4.2 THE SQUARING FUNCTOR

Let $A: S^\infty \to S^\infty$ denote the antipodal map (so $\mathbb{R}P^\infty = S^\infty/A$)

Given a space X, let $\tau: X \times X \to X \times X$, $(x,y) \mapsto (y,x)$ and

$$SQ(X) = X \times X \times S^\infty / \tau \times A$$

$SQ(X) \to \mathbb{R}P^\infty$ is an $X \times X$ - fibre bundle.

The restriction to a fibre of any $S \in H^*(SQ(X))$ is τ^*-invariant.

Given $a = [\alpha] \in H^n(X)$, is there a class $SQ(a) \in H^*(SQ(X))$ that restricts to $\pi_1^* a \cup \pi_2^* a \in H^{2n}(X \times X)$?

Let $\alpha_i := \pi_i^* \alpha \in C^n(X \times X \times S^\infty)$ for $\pi_1, \pi_2 : X \times X \times S^\infty \to X$ the projections.

Cochains on $SQ(X) \leftrightarrow (\tau \times A)$ - invariant cochains on $X \times X \times S^\infty$.

If cup product on cochains were commutative we could take $SQ([\alpha]) = [\alpha_1 \cup \alpha_2]$.
So failure of commutativity is reflected by difficulty defining $SQ(a)$.

Let $\rho \in C^0(S^\infty)$ (ie $\rho : S^\infty \to \mathbb{Z}_2$) st $\rho + A^*\rho \equiv 1$

Abbreviate $A^*\rho \rightsquigarrow \bar\rho$. (Then $d\rho = d\bar\rho$ is A-invariant

$\rightsquigarrow$ representative of generator of $H^1(\mathbb{RP}^\infty)$.)

Now

$$\rho\,\alpha_1 \cup \alpha_2 + \bar\rho\,\alpha_2 \cup \alpha_1 \in C^{2n}(SQ(X))$$

restricts to a representative of $\pi_1^*[\alpha] \cup \pi_2^*[\alpha]$ on each fibre, but is not closed

$$d(\rho\,\alpha_1 \cup \alpha_2 + \bar\rho\,\alpha_2 \cup \alpha_1) = (d\rho)(\alpha_1 \cup \alpha_2 + \alpha_2 \cup \alpha_1)$$

However, one can take

$$SQ(\alpha) = \rho\,\alpha_1 \cup \alpha_2 + \bar\rho\,\alpha_2 \cup \alpha_1 + (d\rho)(\bar\rho\,\alpha_1 \cup_1 \alpha_2 + \rho\,\alpha_2 \cup \alpha_1)$$

$$+ (d\rho)^2(\ \cdot \cup_2 \ \ ) + \dots \qquad (d\rho)^{n}(\ \cup_{n}\ \ )$$

The diagonal map $X \times S^\infty \to X \times X \times S^\infty$ induces a well-defined
$(x, s) \mapsto (x, x, s)$

$\Delta : X \times \mathbb{R}P^\infty \hookrightarrow SQ(X)$

By the Künneth formula, any $b \in H^{2n}(X \times \mathbb{R}P^\infty)$ can be written uniquely as $\sum_{i=0}^{2n} b_i \omega^i$ with $b_i \in H^{2n-i}(X)$ and $\omega \in H^1(\mathbb{R}P^\infty)$ the generator

<u>Theorem A</u>

Given $a \in H^n(X)$, $SQ(a)$ is the unique class in $H^{2n}(SQ(X))$ st

- restriction to any fibre is $\pi_1^* a \cup \pi_2^* a \in H^{2n}(X \times X)$
- $\Delta^* SQ(a) = \sum_{i=0}^{n} b_i \omega^i$ for some $b_i \in H^{2n-i}(X)$ (ie no coefficients of ω^i with $i > n$ vanish)

Writing $\underline{Sq}(a) := \Delta^* SQ(a) \in H^{2n}(X \times \mathbb{R}P^\infty)$, $Sq^k(a)$ can thus be characterised as the coefficient of ω^{n-k} in $\underline{Sq}(a)$

COHOMOLOGY OF $SQ(X)$

To prove Theorem A, first let $\mathrm{Sym}^2 H^*(X)$ and $\mathrm{Alt}^2 H^*(X)$ denote the kernel and image of the swapping map on $H^*(X) \otimes H^*(X)$.

Note that $\mathrm{Sym}^2 H^*(X)$ is the direct sum of $\mathrm{Alt}^2 H^*(X)$ and the image of the <u>linear</u> map

$$H^*(X) \to \mathrm{Sym}^2 H^*(X), \quad a \mapsto a \otimes a$$

Let $P: \mathrm{Sym}^2 H^*(X) \to H^*(X \times X)$ be induced by $a \otimes b \mapsto \pi_1^* a \cup \pi_2^* b$ (so $SQ(a)$ is a preimage of $P(a \otimes a)$).

