3.3 CLASSIFYING SPACES

CLASSIFYING MAPS FOR CX LINE BUNDLES

$\mathbb{C}P^n$ has a tautological cx line bundle $\mathcal{O}_{\mathbb{C}P^n}(-1)$, as does $\mathbb{C}P^\infty = \mathbb{C}^\infty / \mathbb{C}^* = S^\infty / U(1) = \bigcup_{n \geq 0} \mathbb{C}P^n$

Proposition

For any CW complex X

a) for any cx line bundle $E \to X$, $\exists f : X \to \mathbb{C}P^\infty$ st

$$E \cong f^* \mathcal{O}(-1)$$

b) $f^* \mathcal{O}(-1) \cong g^* \mathcal{O}(-1)$ iff $f \simeq g : X \to \mathbb{C}P^\infty$

(If X has finite dimension, can replace $\mathbb{C}P^\infty$ by $\mathbb{C}P^n$ provided that $2n+2 > \dim X$ in (a), or $2n+1 > \dim X$ in (b))

Thus $\left\{ \begin{array}{l} \text{isomorphism classes of cx} \\ \text{line bundles on } X \end{array} \right\} \longleftrightarrow [X, \mathbb{C}P^\infty]$

Corollary

For a CW complex X, the first Chern class defines a bijection

$$\left\{\begin{array}{l}\text{isomorphism classes of } \mathbb{C} \\ \text{line bundles on } X\end{array}\right\} \longleftrightarrow H^2(X;\mathbb{Z})$$

Proof.

Because S^∞ is contractible and S^1 is $K(\mathbb{Z},1)$, the LES for homotopy groups of the fibration $S^1 \to S^\infty \to \mathbb{C}P^\infty$ implies $\mathbb{C}P^\infty$ is a $K(\mathbb{Z},2)$.

Thus $[X, \mathbb{C}P^\infty] \longleftrightarrow H^2(X;\mathbb{Z})$, $f \mapsto f^* u_\mathbb{Z}$

for $u_\mathbb{Z} \in H^2(\mathbb{C}P^\infty;\mathbb{Z})$ the canonical element.

Check that $c_1(\mathcal{O}(-1)) = u_\mathbb{Z}$. $\square$

Proof of proposition: Let $L \to X$ be a cx line bundle.
Given $x \in X$ and $y \in \mathbb{CP}^n$

isomorphism $\phi : L_x \xrightarrow{\sim} \mathcal{O}(-1)_y$

$\Leftrightarrow$ homomorphism $\psi : L_x \to \mathbb{C}^{n+1}$ with image y

So given $x \in X$

pairs (ϕ, y) $\Leftrightarrow$ $\psi \in \mathrm{Hom}(L_x, \mathbb{C}^{n+1}) \setminus \{0\}$

Hence

pair $(f : X \to \mathbb{CP}^n$, bundle isomorphism $\Phi : L \to f^* \mathcal{O}(-1))$

$\Leftrightarrow$ nowhere-vanishing section Ψ of $\mathrm{Hom}(L, \mathbb{C}^{n+1})$,

a cx rank $n+1$ vector bundle over X

$rk_{\mathbb{R}} (\mathrm{Hom}(L, \mathbb{C}^{n+1})) = 2n+2 > \dim X \Rightarrow$ such section Ψ exists

$> \dim X \times [0,1] \Rightarrow$ any two Ψ are homotopic.

That homotopic maps $X \to \mathbb{CP}^n \rightsquigarrow$ isomorphic bundles is intuitive but technical.

