PROOF OF "FUNDAMENTAL THEOREM" ON $K(G,n)$

Theorem

G an abelian group

If X is a CW complex, and Y is a $K(G,n)$
(i.e. $\pi_n Y \cong G$ and all other $\pi_i Y = 0$) then

$$[X,Y] \to H^n(X;G), \quad f \mapsto f^* u_G$$

is a bijection, where $u_G \in H^n(Y;G) \cong \mathrm{Hom}(G,G)$
is the canonical element.

Sophisticated point of view:
Prove that (given G) the sequence $K(G,n)$ forms a "spectrum",
so that the assignment $h^n(X) := [X, K(G,n)]$ defines a
cohomology theory.
Then it suffices to prove that $h^n(X) = H^n(X;G)$ for $X = \{pt\}$.

Starting point for naive approach:

<u>Lemma</u>

If Y is $(n-1)$-connected $(n \geq 1)$ and X is a CW complex then

a) any $f: X \to Y$ is homotopic to a map that is constant on the $(n-1)$-skeleton X_{n-1}

b) if $f, f': X \to Y$ both map X_{n-2} to $y_0 \in Y$ and are homotopic, then they are also homotopic relative to X_{n-2}.

Proof:

a) Apply ~~lem~~ Compression lemma to $f|_{X_{n-1}}$: $X_{n-1} \to (Y, y_0)$
$\Rightarrow f|_{X_{n-1}} \simeq y_0$. Then apply homotopy extension.

b) Let $F: X \times [0,1] \to Y$ be the homotopy.

$F|_{X_{n-2} \times [0,1] \cup X \times \{0,1\}} \simeq G$ (relative to $X \times \{0,1\}$) st
G maps $X_{n-2} \times [0,1]$ to y_0 by compression lemma.
Use homotopy extension to get $F': X \times [0,1] \to Y$ that maps $X_{n-2} \times [0,1]$ to y_0 while agreeing with F on $X \times \{0,1\}$. □

Key observation:

$$d\,\delta_f(g,g') = \omega(g) - \omega(g') \in C^{n+1}(X;\pi_n(Y))$$

Consequences

- Given $f : X_{n-1} \to Y$, its restriction to X_{n-2} extends to X_n iff $\omega(f)$ exact. (not so relevant now)
- If g and $g' : X_n \to Y$ both extend to X_{n+1} then $d\delta_f(g,g') = 0$, so $\delta_f(g,g')$ defines a cohomology class $\Delta_f(g,g') \in H^n(X;\pi_n(Y))$

<u>Proposition</u>

Two extensions $g, g' : X_n \to Y$ of the same $f : X_{n-1} \to Y$ are homotopic relative to X_{n-2} iff $\delta_f(g,g') \in C^n(X;\pi_n(Y))$ is exact. So if g, g' both extend to X_{n+1}, then they are homotopic relative to X_{n-2} iff $\Delta_f(g,g') = 0$.

RUDIMENTARY OBSTRUCTION THEORY

Let X be a CW complex with a single 0-cell x_0, st all attaching maps $\varphi: S^{n-1} \to X_{n-1}$ map a fixed s_0 to x_0.

Given $f: X_{n-1} \to Y$ and $y_0 := f(x_0)$, let

$$\omega \in C^n_{CW}(X; \pi_{n-1}(Y, y_0))$$

be the cellular cochain that assigns to an n-cell e^n in X the composition of its attaching map $\varphi: S^{n-1} \to X_{n-1}$ with f (basically "restriction of f to ∂e^n")

Trivial lemma: f extends to a map $g: X_n \to Y$ iff $\omega(f) = 0$.

Given two extensions $g, g': X_n \to Y$ of the same $f: X_{n-1} \to Y$, let

$$\delta_f(g, g') \in C^n_{CW}(X; \pi_n(Y, y_0))$$

assign to e^n the result of gluing $g|e^n$ and $g'|e^n$ to a map $S^n \to Y$.

Trivial lemma: $g \simeq g'$ relative to X_{n-1} iff $\delta_f(g, g') = 0$.

