, diffeomorphism $M \setminus K \to \mathbb{R}_+ \times \Sigma^6$ and

$$\varphi_C = r^2 dr \wedge \omega + r^3 \alpha \quad \text{for} \quad \omega \in \Omega^2(\Sigma^6), \quad \alpha \in \Omega^3(\Sigma^6)$$

such that

$$\|\nabla^k(\varphi - \varphi_C)\| = O(r^{\nu - k}).$$

The Bryant-Salamon G_2-metrics on $S^3 \times \mathbb{R}^4$, $\Lambda^2_- \mathbb{C}P^2$ and $\Lambda^2_- S^4$ are asymptotically conical with rates -4, -3 and -3 respectively.

If the conical G_2-structure φ_C is torsion-free, then setting $X = -\frac{\lambda}{3} r \frac{\partial}{\partial r}$ defines a "Gaussian" soliton with dilation constant λ.

If merely $d\varphi_C = 0$, then $(\varphi_C, -\frac{\lambda}{3} r \frac{\partial}{\partial r})$ is not a soliton, but does solve the equations to first order, so still a sensible asymptotic model.

Cohomogeneity 1 action

Suppose M^7 has an action by G such that generic orbit Σ has dimension 6. Then there are at most two "special" orbits, at ends of $\Sigma \times$ open interval.

Unique special orbit $X \Rightarrow M$ is a total space of a vector bundle over X.

Looking for G-invariant solutions to PDE reduces to an ODE.

Bryant-Salamon AC G_2 manifolds

- $M = S^3 \times \mathbb{R}^4$ with $SU(2)^3$ action, $\Sigma = S^3 \times S^3$.
- $M = \Lambda^2_- \mathbb{C}P^2$ with $SU(3)$ action, $\Sigma = SU(3)/T^2$.
- $M = \Lambda^2_- S^4$ with $Sp(2)$ action, $\Sigma = \mathbb{C}P^3$.

Only the last two have any chance of deforming to expanders, since non-steady solitons must be exact.

Aside: In the first two, Σ has $\mathbb{Z}_3$ of automorphisms that do not extend to M
$\rightsquigarrow$ 3 different ways to glue in special orbit to resolve conical singularity (**Atiyah-Witten (2001)** "G_2 conifold transition")

3. Main results
Expanders on $\Lambda^2_- S^4$

$\{$ Closed $Sp(2)$-invariant conical G_2-structures on $\mathbb{R}_+ \times \mathbb{C}P^3$ $\} \cong \mathbb{R}$,
contains a unique torsion-free cone (limit of Bryant-Salamon manifold)

Theorem

Up to scale, $\Lambda^2_- S^4$ admits precisely a 1-parameter family of $Sp(2)$-invariant expanding G_2-solitons that are AC with rate -2.

Their asymptotic limits biject with the closed cones "on one side" of the torsion-free one.

Keeping the scale fixed, the family can be parametrised by $\lambda > 0$.

Limit as $\lambda \to 0$ is the Bryant-Salamon G_2-manifold, considered as a static G_2-soliton.

Expanders on $\Lambda^2_- \mathbb{C}P^2$

$\{$ Closed $SU(3)$-invariant conical G_2-structures on $\mathbb{R}_+ \times SU(3)/T^2$ $\} \cong \mathbb{R}^2$, contains a unique torsion-free cone (limit of Bryant-Salamon manifold)

Among $SU(3)$-invariant G_2-structures on $\Lambda^2_- \mathbb{C}P^2$, there is a subset that is in addition anti-invariant under multiplying fibres by -1.

Soliton equations for such G_2-structures reduce to the $Sp(2)$-invariant ones

$\rightsquigarrow$ 1-parameter family of $SU(3) \times \mathbb{Z}_2$-symmetric expanders on $\Lambda^2_- \mathbb{C}P^2$ where we understand the asymptotic limit cones well.

Stability of AC expander ends $\rightsquigarrow$

Theorem

Up to scale, $\Lambda^2_- \mathbb{C}P^2$ admits a 2-parameter family of $SU(3)$-invariant expanding G_2-solitons that are AC with rate -2.

but less clear picture of which closed cones appear as asymptotic limits.