Finding a preimage $\Sigma(a, b) \in H^*(SQ(X))$ of $P(a \otimes b + b \otimes a)$ is much easier: can simply take

$$\Sigma([\alpha], [\beta]) := [\alpha_1 \cup \beta_2 + \alpha_2 \cup \beta_1]$$

(Note: $\Sigma([\alpha], [\beta])\omega = [d(\rho\alpha_1 \cup \beta_2 + \bar{\rho}\alpha_2 \cup \beta_1)] = 0$)

Σ and SQ can be combined into a single linear map

$$\overline{SQ} : \mathrm{Sym}^2 H^*(X) \to SQ(X)$$

induced by

$$[\alpha]\otimes[\beta] \mapsto \bar\rho\alpha_1\cup\beta_2 + \rho\alpha_2\cup\beta_1 + (d\rho)(\bar\rho\alpha_1\cup_1\beta_2 + \rho\alpha_2\cup_1\beta_1) + (d\rho)^2(\ldots$$

$$= \alpha_1\cup\beta_2 + d\left(\rho(\alpha_1\cup_1\beta_2 + (d\rho)(\alpha_1\cup_2\beta_2) + (d\rho)^2(\quad)+\ldots)\right)$$

Theorem A follows from

<u>Lemma B</u>

$\overline{SQ}\times\pi^* : \mathrm{Sym}^2 H^*(X)\otimes H^*(\mathbb{R}P^\infty) \to H^*(SQ(X))$ is surjective,
with kernel $A(H^2 H^*(X))\otimes\omega H^*(\mathbb{R}P^\infty)$.

Indeed, the lemma shows that kernel of restriction to fibre is generated by images of $SQ(a)\omega^i$ with $i>0$. Pulling back such an element of degree $2n$ to $X\times\mathbb{R}P^\infty$ always has non-zero ω^j coefficient for some $j\geq n$ because $Sq^0(a)=a$.

PROPERTIES OF Sq FROM SQ DESCRIPTION

Since

$$\Delta^*(SQ(a)\,SQ(b)) = \Delta^* SQ(a)\; \Delta^* SQ(b)$$

$$= \underline{Sq}(a)\,\underline{Sq}(b) = \sum_{i=0}^{m} Sq^i(a)\,\omega^{m-i} \sum_{j=0}^{n} Sq^j\,\omega^{n-j}$$

$$= \sum_{k=0}^{m+n} \sum_{l=0}^{k} Sq^l(a)\, Sq^{k-l}\, \omega^{m+n-k}$$

the Cartan formula follows immediately from

Lemma

$SQ(ab) = SQ(a)\,SQ(b)$ for any $a \in H^m(X)$, $b \in H^n(X)$

Proof:

By the above $\Delta^*(SQ(a)\,SQ(b))$ has no ω^i terms for $i > m+n$, and the restriction of $SQ(a)\,SQ(b)$ to a fibre is

$(\pi_1^* a \cup \pi_2^* a) \cup (\pi_1^* b \cup \pi_2^* b) = \pi_1^*(ab) \cup \pi_2^*(ab)$. $\square$

Proposition

If $R^k : H^n(X) \to H^{n+k}(X)$ are cohomology operations satisfying the axioms of Sq^k, ie

- R^k homomorphisms
- $R^0 = \mathrm{Id}$ and $R^k(a) = 0$ for $a \in H^n(X)$ with $k > n$
  $= a^2$ if $k = n$
- Cartan formula $R^k(ab) = \sum R^i(a) R^{k-i}(b)$

then $R^k = Sq^k$.

Proof:

Already explained that axioms $\Rightarrow$ stability.

In fact, functoriality implies that

stability $\Leftrightarrow$ for any pair (Y,A), the snake maps

$$\delta : H^n(Y,A) \to H^{n+1}(Y,A) \quad \text{satisfy}$$

$$R^k \circ \delta = \delta \circ R^k : H^n(A) \to H^{n+k+1}(Y,A)$$

Now apply that to $Y = SQ(X)$, $A =$ image of $\Delta : X \times \mathbb{R}P^\infty \hookrightarrow SQ(X)$

Lemma B

$\Rightarrow \left(\text{image of } \Delta^* : H^{2n}(SQ(X)) \to H^{2n}(X \times \mathbb{R}P^\infty)\right) \cap \bigoplus_{i>0} H^{n+i}(X)\,\omega^{n-i} = 0$

$\Leftrightarrow \delta : H^{2n}(X \times \mathbb{R}P^\infty) \to H^{2n}(SQ(X), X \times \mathbb{R}P^\infty)$ injective on $\bigoplus_{i>0} H^{n+i}(X)\,\omega^{n-i}$

~~[illegible]~~ For $a \in H^n(X)$, let $R(a) = \sum_{k=0}^{n} R^k(a)\,\omega^{n-k} \in H^{2n}(X \times \mathbb{R}P^\infty)$.