$\square$

<u>Corollary</u>

$$\left\{\begin{array}{l}\text{characteristic classes of} \\ \text{cx line bundles}\end{array}\right\} \longleftrightarrow H^*(\mathbb{C}P^\infty;\mathbb{Z})$$

$$c \longmapsto c(\mathcal{O}(-1))$$

So is generated as a ring by c_1

Proof:
Given $\alpha \in H^*(\mathbb{C}P^\infty;\mathbb{Z})$, define a characteristic class for cx line bundles by assigning to

- a cx line bundle $E \to X$ over a CW complex X

$$f^*\alpha \in H^*(X;\mathbb{Z}) \quad \text{for any } f: X \to \mathbb{C}P^\infty \text{ st } f^*\mathcal{O}(-1) \cong E$$

- a cx line bundle $E \to X$ over an arbitrary space

the class of its pull-back to any CW approximation of X. $\square$

Lemma

For any $\mathbb{C}$ line bundles $E, E' \to X$

$$c_1(E \otimes E') = c_1(E) + c_1(E') \in H^2(X;\mathbb{Z})$$

Proof:

Suffices to check when X is product of two copies of $\mathbb{C}P^\infty$, ~~and~~ E, E' are pull-backs of $\mathcal{O}(-1)$ from the two factors.

Now $H^2(X;\mathbb{Z}) = H^2(\mathbb{C}P^\infty;\mathbb{Z}) \oplus H^2(\mathbb{C}P^\infty;\mathbb{Z})$,

So suffices to check that $j^* c_1(E \otimes E') = c_1(\mathcal{O}(-1))$

where $j : \mathbb{C}P^\infty \to X$ maps $y \mapsto (y, x_0)$ or $y \mapsto (x_0, y)$ for a fixed x_0.

But clearly $j^*(E \otimes E') \cong \mathcal{O}(-1)$. $\square$

UNIVERSAL BUNDLES

Definition

A principal G-bundle $EG \to BG$ is <u>universal</u> if for any paracompact* X

$$[X, BG] \to \{\text{isomorphism classes of } G\text{-bundles } P \to X\}$$
$$f \mapsto f^* EG$$

is a bijection.

*CW complex good enough for us

Then BG is called a <u>classifying space</u> for G.

Lemma

If a universal bundle $EG \to BG$ exists then

G-characteristic classes in $H^*(\; ; R) \leftrightarrow H^*(BG; R)$

Proof:

Easy because the definition is so strong. $\square$

Theorem

1) $EG \to BG$ is universal $\Leftrightarrow$ EG weakly contractible (i.e. $\pi_i EG = 0$ for all i)
2) For any compact Hausdorff G, there is a principal G-bundle $EG \to BG$ with EG contractible.
3) A universal bundle $EG \to BG$ is unique up to bundle-homotopy-equivalence. In particular, a classifying space BG is unique up to homotopy equivalence.

Proof sketch:

$\Leftarrow$ of 1): Given $P \to X$, can form an associated bundle $P \times_G EG \to X$ whose sections define maps $f: X \to BG$ together with isomorphism $f^* EG \cong P$, analogously to $G = U(1)$ case.
Fibres weakly contractible $\Rightarrow$ section exists, and is unique up to homotopy. $\blacksquare$

2) For spaces A and B, the join $A * B$ is $A \times B \times [0,1]/\sim$ where $(a,b,0) \sim (a',b,0)$ and $(a,b,1) \sim (a,b',1)$.

The n-fold join $G^{*n} = G * \cdots * G$ can be described as $G^n \times \Delta_{n-1}/\sim$, where $(g_1,\dots,g_n,t_1,\dots,t_n) \sim (g_1',\dots,g_n',t_1,\dots,t_n)$ if $g_i = g_i'$ whenever $t_i \neq 0$.

The inclusion of G^{*n} in $G^{*(n+1)}$ is homotopic to a constant map

$\rightsquigarrow$ the infinite join $G^{*\infty} = \bigcup G^{n*}$ is contractible.

It has a free G-action, so can take $EG = G^{*\infty}$ and $BG = EG/G$.

3) If BG' is another classifying space then there are classifying maps in both directions $BG' \to BG$ and $BG \to BG'$.

$\Rightarrow$ of 1) Combine the above. $\square$

GRASSMANNIANS

Let $Gr_n(\mathbb{C}^k) = \{n\text{-dimensional subspaces in } \mathbb{C}^k\}$

It has a tautological cx rank n bundle.