Explicit shrinkers

Spotting an explicit polynomial solution to the ODE (same in both cases) $\rightsquigarrow$

Theorem

There is an $Sp(2)$-invariant AC shrinking G_2-soliton on $\Lambda^2_- S^4$.
There is an $SU(3) \times \mathbb{Z}_2$-symmetric AC shrinking G_2-soliton on $\Lambda^2_- \mathbb{C}P^2$.
Both have rate -2.

Conjecture: This is unique.

The asymptotic cone of the shrinker on $\Lambda^2_- S^4$ does *not* match the asymptotic cone of any $Sp(2)$-invariant expander.

However, numerics suggest that the asymptotic cone of the shrinker on $\Lambda^2_- \mathbb{C}P^2$ *does* match the asymptotic cone of some $SU(3)$-invariant expanders.

$\rightsquigarrow$ potential model for "flowing through" the singularity in a way that cuts out a $\mathbb{C}P^2$ and glues it back in a topologically different way, realising a G_2 conifold transition.

Steady G_2-solitons

Theorem

- *There are no complete $Sp(2)$-invariant steady G_2 solitons on $\Lambda^2_- S^4$ (other than the static one from Bryant-Salamon AC G_2-manifold)*
- *There is precisely a 1-parameter family of $SU(3)$-invariant AC steady G_2 solitons on $\Lambda^2_- \mathbb{C}P^2$, all asymptotic with rate -1 to the unique $SU(3)$-invariant torsion-free cone.*

One limit of the family is the static soliton on Bryant-Salamon.

Other limit is a complete steady G_2 soliton with exponential volume growth:
The cross-section $SU(3)/T^2$ is an S^2-bundle over $\mathbb{C}P^2$, in two other ways then the sphere bundle in $\Lambda^2_- \mathbb{C}P^2$ we started with.
Asymptotically, the fibres have constant size, and the base is the sinh cone over $\mathbb{C}P^2$, i.e. the negative Einstein metric

$$dr^2 + (\sinh r)^2 g_{\mathbb{C}P^2} \quad \text{on} \quad \mathbb{R}_+ \times \mathbb{C}P^2.$$

4. ODE for invariant solitons
Invariant G_2-structures

$SU(3)$-invariant G_2-structures φ on $\mathbb{R}_+ \times SU(3)/T^2$ such that

- $\|\frac{\partial}{\partial r}\| = 1$, and
- restriction to each slice $SU(3)/T^2$ is closed

are parametrised by triples of functions $f_1, f_2, f_3 : \mathbb{R}_+ \to \mathbb{R}_+$.

$$\varphi = f_1^2\omega_1 + f_2^2\omega_2 + f_3^2\omega_3 + f_1 f_2 f_3 \Omega$$

for $\omega_i \in \Omega^2(SU(3)/T^2)$ and $\Omega \in \Omega^3(SU(3)/T^2)$ $SU(3)$-invariant.

f_i = scale of S^2 fibres of one of three possible fibrations $SU(3)/T^2 \to \mathbb{C}P^2$

$\mathrm{Vol}(SU(3)/T^2)$ proportional to $(f_1 f_2 f_3)^2$

Ones with $f_2 = f_3$ have extra $\mathbb{Z}_2$ symmetry, and also define analogous $Sp(2)$-invariant G_2-structures on $\mathbb{R}_+ \times \mathbb{C}P^3$.
(Then f_1 = scale of S^2 fibres of $\mathbb{C}P^3 \to S^4$, and $f_2 = f_3$ a scale of base.)

Closure and cones

$$d\varphi = 0 \iff 2\frac{d}{dr}(f_1 f_2 f_3) = f_1^2 + f_2^2 + f_3^2$$

Then the torsion of φ is captured by

$$d^*\varphi = \tau_1\omega_1 + \tau_2\omega_2 + \tau_3\omega_3$$

for

$$\tau_i = (f_i^2)' + \frac{f_i^2}{f_1 f_2 f_3}\left(f_i^2 - f_j^2 - f_k^2\right),$$

and satisfies the algebraic constraint

$$\text{“}d^*\varphi \text{ has type } 14\text{”} \iff \varphi \wedge d^*\varphi = 0 \iff \frac{\tau_1}{f_1^2} + \frac{\tau_2}{f_2^2} + \frac{\tau_3}{f_3^2} = 0.$$

Cones $\leftrightarrow$ $f_i = c_i r$ linear

Then $d\varphi = 0 \iff 6c_1c_2c_3 = c_1^2 + c_2^2 + c_3^2$ is a scale-normalisation:
for homothety class on $SU(3)/T^2$, there is a unique choice of “cone angle” to make a closed cone.