Since $R^0(a) = Sq^0(a) = a$,

$$R(a) - Sq(a) \in \bigoplus_{i>0} H^{n+i}(X)\,\omega^{n-i}$$

So it suffices to prove $\delta(R(a)) = \delta(Sq(a))$.

If $a = [\alpha]$ then $\rho\alpha_1 + \bar{\rho}\alpha_2$ is an invariant cochain on ~~[illegible]~~ $X \times X \times S^\infty$ that restricts to α, so

$$\delta(a) = [d(\rho\alpha_1 + \bar{\rho}\alpha_2)] = [(d\rho)(\alpha_1 + \alpha_2)] = \omega[\alpha_1 + \alpha_2] \in H^{2n+1}(SQ(X))$$

Hence

$$\delta(R(a)) = \sum \delta(R^k(a)\,\omega^{n-k}) = \sum \delta(R^k(a))\,\omega^{n-k} = \sum R^k(\delta(a))\,\omega^{n-k}$$

$$= \sum R^k(\omega[\alpha_1 + \alpha_2])\,\omega^{n-k} = \sum (\omega^2 R^{k-1}[\alpha_1 + \alpha_2] + \omega R^k[\alpha_1 + \alpha_2])$$

$$= \omega R^n[\alpha_1 + \alpha_2] + \omega^{n+1} R^0[\alpha_1 + \alpha_2] = \omega\, Sq^n[\alpha_1 + \alpha_2] + \omega^{n+1} Sq^0[\alpha_1 + \alpha_2] = \delta(Sq(a)).$$

$\square$

<u>Lemma</u> For any $a \in H^n(X)$,

$\underline{Sq}(\underline{Sq}(a)) \in H^{4n}(X \times \mathbb{R}P^\infty \times \mathbb{R}P^\infty)$ is invariant under swapping the two $\mathbb{R}P^\infty$ factors.

Proof sketch:

$\underline{Sq}(\underline{Sq}(a))$ is the pull-back of $SQ(SQ(a)) \in H^{4n}(SQ(SQ(X)))$.

Factor the relevant map $X \times \mathbb{R}P^\infty \times \mathbb{R}P^\infty \longrightarrow SQ(SQ(X))$ through a $\mathbb{Z}_2^3$-quotient of $X \times X \times X \times X \times S^\infty \times S^\infty$, and check that the class on that space is invariant under an involution.

The lemma potentially imposes relations between $\underline{Sq}^k \underline{Sq}^l$, which can be unravelled to the Adem relations.

4.3 Sq, SW AND WU

Two important links between Sq and SW classes were first noted by Wu.

"THE WU FORMULA"

$Sq^k w_m$ is a $\mathbb{Z}_2$ characteristic class of real vector bundles, so must be some polynomial in SW classes.

Proposition

$$Sq^k w_m = w_k w_m + \binom{k-m}{1} w_{k-1} w_{m+1} + \cdots + \binom{k-m}{k} w_0 w_{m+k}$$

where

$$\binom{p}{q} = \frac{p(p-1)\cdots(p-q+1)}{q!}$$ regardless of whether $p<0$ or $p>q$

Proof:
Prove for direct sums of line bundles, by induction on rank. $\square$

<u>Observe</u>

If $n = 2^r s$ with s odd ~~[illegible]~~ then

taking $k = 2^r$, $m = n-k$ $\Rightarrow$ $\binom{k-m}{k} = 1$

$$\rightsquigarrow w_n = Sq^k w_m + w_k w_m + \cdots + \binom{k-m}{k-1} w_1 w_{m-1}$$

Provided that n is not a power of 2 (so $s \neq 1$, and $m > 0$), this expresses w_n in terms of SW classes of lower degree.

$\Rightarrow$ all SW classes determined by ones in power-of-2 degrees + Steenrod operations.

(In fact, if $n = 2^r s$ with s odd then w_n determines w_i for $i = n+1, \dots, n+2^r-1$.)

