GROTHENDIECK DEFINITION OF CHERN CLASSES

For a cx rank n bundle $E \to X$, consider the tautological line bundle $T \to \mathbb{P}(E)$ of $\pi : \mathbb{P}(E) \to X$ as above, and

$$u := -c_1(T) \in H^2(\mathbb{P}(E); \mathbb{Z})$$

By Leray-Hirsch, $1, \dots, u^{n-1}$ is a basis for $H^*(\mathbb{P}(E); \mathbb{Z})$ over $H^*(X; \mathbb{Z})$. In particular, $u^n \in H^{2n}(\mathbb{P}(E); \mathbb{Z})$ can be written in terms of that basis.

Definition

The Chern classes of E are the unique $c_k(E) \in H^{2k}(X; \mathbb{Z})$ st

$$u^n = - \sum_{i=1}^{n} \pi^* c_i(E) \cup u^{n-i}$$

Proposition

Thus defined, the Chern classes satisfy the Whitney sum formula

$$c_k(E \oplus E') = \sum c_i(E)\, c_{k-i}(E')$$

Proof:

Line splitting principle + $\mathbb{C}P^\infty$ is classifying space for complex line bundles $\Rightarrow$ it suffices to check that if X is the product of n copies of $\mathbb{C}P^\infty$ and E is the direct sum $L_1 \oplus \cdots \oplus L_n$ of their tautological line bundles, then the total Chern class of E is

$$c(E) = (1+t_1)(1+t_2)\cdots(1+t_n)$$

for $t_i = c_1(L_i) \in H^2(X;\mathbb{Z}) \cong \bigoplus_{i=1}^{n} H^2(\mathbb{C}P^\infty;\mathbb{Z})$.

Observe: each summand L_i defines a section s_i of $P(E) \to X$

$$s_i^* T \cong L_i, \text{ so } s_i^* u = s_i^*(-c_1(T)) = -c_1(L_i) = -t_i.$$

$$\sum_{k=0}^{n} u^k \pi^* c_{n-k}(E) = 0 \Rightarrow \sum_{k=0}^{n} (-t_i)^k c_{n-k}(E) = 0 \text{ for each } i$$

Thus the polynomial $\sum_{k=0}^{n} t^k c_{n-k}(E) \in H^*(X;\mathbb{Z})[t]$ factorises as $(t+t_1)(t+t_2)\cdots(t+t_n)$. $\square$

COHOMOLOGY OF BU(n)

Theorem

$$H^*(BU(n);\mathbb{Z}) = \mathbb{Z}[c_1,\dots,c_n]$$

Recall that for any $U(n)$-bundle $E \xrightarrow{q} X$, the flag bundle $\pi : Fl\,E \to X$ has fibres {n-tuples of orthogonal lines in E_x}.
The total space of $Fl\,E$ has n tautological line bundles $L_1,\dots,L_n \to Fl\,E$, and $\pi^* E \cong L_1 \oplus \dots \oplus L_n$.
Moreover $\pi^* : H^*(X;\mathbb{Z}) \hookrightarrow H^*(Fl\,E;\mathbb{Z})$

Applying this to $EU(n) \to BU(n) = Gr_n(\mathbb{C}^\infty)$, the total space of $Fl\,EU(n)$ is $F_n :=$ {n-tuples of orthogonal lines in $\mathbb{C}^\infty$}.

$F_1 = \mathbb{CP}^\infty$, and the obvious map $F_n \to F_{n-1}$ is a $\mathbb{CP}^\infty$-fibre bundle

Repeated application of Leray–Hirsch

$\Rightarrow$ $H^*(F_n;\mathbb{Z}) = \mathbb{Z}[t_1,\dots,t_n]$ for $t_i = c_1(L_i)$

Clearly $\pi^* : H^*(BU(n);\mathbb{Z}) \hookrightarrow H^*(F_n;\mathbb{Z})$ is invariant under the action of the symmetric group S_n on F_n that permutes the n lines in the n-tuple (and hence the t_i), so the image is contained in

$$\mathbb{Z}[t_1,\dots,t_n]^{S_n} = \mathbb{Z}[\sigma_1,\dots,\sigma_n]$$

where σ_i is the i^{th} elementary symmetric function in $t_1,\dots,t_n$.