Equipping that with the hermitian metric from the standard metric on $\mathbb{C}^k$, its $U(n)$ frame bundle is the complex Stiefel manifold

$$V_n(\mathbb{C}^k) = \{\text{orthonormal } n\text{-tuples in } \mathbb{C}^k\}$$

$V_n(\mathbb{C}^k)$ is $2(k-n)$-connected $\Rightarrow$

$$\left\{\begin{array}{l}\text{isomorphism classes of}\\ \text{rank } n \text{ bundles on CW}\\ \text{complex } X \text{ of dim} \le 2(n-k)\end{array}\right\} \leftrightarrow [X, V_n(\mathbb{C}^k)]$$

$V_n(\mathbb{C}^\infty) = \bigcup_k V_n(\mathbb{C}^k)$ is contractible, so one can take

$$EU(n) = V_n(\mathbb{C}^\infty), \quad BU(n) = Gr_n(\mathbb{C}^\infty)$$

Similarly $EO(n) = V_n(\mathbb{R}^\infty)$, $BO(n) = Gr_n(\mathbb{R}^\infty)$.

See later that

$H^*(BU(n);\mathbb{Z})$ is a free polynomial ~~[illegible]~~ ring $\mathbb{Z}[c_1,\dots,c_n]$ with one generator $c_k \in H^{2k}(BU(n);\mathbb{Z})$ for each $1 \le k \le n$, the <u>Chern classes</u>.

$H^*(BO(n);\mathbb{Z}/2)$ is a free polynomial algebra $(\mathbb{Z}/2)[w_1,\dots,w_n]$ with one generator $w_k \in H^k(BO(n);\mathbb{Z}/2)$ for each $1 \le k \le n$, the <u>Stiefel-Whitney classes</u>.

3.4 OBSTRUCTION CLASSES

Let $p: E \to X$ be a fibre bundle over a CW complex, with fibre F, and $y_0 \in F$ a base point. (and F path-connected)

Does E have a section?
Or what is the biggest n st $\exists$ section $f: X_n \to E$ over the n-skeleton?

Given a section $f: X_n \to E$ and any map $J: (D^{n+1}, S^n) \to (X, X_n)$ (eg the characteristic map of a $(n+1)$-cell e^{n+1}, ie the map whose restriction to S^n is the attaching map while interior of D^{n+1} is mapped homeomorphically to e^{n+1}), picking any trivialisation of $J^*E \to D^{n+1}$ turns $f \circ J|_{S^n}$ into a map $J_f : S^n \to F$.

Given a base point $s_0 \in S^n$, $[J_f] \in \pi_n(F, J(s_0)) \cong \pi_n(F, y_0)$.

$f \circ J|_{S^n}$ extends to a section of $J^*E \to D^{n+1}$ iff $[J_f] = 0$

In particular

- $f : X_n \to E$ extends to X_{n+1} iff $[\mathfrak{I}_f] = 0$ for the characteristic map $\mathfrak{I}$ at each $(n+1)$-cell
- $\pi_n F = 0 \Rightarrow$ any $f : X_n \to E$ extends to X_{n+1}
- F (weakly) contractible $\Rightarrow$ a section $X \to E$ always exists!

To relate these obstructions to cellular cochains, suppose $\alpha : \pi_n(F, y_0) \to K$ is a group homomorphism that is invariant under the action of $\pi_1(F, y_0)$ on $\pi_n(F, y_0)$, and further that the F-fibre bundle $p : E \to X$ has local trivialisations st all transition functions $F \to F$ preserve α.

<u>Example</u>

- If E is a principal G-bundle with G simply-connected we could take $\alpha = id$ (ie $K = \pi_n F$)

- For the unit sphere bundle in a real rank $n+1$ bundle we could take
  - $\alpha : \pi_n S^n \cong \mathbb{Z}$ if orientable
  - $\alpha : \pi_n S^n \to \mathbb{Z}/2$ otherwise

Then given a section $f : X_n \to E$, define $\omega(f) \in C^{n+1}_{cw}(X;K)$ by sending an $(n+1)$-cell e^{n+1} to $\alpha([\mathcal{J}_f]) \in K$ for $\mathcal{J}$ the characteristic map of the cell.