Soliton ODE

The soliton condition for $\varphi = f_1^2\omega_1 + f_2^2\omega_2 + f_3^2\omega_3 + f_1f_2f_3\Omega$ and $X = u\frac{\partial}{\partial r}$ is naively a 2nd-order ODE system for (f_1,f_2,f_3,u) (with some constraints).

Useful to rewrite it as a 1st-order ODE in $(f_1, f_2, f_3, \tau_1, \tau_2, \tau_3)$;
system in 5 variables after taking into account type 14 constraint.

If we impose $f_2 = f_3$ we are left with a system in 3 variables (f_1, f_2, τ_2):

$$\begin{aligned}
\text{Closure} &\Leftrightarrow \frac{d}{dr}(f_1f_2^2) = f_1^2 + 2f_2^2 \\
\text{Definition of torsion} &\Leftrightarrow \frac{d}{dr}\log\frac{f_1}{f_2} = \frac{f_2^2 - f_1^2 - \frac{3}{2}f_1\tau_2}{f_1f_2^2} \\
\text{Soliton condition} &\Leftrightarrow \qquad \frac{d\tau_2}{dr} = \frac{4(\lambda f_1f_2^2 - 3\tau_2)(f_2^2 - f_1^2 - \frac{3}{2}f_1\tau_2)}{3f_1(f_1^2 + 2f_2^2)}
\end{aligned}$$

Qualitative features

- Homogeneous if we consider r, f_1, f_2, τ_2 to have weight 1, λ weight -2, but inhomogeneous if we consider $\lambda \neq 0$ unweighted.
- $S := f_2^2 - f_1^2 - \frac{3}{2}f_1\tau_2$ plays a distinguished role (and $\equiv 0$ on a closed cone)

5. Initial value problem and infinite lifetime
General strategy

Complete solutions to ODE for G-invariant structures on $\mathbb{R}_+ \times G/H$?

Suppose that $H \subset G$, that H acts on vector space V, and that the action is transitive on unit sphere in V, with stabiliser $K \subset H$.

Then think of the vector bundle $G \times_K V := (G \times V)/K \to G/K$ as

$$\text{zero section } G/K \quad \sqcup \quad \mathbb{R}_+ \times G/H.$$

To find AC structures on $G \times_K V$, can consider

(1) Solutions on $(0, \epsilon) \times G/H$ that extend smoothly over G/K at $r = 0$?

(2) Solutions on $(R, \infty) \times G/H$ asymptotic to prescribed cone?

(3) Do they fit together?

Alternatively "Solve (1), evolve forward and see what happens"

Picture for (1) is clearest, applying methods of **Eschenburg-Wang (2000)**.

Initial value problem on $\Lambda^2_- \mathbb{C}P^2$

- Identify conditions on $f_i : [0, \epsilon) \to \mathbb{R}_+$ ensuring smooth extension of $\varphi = f_1^2\omega_1 + f_2^2\omega_2 + f_3^2\omega_3 + f_1 f_2 f_3 \Omega$ from $(0, \epsilon) \times SU(3)/T^2$ to $\mathbb{C}P^2$.
  - f_1 odd with $f_1'(0) = 1$ (so that S^2 fibres shrink to zero at right rate)
  - f_2 and f_3 even with $f_2(0) = f_3(0) = b = \sqrt[4]{\mathrm{Vol}(\mathbb{C}P^2)} > 0$
- Solve resulting singular initial value problem by power series in r.

Proposition

For each $\lambda \in \mathbb{R}$, there is a 2-parameter family $\varphi_{b,c}$ of solutions to the G_2-soliton equation with dilation constant λ defined for small r that extend smoothly to (a neighbourhood of zero section in) $\Lambda^2_- \mathbb{C}P^2$.