But $\pi^* c_k = \sigma_k$ by the Whitney sum formula. Thus

$$\pi^* : H^*(BU(n);\mathbb{Z}) \cong \mathbb{Z}[\sigma_1,\dots,\sigma_n]$$

and $H^*(BU(n);\mathbb{Z})$ is the free polynomial ring generated by the c_k. $\square$

3.6 CHARACTERISTIC CLASSES OF REAL VECTOR BUNDLES

Real vector bundles of rank n $\leftrightarrow$ principal $GL(n,\mathbb{R})$-bundles
$\leftrightarrow$ principal $O(n)$-bundles

Orientation $\leftrightarrow$ $GL_+(n,\mathbb{R})$-reduction $\leftrightarrow$ $SO(n)$-reduction

Since each fibre of $E\to X$ has two possible orientations, an orientation can also be described as a section of a double cover (equivalently, that double cover is the $SO(n)$-quotient of the $O(n)$ frame bundle).

Double covers of X $\leftrightarrow$ index ≤ 2 subgroups of $\pi_1 X$
$\leftrightarrow$ $\mathrm{Hom}(\pi_1 X, \mathbb{Z}/2) \cong H^1(X;\mathbb{Z}/2)$ Hurewicz + universal coeffs.

Thus we can assign to each real $E\to X$ a class $w_1(E) \in H^1(X;\mathbb{Z}/2)$ st E orientable $\Leftrightarrow$ $w_1(E)=0$.

Note
- $w_1(\mathcal{O}_{\mathbb{R}P^n}(-1))$ is the non-zero element in $H^1(\mathbb{R}P^n;\mathbb{Z}/2)\cong\mathbb{Z}/2$
- If base X is a dim 1 CW complex then orientable $\Leftrightarrow$ trivial, so w_1 is also primary obstruction to triviality.

STIEFEL-WHITNEY CLASSES

Theorem

- For real rank n bundles $E \to X$ there are characteristic classes $w_k \in H^k(X; \mathbb{Z}_2)$, $k = 1, \dots, n$, called the *Stiefel-Whitney classes*, st
  - the Whitney sum formula holds
  $$w_k(E \oplus E') = \sum_{i=0}^{k} w_i(E) w_{k-i}(E')$$
  - w_1 of line bundles is as defined above
- These conditions uniquely determine the SW classes
- $H^*(BO(n); \mathbb{Z}_2)$ is the free polynomial algebra $\mathbb{Z}_2[w_1, \dots, w_n]$

The proof is entirely analogous to that for Chern classes. In particular, one uses a real line splitting principle:

For any real $E \to X$, $\exists f: Y \to X$ st

- $f^*: H^*(X; \mathbb{Z}_2) \hookrightarrow H^*(Y; \mathbb{Z}_2)$ (not with $\mathbb{Z}$ coefficients)
- f^*E is isomorphic to a direct sum of line bundles on Y.

Other consequences of splitting principles

- $w_1(E) = w_1(\det E)$, so agrees with the definition in terms of orientations
- If E has rank n, then $w_n(E)$ is the pull-back of the mod 2 Thom class $u \in H^n(E, \dot{E}; \mathbb{Z}_2)$ by any section. In particular, if E is oriented then $w_n(E) = e(E) \bmod 2$.
- For a complex vector bundle $E \to X$ of rank n, consider the real rank $2n$ vector bundle $E_{\mathbb{R}} \to X$ that forgets the complex structure.

$$w_{2k+1}(E) = 0 \qquad w_{2k}(E) = c_k(E) \bmod 2.$$

THE ORIENTED GRASSMANNIAN

Because $SO(n) < O(n)$, one can take $ESO(n) = EO(n)$, and $BSO(n) = EO(n)/SO(n)$. That can be realised explicitly as

$$\widetilde{Gr}_n(\mathbb{R}^\infty) = \{n\text{-planes in } \mathbb{R}^\infty \text{ with a choice of orientation}\}.$$

The obvious double cover $p: \widetilde{Gr}_n(\mathbb{R}^\infty) \to Gr_n(\mathbb{R}^\infty)$ can also be thought of as the classifying map for $ESO(n) \to BSO(n)$ considered as an $O(1)$-bundle.

<u>Proposition</u>

$p^*: H^*(BO(n); \mathbb{Z}_2) \to H^*(BSO(n); \mathbb{Z}_2)$ is surjective, and the kernel is the ideal generated by w_1, so $H^*(BSO(n); \mathbb{Z}_2)$ is the free polynomial algebra

$\mathbb{Z}_2[w_2, \dots, w_n]$.