CONSTRUCTING SW CLASSES BY Sq AND THOM CLASS

Recall: For any real rank n vector bundle $E \to X$ (oriented or not) there is a mod 2 Thom class

$$u(E) \in H^n(E, \dot{E}; \mathbb{Z}_2)$$

st
$$\begin{aligned} H^k(X) &\to H^{n+k}(E, \dot{E}) \\ a &\mapsto \pi^* a \cup u(E) \end{aligned}$$
is an isomorphism.

Now there is a well-defined $Sq^k(u(E)) \in H^{n+k}(E, \dot{E})$, and hence a unique $\tilde{w}_k(E) \in H^k(X)$ st

$$u(E) \cup \pi^* \tilde{w}_k(E) = Sq^k(u(E))$$

Lemma

$\tilde{w}_k$ satisfies the Whitney sum formula.

$$\tilde{w}_k(E \oplus E') = \sum \tilde{w}_i(E) \tilde{w}_{k-i}(E')$$

Proof:
Apply the Cartan formula to $Sq^k(u(E \oplus E')) = Sq^k(u(E)u(E'))$ $\square$

Since $\tilde{w}_1$ of line bundles agrees with w_1 almost by definition, $\tilde{w}_k = w_k$ by uniqueness of SW classes proved before.

Indeed one could use this instead of Grothendieck argument to prove existence of SW classes.

Only advantage of Grothendieck argument is expedience if one has already used it for Chern classes, while description in terms of Sq has other useful consequences.

WU'S THEOREM

Let M be a closed n-dimensional manifold. Whether M is oriented or not, there is a fundamental class $[M] \in H_n(M;\mathbb{Z}_2)$, and by Poincaré duality

$$\begin{aligned} H^k(M) \times H^{n-k}(M) &\to \mathbb{Z}_2 \\ (a, b) &\mapsto (a \cup b)[M] \end{aligned}$$

is a perfect pairing, ie induces an isomorphism

$$H^k(M) \to \mathrm{Hom}(H^{n-k}(M), \mathbb{Z}_2)$$

In particular, $\exists$ unique $v_k(M) \in H^k(M)$ mapping to

$$(b \mapsto Sq^k(b)[M]) \in \mathrm{Hom}(H^{n-k}, \mathbb{Z}_2)$$

Definition

$v_k(M)$ are the <u>Wu classes</u> of the closed manifold M.

Note:

- $v_k(M) = 0$ if $2k > n$, since then Sq^k vanishes on H^{n-k}.
- If $n = 2k$ then $v_k(M)$ is "characteristic" for the mod 2 intersection form on M, ie
$$a^2 = v_k(M)a \quad \text{for any} \quad a \in H^k(M)$$

The total Wu class is $v(M) = 1 + v_1(M) + \dots + v_n(M) \in H^*(M)$.

If M is smooth, then it has a tangent bundle TM, and we can also consider the SW classes $w_k(M) \equiv w_k(TM)$.

Theorem

The total SW class $w(M) = 1 + w_1(M) + \dots + w_n(M)$ equals the total Steenrod square $Sq(v(M)) = \sum_{i \geq 0} Sq^i(v(M))$, ie
$$w_k(M) = \sum_{i=0}^{k} Sq^i(v_{k-i}(M))$$

Corollary

SW classes of closed smooth manifolds are invariant under homotopy equivalences (and not just diffeomorphisms)

PROOF OF WU'S THEOREM

Sneaky trick:
The normal bundle of the diagonal $\Delta \subset M \times M$ is isomorphic to TM, so by excision

$$H^*(TM, \dot{TM}) \cong H^*(M \times M, M \times M \setminus \Delta)$$

That maps to $H^*(M \times M) \cong H^*(M) \otimes H^*(M)$. Let $u' \in H^*(M \times M)$ be the image of the Thom class $u \in H^n(TM, \dot{TM})$.

Lemma

Let e_i be a basis for $H^*(M)$, and let $\hat{e}_i$ be the dual basis (wrt the perfect cup product pairing).
Let $p, q : M \times M \to M$ be the projections. Then

$$u' = \sum_i p^* e_i \cup q^* \hat{e}_i$$

Proof:

$u' \in H^n(M \times M)$ is characterised by

$$(u' \cup a)[M \times M] = (a|_\Delta)[\Delta] \quad \text{for all } a \in H^*(M \times M)$$

Any a can be written as a sum of $p^*x \cup q^*y$ for $x, y \in H^*(M)$, so equivalently

$$(u' \cup p^*x \cup q^*y)[M \times M] = (p^*x \cup q^*y)|_\Delta [\Delta] = (x \cup y)[M]$$

for all $x, y \in H^*(M)$ (of degrees adding to n). $\sum p^* e_i \cup q^* \hat{e}_i$ satisfies this. $\square$

Now let $R : H^*(M) \to H^*(M \times M)$ be the composition of the Thom isomorphism ϕ, excision, and push-forward

$$\begin{tikzcd} H^*(M) \arrow[r, "\phi"] \arrow[rrr, bend left, "R"] & H^*(TM, T\dot{M}) \arrow[r, phantom, "\cong"] & H^*(M \times M, M \times M \setminus \Delta) \arrow[r] & H^*(M \times M) \arrow[lll, bend left, "L"] \end{tikzcd}$$

and seek a left inverse $L : H^*(M \times M) \to H^*(M)$.