<u>Proposition</u>

a) $\omega(f)$ is closed, so defines a class $ob(f) \in H^{n+1}(X;K)$

b) $ob(f)$ depends only on $f|_{X_{n-1}}$

c) If there is any extension of $f|_{X_{n-1}}$ to X_{n+1} then $ob(f) = 0$

d) If $\alpha = id$ and $ob(f) = 0$ then $f|_{X_{n-1}}$ extends to X_{n+1}

Proof:

a) We need to show that $\omega(f)$ vanishes on any sum $\sum e_k^{n+1}$ of $(n+1)$-cells that is a boundary in the cellular chain complex.

$$n\text{-cells} \leftrightarrow \text{generators of } C_n^{CW}(X) := H_n(X_n, X_{n-1}) \underset{\text{Hurewicz}}{\cong} \pi_n(X_n, X_{n-1})$$

by the map $e^n \mapsto [\text{characteristic map } J : (D^n, S^{n-1}) \to (X_n, X_{n-1})]$

The boundary map for $C_*^{CW}(X)$ can thus be identified with a homomorphism $\pi_{n+1}(X_{n+1}, X_n) \to \pi_n(X_n, X_{n-1})$. That is precisely the snake map in the LES of the triple (X_{n+1}, X_n, X_{n-1}) (generalisation of LES of a pair)

$$\pi_{n+1}(X_{n+1}, X_n) \to \pi_n(X_n, X_{n-1}) \to \pi_n(X_{n+1}, X_{n-1}) \to \pi_n(X_{n+1}, X_n) \to$$

Thus $\sum e_k^{n+1}$ is a boundary $\Leftrightarrow \sum [(J_k)_f] = 0$ in $\pi_{n+1}(X_{n+1}, X_n)$

$$\Rightarrow \sum \omega(f)(e_k^{n+1}) = \sum \alpha[(J_k)_f] = \alpha([J_f]) \text{ for some } J : D^{n+1} \to X_n$$
$$= 0$$

$\square$

b) Like in proof of main theorem on $K(G, n)$

pair $f, f' : X_n \to E$ that agree on X_{n-1}

$\rightsquigarrow \delta(f, f') \in C^n_{CW}(X; K)$ st $d\,\delta(f, f') = \omega(f') - \omega(f)$

($\delta(f, f') = 0$ if f is homotopic to f' rel X_{n-1})

c) By construction

$f : X_n \to E$ extends to X_{n+1}

$\Leftrightarrow$ $[J_f] = 0$ for characteristic map of each $(n+1)$-cell

$\underset{(*)}{\Rightarrow}$ $\omega(f) = 0$

d) If $\omega(f) = -d\delta$, pick $f' : X_n \to E$ that agrees with f on X_{n-1} st $\delta(f, f') = \delta$, so that $\omega(f') = 0$.

If $\alpha = \mathrm{Id}$ then converse to $(*)$ holds, so f' extends to X_{n+1}. $\square$

Theorem

Let F $(n-1)$-connected, and $\alpha : \pi_n F \to K$ a homomorphism to an abelian group K. Let $p : E \to X$ be an F-fibre bundle over a CW complex, with local trivialisations such that transition functions preserve α.

Then there is a class $ob(E) \in H^{n+1}(X;K)$ st

- Section over X_{n+1} exists $\Rightarrow ob(E) = 0$
- If $\alpha = id$ and $ob(E) = 0$ then section over X_{n+1} exists
- For any $g : Y \to X$, $ob(g^* E) = g^* ob(E) \in H^{n+1}(Y;K)$

Proof: Section $f : X_n \to E$ over X_n exists and any two are homotopic on X_{n-1}, so can define $ob(E) = ob(f)$ for any such f.
By cellular approximation, it suffices to prove functoriality when $g : Y \to X$ is cellular. Then the induced map on cellular chain complexes can be interpreted in terms of homomorphisms $g_* : \pi_n(Y_n, Y_{n-1}) \to \pi_n(X_n, X_{n-1})$. $\square$

$ob(E) \in H^{n+1}(X;K)$ is the "primary obstruction class".