Proof:

If E is the real line bundle $\det EO(n) \to BO(n)$ and $\dot{E}$ is the complement to the zero section, then $\dot{E} \simeq BSO(n)$, so

$$H^*(BSO(n); \mathbb{Z}_2) \cong H^*(\dot{E}; \mathbb{Z}_2)$$

can be understood by Thom isomorphism theorem + LES for cohomology of $(E, \dot{E})$. ($\rightsquigarrow$ Gysin sequence)

<u>A warning</u>

That $\mathbb{Z}_2[w_1, \dots, w_n]$ is a free polynomial algebra only means $w_1, \dots, w_n$ are independent with respect to cup product structure, not with respect to cohomology operations. Eg consider the Bockstein $\beta : H^n(X; \mathbb{Z}_2) \to H^{n+1}(X; \mathbb{Z}_2)$ of the sequence of coefficients $0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$

Line splitting principle $\Rightarrow \beta(w_{2k}) = w_{2k+1} + w_1 w_{2k}$ (and $\beta(w_{2k+1}) = w_1 w_{2k+1}$)

Thus $w_{2k} = 0 \Rightarrow w_{2k+1} = 0$!

Considering Steenrod squares, $w_{2^i}, w_{2^{i+1}}, \dots, w_{2^j}$ determine w_k for all $2^i \leq k < 2^{j+1}$!

PONTRJAGIN CLASSES

For a real rank n bundle $E \to X$, consider the complex rank n bundle $E_{\mathbb{C}} := E \otimes_{\mathbb{R}} \mathbb{C}$.

Exercise: $E_{\mathbb{C}} \cong \overline{E_{\mathbb{C}}}$ as complex vector bundles, so odd Chern classes of $E_{\mathbb{C}}$ are 2-torsion

$c_k(E) = (-1)^k c_k(\overline{E}) \Rightarrow$ odd Chern classes of $E_{\mathbb{C}}$ are 2-torsion (so Bockstein of some combination of SW classes)

Definition

The Pontrjagin classes of a real rank n bundle $E \to X$ are

$$p_k(E) := (-1)^k c_{2k}(E_{\mathbb{C}}) \in H^{4k}(X;\mathbb{Z}) \quad \text{for } 2k \leq n$$

Easy consequences:

- Whitney sum formula modulo 2-torsion

$$2p_k(E \oplus E') = 2\sum_{i=0}^{k} p_i(E)\, p_{k-i}(E')$$

- For an orientable bundle of even rank $2n$

$$p_n(E) = e(E)^2 \in H^{4n}(X;\mathbb{Z})$$

Theorem

For $B = BO(n)$ or $BSO(n)$, the quotient of $H^*(B;\mathbb{Z})$ by the 2-torsion subgroup is a free polynomial algebra.

- $\mathbb{Z}[p_1, \dots, p_k]$ for $B = BO(2k)$, $BO(2k+1)$ or $BSO(2k+1)$
- $\mathbb{Z}[p_1, \dots, p_{k-1}, e]$ for $B = BSO(2k)$

Proof uses

Oriented plane splitting principle

For any real [redacted] $E \to X$, $\exists f : Y \to X$ st

- f^*E is a sum of oriented bundles of rank ~~4~~ 2 and real line bundles (which can be taken trivial if E is oriented)
- the kernel of $f^* : H^*(X;\mathbb{Z}) \to H^*(Y;\mathbb{Z})$ consists only of 2-primary torsion elements.

Proof idea:

To split off a single 2-plane, take $Y = \widetilde{Gr}_2(E)$.

Show that $\ker H^*(X;\mathbb{Z}) \to H^*(\widetilde{Gr}_2(E);\mathbb{Z})$ is 2-torsion by identifying fibre $\widetilde{Gr}_2(E_x)$ with a degree 2 hypersurface in $\mathbb{P}(E_x \otimes_{\mathbb{R}} \mathbb{C})$.

Ch 4 STEENROD SQUARES

4.1 AXIOMS AND CONSTRUCTION

Abbreviate $H^*(\ ; \mathbb{Z}_2)$ as $H^*(\ )$.

Theorem

For each n and $k \geq 0$ there is a cohomology operation

$$Sq^k : H^n(X) \to H^{n+k}(X)$$

st

i) Sq^k is a homomorphism (of $\mathbb{Z}_2$ vector spaces)

ii) $Sq^0 = \mathrm{Id}$

iii) For $k=n$, $Sq^n : H^n(X) \to H^{2n}(X)$ maps $\alpha \mapsto \alpha^2$

iv) $Sq^k = 0$ for $k > n$

v) The Cartan formula holds: $Sq^k(\alpha\beta) = \sum_{i=0}^{k} Sq^i(\alpha)\, Sq^{k-i}(\beta)$

Moreover, these properties completely characterise Sq^k.