Two scale-invariant parameters: λb^2 and $\frac{c}{b}$.

$\rightsquigarrow$ up to scale there are 2-parameter families of local expanders and shrinkers on $\Lambda^2_- \mathbb{C}P^2$, and 1-parameter family of local steady solitons.

The parameter c controls the leading term in $f_2 - f_3$.

The subfamily $\varphi_{b,0}$ has $f_2 = f_3$, so has extra $\mathbb{Z}_2$-symmetry.

Complete solutions on $\Lambda^2_- S^4$

The subfamily $\varphi_{b,0}$ also defines solution near zero section of $\Lambda^2_- S^4$.

$\rightsquigarrow$ up to scale there are 1-parameter families of local expanders and shrinkers on $\Lambda^2 S^4$, and a unique local steady soliton.

The latter defines the static soliton on the Bryant-Salamon G_2-manifold.

Hence there are no non-trivial $Sp(2)$-invariant steady solitons on $\Lambda^2_- S^4$.

Theorem

For $\lambda = -1$, $\varphi_{\frac{3}{2},0}$ is the explicit solution

$$f_1 = r, \quad f_2^2 = f_3^2 = \frac{9}{4} + \frac{r^2}{4}, \quad u = \frac{r}{3} + \frac{4r}{9 + r^2}.$$

AC with rate -2 to cone with $c_1 = 1$, $c_2 = \frac{1}{2}$.

Theorem

For $\lambda > 0$, every local solution $\varphi_{b,0}$ extends to an AC solution on $\Lambda^2_- S^4$.

Moreover, $b \mapsto \lim_{r\to\infty} \frac{f_1}{f_2}$ is a continuous bijection $(0,\infty) \to (1,\infty)$.

Long-time behaviour on $\mathbb{R}_+ \times \mathbb{C}P^3$

$$\frac{d}{dr}(f_1 f_2^2) = f_1^2 + 2f_2^2 \tag{1}$$

$$\frac{d}{dr}\log\frac{f_1}{f_2} = \frac{S}{f_1 f_2^2} \tag{2}$$

$$\frac{d\tau_2}{dr} = \frac{4(\lambda f_1 f_2^2 - 3\tau_2)S}{3f_1(f_1^2 + 2f_2^2)} \tag{3}$$

Solutions are forward-complete (infinite lifetime) unless $\log\frac{f_1}{f_2}$ unbounded.

If $\frac{f_1}{f_2}$ converges then

- $(1) \Rightarrow \frac{f_i}{r} \to c_i$ for solution of closed cone equation $c_1^2 + 2c_2^2 = 6c_1c_2^2$
- $(2) \Rightarrow \liminf \frac{S}{r^2} = 0$

In fact for $\lambda \neq 0$, factor of S in only λ term in (3) causes

$$\log\frac{f_1}{f_2} \text{ bounded } \Rightarrow S \text{ converges } \Rightarrow \text{ AC with rate } -2$$

6. Stability and rigidity for the end problem
Decoupling

While the initial value problem (1) was essentially the same regardless of λ, the end problem (2) is very different.

First, the steady case $\lambda = 0$ has a very different character because the scale

$$g := \sqrt[3]{f_1 f_2 f_3} = \sqrt[6]{\mathrm{vol}(\Sigma)}$$

essentially decouples from the homothety class

$$\left(\frac{f_1}{g}, \frac{f_2}{g}, \frac{f_3}{g}\right)$$

The latter evolves in a surface under a 2nd order autonomous ODE
$\Leftrightarrow$ 1st order ODE in 4 parameters
(Reduces to 2 parameters thanks to conserved quantities $\tau_i - u f_i^2$.)

Torsion-free cone $c_1 = c_2 = c_3 = \frac{1}{2}$ is unique fixed point
$\Rightarrow$ Solutions with $\frac{f_i}{g}$ bounded are asymptotic to the torsion-free cone.
Eigenvalues of linearisation at fixed point give rate -1 (and $S = O(r)$).