Proof:

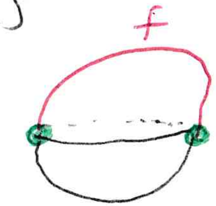
Given a homotopy from g to g' relative to X_{n-2},

restriction to X_{n-1}

$\rightsquigarrow$ homotopy $F : X_{n-1} \times [0,1] \to Y$ relative to X_{n-2}
from f to itself

$\rightsquigarrow$ $\varepsilon_f(F) \in C^{n-1}(X; \pi_n Y)$ that assigns to an $(n-1)$-cell e^{n-1}
the map $S^n \to Y$ induced from $F : e^{n-1} \times [0,1] \to Y$
by gluing boundary pieces

f on $e^{n-1} \times \{1\}$

$\boxed{F|_{e^{n-1}\times[0,1]}}$ doesn't move $\partial e^{n-1} \subset X_{n-2}$ $\rightsquigarrow$ 

$\Rightarrow \delta_f(g, g') = d\varepsilon_f(F)$

f on $e^{n-1} \times \{0\}$

Conversely, if $\delta_f(g, g')$ is exact, say $d\varepsilon$, then ε can be realised as $\varepsilon = \varepsilon_f(F)$ for a self-homotopy F of $f : X_{n-1} \to Y$ relative to X_{n-2}.

Homotopy extension $\rightsquigarrow$ $g \simeq g'' : X \to Y$ (relative to X_{n-2}) st
$\delta_f(g'', g') = 0$, so $g'' \simeq g'$ relative to X_{n-1}.

$\square$

Corollary

If X is a CW complex and Y is a $K(G,n)$ then there is a well-defined bijection

$$[X,Y] \to H^n(X;G)$$

that sends the class of any $f: X \to Y$ that sends X_{n-1} to the base point $y_0 \in Y$ to $\Delta(f, f_0) \in H^n(X;G)$ (where $f_0: X \to Y$ is the constant map to y_0).

Proof:

We have argued

- $[X,Y] \Leftrightarrow$ homotopy-rel-X_{n-2} classes of maps $X \to Y$ that send X_{n-1} to y_0
- homotopy-rel-X_{n-2} classes of maps $X_n \to Y$ that are fixed on X_{n-1} and can be extended to X_{n+1} $\Leftrightarrow H^n(X;G)$

Finally note there is no obstruction to extending maps from X_{n+1} to all of X because $\pi_i Y = 0$ for $i > n$. $\square$

and homotopies from X_n

Remark

When Y is arbitrary, one could in principle approach the question of whether $f \simeq g : X \to Y$ as follows

- Restrictions to X_1 homotopic iff $f_* = g_* : \pi_1 X \to \pi_1 Y$.

  Then homotopy extension $\rightsquigarrow$ $g \simeq g_1$ st $g_1|_{X_1} = f|_{X_1}$

- Restriction of f and g_1 to X_2 homotopic rel X_0 iff $\Delta(f, g_1) \in H^2(X; \pi_2 Y)$ is zero.

  Then $g_1 \simeq g_2$ st $g_2|_{X_2} = f|_{X_2}$

Etc

But in general, obstruction in n^{th} step could depend on choices in previous steps.

3 CHARACTERISTIC CLASSES

3.1 PRINCIPAL G-BUNDLES AND SETUP

Definition

Let G be a topological group (or a Lie group)

A principal G-bundle is a fibre bundle $\pi : P \to X$ with a free G-action that preserves and is transitive on the fibres (so $P/G \cong X$). Usually take G to act on the right.

(Note: a principal G-bundle is trivial, ie $P \cong X \times G$, iff it has a section)

If $f : Y \to X$, then the pull-back bundle $f^*P \to Y$ is also a principal G-bundle.

Definition

A G-characteristic class c assigns to every principal bundle $P \to X$ (on every topological space X) a class $c(P) \in H^k(X;R)$ (for some k and coefficient group R) st for any $f : Y \to X$

$$c(f^*P) = f^*(c(P)) \in H^k(Y;R)$$

ASSOCIATED BUNDLES, REDUCTIONS AND LIFTS

Definition

Let ρ be a left action of G on a space K.
For a principal G-bundle $\pi : P \to X$ the associated K-bundle is $P \times_\rho K := P \times K / G$ where $g \in G$ acts by $(p, k) \mapsto (pg, \rho(g^{-1})k)$. (So $P \times_\rho K \to X$ is a K-fibre bundle).