If it vanishes and $\alpha = id$, then the next obstruction comes from the next non-trivial $\pi_m F$, but may depend on choice of section over the n-skeleton.

If F is a $K(G,n)$ then there are no further obstructions so one can replace "sections over X_{n+1}" by "section over X" in statement of theorem.

Examples

- For an oriented real rank n bundle $E \to X$, the primary obstruction in $H^n(X;\mathbb{Z})$ to existence of a section of its unit S^{n-1}-bundle coincides with the Euler class $e(E)$.

Since $SO(2) = K(\mathbb{Z},1)$ for an oriented rank 2 bundle, a non-vanishing section over all of X exists iff $e(E) = 0 \in H^2(X;\mathbb{Z})$.

Equivalently, a $\mathbb{C}$ line bundle is trivial iff $c_1 = 0$.

• For a non-oriented real rank n bundle, the primary obstruction to existence of a nowhere-vanishing section only defines a class in $H^n(X;\mathbb{Z}/2)$.

More generally, one could ask about the existence of k sections that are linearly independent at each point. That is equivalent to asking for a section of $V_k(E)\to X$, a bundle whose fibres are the Stiefel manifold

$$V_k(\mathbb{R}^n) = \{k\text{-tuple of orthonormal vectors in } \mathbb{R}^n\}$$

Exercise: $V_k(\mathbb{R}^n)$ is $(n-k-1)$-connected, with $\pi_{n-k} \cong \begin{cases}\mathbb{Z} & \text{for } n-k \text{ even}\\ \mathbb{Z}/2 & \text{for } n-k \text{ odd}\end{cases}$ (and $k>1$).

Either way, one can take $\alpha: \pi_{n-k}V_k(\mathbb{R}^n)\to\mathbb{Z}/2$ to obtain a primary obstruction in $H^{n-k+1}(X;\mathbb{Z}/2)$. It coincides with the Stiefel-Whitney class w_{n-k+1}, more later.

- For an orientable ~~rank~~ n bundle

$\exists$ $(n-1)$ pointwise LI sections $\iff$ trivial

So $w_2 \in H^2(X;\mathbb{Z}/2)$ can also be interpreted as the primary obstruction to triviality of an $SO(n)$-bundle.

Indeed, for $n > 2$

$w_2 = 0 \iff$ trivial over X_2

This can also be seen from $\pi_1 SO(n) = \mathbb{Z}/2$ for $n > 2$.

Also

$w_2 = 0 \iff \exists$ $Spin(n)$-lift of $SO(n)$-bundle.

$\pi_2 SO(n) = \pi_2 Spin(n) = 0$ and $\pi_3 SO(n) = \pi_3 Spin(n) = \mathbb{Z}$, so the next obstruction to triviality of an oriented bundle (and primary obstruction to triviality of a spin bundle) is a class in $H^4(X;\mathbb{Z})$.

3.5 CHERN CLASSES

Complex vector bundles of rank n $\Leftrightarrow$ principal $GL(n,\mathbb{C})$-bundles

$\Leftrightarrow$ principal $U(n)$-bundles

Theorem

- For complex rank n bundles, there are characteristic classes called the Chern classes $c_k \in H^{2k}(\;;\mathbb{Z})$, $k = 1, \dots, n$, st
  - Chern classes of bundles of different ranks are related by the Whitney sum formula

    $$c_k(E \oplus E') = \sum_{i=0}^{k} c_i(E)\, c_{k-i}(E')$$

    (where $c_0 = 1$, and $c_i(E) = 0$ if $i > \mathrm{rk}\, E$)
  - c_1 of rank 1 bundles coincides with Euler class
- These conditions completely characterise the Chern classes
- $H^*(BU(n);\mathbb{Z})$ is the free polynomial ring $\mathbb{Z}[c_1, \dots, c_n]$.