TWO EASY CONSEQUENCES OF AXIOMS

Lemma
$Sq^1 : H^n(X) \to H^{n+1}(X)$ coincides with the Bockstein map of the SES of coefficients $0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$.

Proof:
$K(\mathbb{Z}_2, n)$ can be constructed explicitly as a CW complex with a single 0-cell, no i-cells for $0 < i < n$, a single n-cell, and a single $(n+1)$-cell with attaching map (of degree 2).
Therefore $H^{n+1}(K(\mathbb{Z}_2, n); \mathbb{Z}_2)$ can have rank at most 1, so there cannot be two different non-zero cohomology operations $H^n(X) \to H^{n+1}(X)$ $\square$

Any cohomology operation $\alpha : H^m(X; G) \to H^n(X; K)$ has a relative version $H^m(X, A; G) \to H^n(X, A; K)$ (because $H^m(X, A; G) \leftrightarrow$ homotopy classes of maps $X \to K(G, n)$ that map A to base point).

Given X, consider the suspension pair $(X \times I, X \times \{0,1\})$.
LES $\rightsquigarrow$ natural isomorphism $\delta : H^m(X;G) \to H^{m+1}(X \times I, X \times \{0,1\}; G)$.

For any cohomology operation $\alpha : H^m(X;G) \to H^n(X;K)$ one can thus define another one by the composition

$$H^{m-1}(X;G) \xrightarrow{\delta} H^m(X \times I, X \times \{0,1\}; G) \xrightarrow{\alpha} H^n(X \times I, X \times \{0,1\}; K) \xrightarrow{\delta^{-1}} H^{n-1}(X;K)$$

That is the <u>suspension</u> of α.
Sq^k is a <u>stable</u> cohomology operation in the sense that it is really a sequence of operations related by suspension.

<u>Lemma</u>
$\delta \circ Sq^k = Sq^k \circ \delta : H^n(X) \to H^{n+k+1}(X \times I, X \times \{0,1\})$.

Proof:
Because the coefficients $G = \mathbb{Z}_2$ are a ring, δ can be described as cup product with a degree 1 class ω pulled back from $H^1(I, \{0,1\}) \cong \mathbb{Z}_2$.
Claim follows from Cartan formula and $Sq^1(\omega) = \omega^2 = 0$.

$\square$

SOME LESS OBVIOUS PROPERTIES

Adem relations

For $k < 2l$

$$Sq^k Sq^l = \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \binom{l-i-1}{k-2i} Sq^{k+l-i} Sq^i$$

Consequence:

Any composition of Steenrod squares is a linear combination of admissible ones, ie one of the form

$$Sq^{k_1} Sq^{k_2} \cdots Sq^{k_j} \quad \text{with} \quad k_i \geq 2k_{i+1}, \ \forall i$$

Theorem

- The admissible compositions form a basis for the $\mathbb{Z}_2$ vector space of stable $\mathbb{Z}_2$ cohomology operations
- For each n, the algebra of $\mathbb{Z}_2$ cohomology operations on $H^n(X;\mathbb{Z}_2)$ (ie $H^*(K(\mathbb{Z}_2,n);\mathbb{Z}_2)$) is a free polynomial algebra generated by some of the admissible compositions.

Lemma

For $R = \mathbb{Z}_2$

$$C^m(X) \to C^{m+k}(X), \quad \alpha \mapsto \alpha \cup_{m-k} \alpha$$

induces a well-defined homomorphism

$$Sq^k : H^m(X) \to H^{m+k}(X)$$

So for $[\alpha] \in H^m(X)$, $Sq^m([\alpha]) = [\alpha \cup_0 \alpha] = [\alpha]^2$.

Less obvious to verify $Sq^0 = \mathrm{Id}$ and Cartan formula.

DEFINITION OF Sq VIA CUP-i PRODUCTS

Slogan: Sq comes from failure of cup product to commute on singular cochains with $\mathbb{Z}_2$ coefficients (or any other cochain complex that produces $H^*(\,;\mathbb{Z}_2)$).

For any ring R, one way to measure failure of graded commutativity on $C^*(X) = \{$singular cochains with R coefficients$\}$ is the cup-1 product.
But that is not commutative...

Theorem

For each $i \geq 0$ there exists a bilinear product $C^m(X) \times C^n(X) \to C^{m+n-i}(X)$ s.t.
$(\alpha, \beta) \mapsto \alpha \cup_i \beta$

- $\cup_0$ is the usual cup product
- $d(\alpha \cup_i \beta) = (-1)^i (d\alpha) \cup_i \beta + (-1)^{i+m} \alpha \cup_i (d\beta) + (-1)^{i+1} \alpha \cup_{i-1} \beta + (-1)^{mn+1} \beta \cup_{i-1} \alpha$

(Sign conventions vary, but even once fixed that does not determine $\cup_i$ completely.)