Note

If K is a group and $\rho : G \to K$ is a homomorphism and we let $g \in G$ act on K as left-multiplication by $\rho(g)$, then right multiplication by K on second factor of $P \times K$ descends to $P \times_\rho K$, making it a principal K-bundle.

Given a homomorphism $\rho : G \to K$, a <u>G-structure</u> on a principal K-bundle Q is a principal G-bundle P together with an isomorphism $P \times_\rho K \cong Q$.

"reduction" : ρ injective

"lift" : ρ surjective

Let $\pi : E \to X$ be a $\mathbb{F}$-vector bundle of rank n.

The <u>frame bundle</u> of E is the principal $GL(n, \mathbb{F})$-bundle

$$P_E := \{ (x, \phi) : x \in X, \ \phi : E_x \to \mathbb{F}^n \text{ isomorphism} \}$$

where $g \in GL(n, \mathbb{F})$ acts by $\phi \mapsto \phi \circ g$.

<u>Note</u> E is associated to P_E by the standard representation of $GL(n, \mathbb{F})$ on $\mathbb{F}^n$.

Various structures on E can be encoded by G-structures on P_E.

Eg for a real rank n bundle

metric $\Leftrightarrow$ $O(n)$-reduction of $GL(n,\mathbb{R})$-bundle

(always exists if base is paracompact (ie partitions of unity exist))

orientation $\Leftrightarrow$ $GL_+(n,\mathbb{R})$-reduction of $GL(n,\mathbb{R})$-bundle

$\Leftrightarrow$ $SO(n)$-reduction of $O(n)$-bundle

"spin structure" $\Leftrightarrow$ $Spin(n)$ lift of $SO(n)$-bundle, where

$\rho : Spin(n) \to SO(n)$ is the double cover.

CX structure $J \in End(E)$ st $J^2 = -Id$ $\Leftrightarrow$ $GL(\frac{n}{2}, \mathbb{C})$-reduction of $GL(n,\mathbb{R})$-bundle

Similarly for a complex rank n bundle

hermitian metric $\Leftrightarrow$ $U(n)$-reduction of $GL(n,\mathbb{C})$-bundle.

THE FIRST CHERN CLASS

Essentially

complex line bundles $\Leftrightarrow$ principal $U(1)$-bundles
$\Leftrightarrow$ principal $SO(2)$-bundles $\Leftrightarrow$ oriented real rank 2 bundles

Theorem

For complex line bundles $E \to X$ there is a characteristic class $c_1(E) \in H^2(X;\mathbb{Z})$ st

- c_1 generates the ring of characteristic classes (any other char class of cx line bundles is a polynomial in c_1)
- If X is a CW complex then $\{$isomorphism classes of cx line bundles on $X\} \to H^2(X;\mathbb{Z})$, $E \mapsto c_1(E)$ is a bijection.

This uniquely determines c_1 up to sign. Can fix the sign by demanding that tautological bundle on $\mathbb{CP}^1$ has $c_1 = -1$.

Different perspectives
- describe a line bundles via maps to "classifying space" $\mathbb{C}P^\infty$
- obstruction to triviality of principal G-bundle
- obstruction to existence of section of S^n-bundle (Euler class)
- Čech cohomology class of transition functions
- for a smooth $\mathbb{C}$ line bundle on a smooth mfd, the image of c_1 in $H^2(X;\mathbb{R}) \cong H^2_{dR}(X)$ can also be described as de Rham class of curvature of any connection $\cdot \frac{i}{2\pi}$. (Chern-Weil theory)

Remark
It is not typical for the isomorphism class of a principal G-bundle to be determined by its characteristic classes. Reflected by:
- the classifying space happens to be an Eilenberg-MacLane space
- primary obstruction to triviality happens to be a complete obstruction
- Čech cohomology more workable because structure group is abelian

3.2 THE EULER CLASS

Define a characteristic class for oriented real rank n vector bundles $\pi : E \to X$, or equivalently for principal $SO(n)$-bundles.