Define $I : H^*(M) \to \mathbb{Z}_2$ as evaluation on $[M]$ on $H^n(M)$, and 0 on lower degrees. Note that

$$x = \sum e_i \, I(x \cup \hat{e}_i) \quad \text{for any } x \in H^*(M)$$

Lemma

If we set $L(p^*x \cup q^*y) = x\, I(y)$, then $L : H^*(M \times M) \to H^*(M)$ satisfies $LR = \mathrm{Id} : H^*(M) \to H^*(M)$

Proof:

$R(x) = u' \cup p^*x = u' \cup q^*x = \sum p^*e_i \cup q^*(x \cup \hat{e}_i)$, so

$LR(x) = \sum e_i \, I(x \cup \hat{e}_i) = x.$ $\square$

Hence

$$w(M) = \phi^{-1} Sq(u) = L\, Sq(u')$$

$$= L\, Sq\left(\Sigma p^* e_i \cup q^* \hat{e}_i\right)$$

Cartan $= L \sum p^*(Sq\, e_i) \cup q^*(Sq\, \hat{e}_i)$

$$= \sum Sq\, e_i \; I(Sq\, \hat{e}_i)$$

def of v_k $= \sum Sq\, e_i \; I(v \cup \hat{e}_i)$

$$= Sq\left(\Sigma e_i \; I(v \cup \hat{e}_i)\right) = Sq(v)$$

$\square$

4.4 SOME APPLICATIONS

Using Wu's theorem, one can also determine the Wu classes from the Stiedel-Whitney classes, eg

$$v_1 = w_1$$

$$w_2 = v_2 + v_1^2 \Rightarrow v_2 = w_2 + w_1^2$$

$$\vdots$$

In particular, for any orientable closed manifold

$$v_2 = w_2$$

<u>Corollary</u>

- The tangent bundle of a closed 3-dimensional mfd is trivial iff it is orientable (since $v_2 = 0$)
- For an oriented closed 4-mfd M, $w_2(M)$ is characteristic for the intersection form, ie $a^2 = w_2(M)a$ mod 2 for all $a \in H^2(M;\mathbb{Z})$. In particular, the intersection form is even iff M is spin.

THE PRIMARY OBSTRUCTION FOR SPIN BUNDLES

$SU(n)$ is simply-connected, so the natural map $SU(n) \hookrightarrow SO(2n)$ lifts to the double cover $Spin(2n) \to SO(2n)$.

In particular, $SU(2) \hookrightarrow Spin(4) \hookrightarrow Spin(m)$ for any $m \geq 4$.

<u>Lemma</u>

For $m \geq 5$, $SU(2) \hookrightarrow Spin(m)$ is a 4-equivalence.

In particular, $\pi_3 Spin(m) \cong \pi_3 SU(2) \cong \pi_3 S^3 \cong \mathbb{Z}$.

Thus the primary obstruction to triviality of a $Spin(m)$-bundle $E \to X$ is a class $p(E) \in H^4(X; \mathbb{Z})$.

<u>Lemma</u>

i) $p_1(E) = 2p(E)$, and $w_4(E) = p(E)$ mod 2

ii) If E is a complex vector bundle with $c_1(E) = 0$

then $p(E) = -c_2(E)$

Proof:

ii) $SU(2) \hookrightarrow Spin(m)$ ~~4~~-equivalence $\Rightarrow$

$BSU(2) \to BSpin(m)$ 5-equivalence,

So generator $p \in H^4(BSpin(m); \mathbb{Z})$ pulls back to

generator $-c_2 \in H^4(BSU(2); \mathbb{Z}) \cong \mathbb{Z}$

i) Same relation exists, coefficients fixed by ii). $\square$

Corollary

- For a closed spin manifold of dim ≤ 7

$$p(M) = w_4(M) = v_4(M) = 0 \mod 2$$

so $p_1(M)$ is divisible by 4.

- For a closed spin 8-manifold, $p(M)$ is characteristic for the intersection form.

$$\Rightarrow \quad p(M)^2 = \text{signature modulo } 8.$$