Concrete point of view:

For $\cup_i$ to have a nice description in terms of simplices, we should define D_i on the standard simplices Δ_n and then extend to all singular simplices functorially.

(In the context of simplicial complexes and their simplicial chain complex, one must choose an ordering of all vertices to make sense of this.)

Eg the standard cup product can be described by specifying

$$D_0(\Delta_n) = \sum_{j=0}^{n} (-1)^j \,(\text{front } j\text{-simplex}) \otimes (\text{back } (n-j)\text{-simplex})$$

$$\in C_*(\Delta_n) \otimes C_*(\Delta_n)$$

CONSTRUCTING CUP-i PRODUCTS BY BARE HANDS

A product $\cup_i : C^*(X) \times C^*(X) \to C^*(X)$ is equivalent to a "coproduct" $D_i : C_*(X) \to C_*(X) \otimes C_*(X)$ by requiring $(\alpha \cup_i \beta)(\sigma) = (\alpha \otimes \beta)(D_i \sigma)$ for any $\sigma \in C_*(X)$.

The conditions on $\cup_i \iff D_i$ increases degree by i and

$$D_i \partial - \partial D_i = D_{i-1} + \tau \circ D_{i-1} \qquad (*)$$

for τ the swapping automorphism on $C_*(X) \otimes C_*(X)$ and ∂ on $C_*(X) \otimes C_*(X)$ defined by Leibniz rule.

Sophisticated point of view:
Can use "Acyclic carrier theorem" to ensure existence of chain maps with nice properties.

In these terms, we need to find a collection of $D_i^n \in C_*(\Delta_n) \otimes C_*(\Delta_n)$ of degree $n+i$ with D_0^n prescribed and satisfying a relation equivalent to (*).

Let $I : C_k(\Delta_n) \to C_k(\Delta_{n+1})$ be the obvious inclusion, and
$C : C_k(\Delta_n) \to C_{k+1}(\Delta_{n+1})$ the "cone map" that adds the vertex $n+1$ to each simplex.

Lemma

Recursively defining

$$D_i^n = (C \otimes C)(I D_{i-1}^{n-1}) + (I \otimes C)(D_i^{n-1}) \in C_*(\Delta_n) \otimes C_*(\Delta_n)$$

works.

Easy consequence:

$D_n^n = \Delta_n \otimes \Delta_n$ for each n.

So for $\alpha, \beta \in C^n(X)$ and $\sigma \in C_n(X)$, define $\alpha \cup_n \beta \in C^n(X)$ by

$$(\alpha \cup_n \beta)(\sigma) = \alpha(\sigma)\beta(\sigma) \in R$$

In particular, if coefficients are $\mathbb{Z}_2$ then

$$(\alpha \cup_n \alpha)(\sigma) = \alpha(\sigma)^2 = \alpha(\sigma) \in \mathbb{Z}_2$$

so $\alpha \cup_n \alpha = \alpha \in C^n(X)$, so

$$Sq^n[\alpha] = [\alpha \cup_n \alpha] = [\alpha], \text{ ie } Sq^n = Id \text{ on } H^n(X).$$

4.2 THE SQUARING FUNCTOR

Let $A : S^\infty \to S^\infty$ denote the antipodal map (so $\mathbb{R}P^\infty = S^\infty/A$)

Given a space X, let $\tau : X \times X \to X \times X$, $(x,y) \mapsto (y,x)$ and

$$SQ(X) = X \times X \times S^\infty / \tau \times A$$

$SQ(X) \to \mathbb{R}P^\infty$ is an $X \times X$-fibre bundle.

The restriction to a fibre of any $S \in H^*(SQ(X))$ is τ^*-invariant.

Given $a = [\alpha] \in H^n(X)$, is there a class $SQ(a) \in H^*(SQ(X))$ that restricts to $\pi_1^* a \cup \pi_2^* a \in H^{2n}(X \times X)$?

Let $\alpha_i := \pi_i^* \alpha \in C^n(X \times X \times S^\infty)$ for $\pi_1, \pi_2 : X \times X \times S^\infty \to X$ the projections.

Cochains on $SQ(X) \leftrightarrow (\tau \times A)$-invariant cochains on $X \times X \times S^\infty$.

If cup product on cochains were commutative we could take $SQ([\alpha]) = [\alpha_1 \cup \alpha_2]$

So failure of commutativity is reflected by difficulty defining $SQ(a)$.