Remark

- If we define the <u>total Chern class</u> of E to be

$$c(E) = 1 + c_1(E) + \cdots + c_n(E) \in H^*(X;\mathbb{Z})$$

then the Whitney sum formula is equivalent to

$$c(E \oplus E') = c(E)\,c(E')$$

"Total Chern class is exponential"

- The Whitney sum formula implies that Chern classes are <u>stable</u>:

$$c_k(E \oplus E') = c_k(E) \quad \text{for any trivial bundle } E'$$

(in contrast to Euler class)

LINE SPLITTING PRINCIPLE AND UNIQUENESS

To prove that the Whitney sum formula on its own determines the Chern classes up to scale we use

Line splitting principle

For any $\mathbb{C}$ vector bundle $E \to X$, there is a cts $f: Y \to X$ st

- $f^*: H^*(X; R) \to H^*(Y; R)$ is injective for any ring R
- f^*E is isomorphic to a direct sum of line bundles over Y

Consequence: any relation between characteristic classes of $\mathbb{C}$ vector bundles that holds for sums of line bundles holds for all bundles.

Easy application: $c_1(E) = c_1(\det E)$

Corollary

If $\tilde{c}_k^{(n)} \in H^{2k}(BU(n); \mathbb{Z})$ are classes st

$$\tilde{c}_k^{(m+n)}(E \oplus E') = \sum_{i=0}^{k} \tilde{c}_i^{(m)}(E)\, \tilde{c}_{k-i}^{(n)}(E')$$

for any $BU(m)$-bundle E and $BU(n)$-bundle E'

and $\tilde{c}_1^{(1)} = \alpha c_1$ for some $\alpha \in \mathbb{Z}$ then

$\tilde{c}_k^{(n)}(E) = \alpha^k c_k(E)$ for all n and k

Proof:

If E is a direct sum of line bundles this is a obvious induction on n. $\square$

Line splitting principle follows from induction on

__Lemma__

Consider the bundle $\pi: \mathbb{P}(E) \to X$ whose fibre over $x \in X$ is $\mathbb{P}(E_x) = \{\text{lines in } E_x\}$. Let $T = \mathcal{O}_{\mathbb{P}(E)}(-1)$ be the tautological line bundle, whose fibre over a line ℓ in E_x is ℓ itself. Then

- $\pi^* E$ is isomorphic to a direct sum of T and another bundle
- $\pi^*: H^*(X; \mathbb{R}) \to \pi^*(\mathbb{P}(E); \mathbb{R})$ is injective.

Proof:
Tautologically $T \hookrightarrow \pi^* E$. If we put a hermitian metric on E, then T has an orthogonal complement $T^\perp$, and

$$\pi^* E = T \oplus T^\perp.$$

The restriction of T to a fibre $\mathbb{P}(E_x) \cong \mathbb{C}P^{n-1}$ is $\mathcal{O}(-1)$

So the restriction of $u := -c_1(T) \in H^2(\mathbb{P}(E); \mathbb{Z})$ to each fibre is a generator of $H^2(\mathbb{C}P^{n-1}; \mathbb{Z})$.

Thus $1, u, \dots, u^{n-1} \in H^*(\mathbb{P}(E); R)$ restricts to a basis of cohomology of each fibre, so Leray-Hirsch theorem

$\Rightarrow$ that is also a basis for $H^*(\mathbb{P}(E); R)$ as a $H^*(X; R)$-module.

In particular, $\pi^*: H^*(X; R) \to H^*(\mathbb{P}(E); R)$ is injective. $\square$

More explicitly, if $E \to X$ has a hermitian metric we can always split E by $Y \to X$ for Y the flag bundle $Fl\, E$ whose fibre over $x \in X$ is

$\{n\text{-tuples of } \mathbb{C} \text{ lines in } E_x\}$