Stability of steady end

The fixed point corresponding to the torsion-free cone $c_1 = c_2 = c_3$ is stable
$\Rightarrow$ the torsion-free cone is stable considered as a steady soliton end
$\Rightarrow$ for c in an open neighbourhood of 0, the local solution $\varphi_{b,c}$ extends to an AC solution asymptotic to the torsion-free cone

Actually get more decisive results thanks to spotting an explicit solution with exponential volume growth:

$$f_1 = 2\sinh\tfrac{r}{2},\ f_2 = \sqrt{1+e^r},\ f_3 = \sqrt{1+e^{-r}}, \quad u = \tanh\tfrac{r}{2},$$

which corresponds to the local solution $\varphi_{\sqrt{2},3}$.

Theorem

For $\lambda = 0$, the local solution $\varphi_{b,c}$

- *extends to an AC solution for $\frac{c^2}{b^2} < \frac{9}{2}$.*
- *is incomplete for $\frac{c^2}{b^2} > \frac{9}{2}$.*

Stability of AC expander ends

Given $\lambda > 0$ and any $SU(3)$-invariant closed cone (c_1, c_2, c_3)

- There is a 2-parameter family of solutions defined for large r asymptotic to the given cone, with rate -2
- Difference between two solutions is of order $\exp(-\frac{\lambda}{6}r^2) * \text{polynomial}$.

Flow lines of this 4-parameter family of solutions fill open subset of 5-dimensional phase space, so AC expander ends are **stable**.

Hence the set $U \subseteq \mathbb{R}_+ \times \mathbb{R}$ of parameters (b, c) such that the local solutions $\varphi_{b,c}$ extends to an AC expander is open.

Results from $Sp(2)$-invariant case show that U contains $\mathbb{R}_+ \times \{0\}$.

Thus (b, c) in a neighbourhood U of $\mathbb{R}_+ \times \{0\}$ yield $SU(3)$-invariant AC expanders on $\Lambda^2_-\mathbb{C}P^2$, but we don't know how big U is or what the asymptotic cones are.

Rigidity of AC shrinker ends

For $\lambda < 0$, for each closed cone (c_1, c_2, c_3) there is a unique solution defined for large r asymptotic to the given cone; shrinker ends are **rigid**.

For AC Ricci solitons, **Kotschwar-Wang (2015)** prove an analogous rigidity statement without any homogeneity assumption.

Haskins-Khan-Payne (2022) adapt this to Laplacian flow.

In the special case of cohomogeneity one G_2 solitons, the sign of λ is significant in an ODE analogous to

$$\frac{dx}{du} = \lambda \frac{x}{u^3} + \frac{x}{2u} + 1 \qquad \text{with } x \to 0 \text{ as } u \to 0.$$

$\lambda < 0$: Unique smooth solution $x(u) = \sqrt{u} \exp\left(\frac{-\lambda}{2u^2}\right) \int_0^u \exp\left(\frac{\lambda}{2s^2}\right) \frac{ds}{\sqrt{s}}$

$\lambda = 0$: General solution $x(u) = 2u + C\sqrt{u}$

$\lambda > 0$: Any two smooth solutions differ by multiple of $\sqrt{u} \exp\left(-\frac{\lambda}{2u^2}\right)$

Consequence of rigidity for AC shrinker ends

Heuristic for $\lambda < 0$:

Invariant shrinkers on $\mathbb{R}_+ \times SU(3)/T^2$ are flow lines in 5-dim phase space.
In 4-dimensional space of flow lines

- 2-dimensional submanifold extends across zero section $\mathbb{C}P^2 \subset \Lambda^2_- \mathbb{C}P^2$
- 2-dimensional submanifold has AC behaviour

Expect isolated intersections $\rightsquigarrow$ *finitely* many AC shrinkers on $\Lambda^2_- \mathbb{C}P^2$.

Similarly, restricting attention to solutions with $f_2 = f_3$:

In 2-dimensional space of flow lines

- 1-dimensional submanifold extends over special orbit
- 1-dimensional submanifold has AC behaviour.

Expect isolated intersections $\rightsquigarrow$ *finitely* many AC shrinkers on $\Lambda^2_- S^4$.

Conjecture: The explicit shrinker is unique.