Recall If $\dot{E} \subset E$ denotes the complement to the zero section, there is a unique Thom class $u(E) \in H^n(E, \dot{E}; \mathbb{Z})$ s.t.

$$H^k(X;\mathbb{Z}) \to H^{n+k}(E, \dot{E}; \mathbb{Z})$$
$$\alpha \mapsto \pi^\# \alpha \cup u(E)$$

is an isomorphism.

Definition
The Euler class of E is the preimage $e(E) \in H^n(X;\mathbb{Z})$ of $u(E)^2 \in H^{2n}(E, \dot{E}; \mathbb{Z})$

Equivalently, $e(E) = s^\# u(E)$ for any section $s : X \to E$

Basic properties of e

a) e is a characteristic class, ie $e(f^*E) = f^* e(E)$ for any $E \to X$ and $f : Y \to X$

b) e is multiplicative under direct sums, ie if E and E' are oriented bundles over X of rank m and n then
$$e(E \oplus E') = e(E) \cup e(E') \in H^{m+n}(X; \mathbb{Z})$$

c) Reversing orientation of E multiplies $e(E)$ by -1

d) If n is odd then $e(E) \in H^n(X; \mathbb{Z})$ is a 2-torsion element

e) If E has a nowhere-vanishing section then $e(E) = 0$.

Proof:

a)-c) immediate from properties of Thom class.

d): n odd $\Rightarrow 2u(E)^2 = 0 \in H^{2n}(E, \dot{E}; \mathbb{Z})$ by graded commutativity of cup product. (Or deduce from (c) since bundle of odd rank has orientation-reversing isomorphism $x \mapsto -x$).

e): If $s : X \to \dot{E}$ then $s^* u = 0$ for any $u \in H^n(E, \dot{E}; \mathbb{Z})$.

$\square$

GEOMETRIC INTERPRETATION

If X is a smooth mfd and $E \to X$ is a smooth real rank r bundle and $s : X \to E$ is a transverse section (ie $Ds_x : T_xX \to E_x$ surjective, $\forall x \in s^{-1}(0)$) then $e(E)$ is Poincaré dual to $s^{-1}(0)$.

In particular, if $\dim X = n$ and X is closed and oriented then

$$e(E)[X] = \#\text{zeros of } s \text{ counted with signs} \in \mathbb{Z}$$

Reinterpretation of Poincaré-Hopf index theorem

For a smooth closed oriented n-manifold X

$$e(TX)[X] = \chi(X) = \sum_{i=0}^{n} b_i(X) \quad \text{(the Euler characteristic)}$$

Proof:
Both sides $=$ # zeros of vector field with signs. $\square$

EULER CLASS AS OBSTRUCTION TO SECTIONS

For $\pi: E \to X$ oriented rank n bundle, property (e) stated existence of nowhere-vanishing section $s: X \to E \Rightarrow e(E) = 0 \in H^n(X;\mathbb{Z})$

Converse holds for CW complexes X of dim $\leq n$.

For now, consider the case when X is a smooth manifold.

Easy if $\dim X = k < n$. Then graphs of zero section and another section s are both dim k submfds in dim $n+k > 2k$ total space E.

If $\dim X = n$, assume WLOG X connected.

If X is closed, then $e(E) = 0 \Rightarrow$ transverse section s has equal number of zeros with positive and negative sign. Connect a positive zero to a negative zero by a path γ in X, and deform section in tubular nbhd of γ to cancel those two out. Repeat.

If X is not compact, then "push zeros off the edge".

Deal with obstructions to existence ~~of~~ of sections of fibre bundles over CW complexes X more systematically later, finding in particular that for oriented real rank 2 bundles there is no obstruction to extending a nowhere-vanishing section from 2-skeleton X_2 (which exists iff $e=0$) to all of X.

Definition
For a cx line bundle $E \to X$, let $c_1(E)$ ~~~~ $\in H^2(X;\mathbb{Z})$ be the Euler class of E considered as an oriented real rank 2 bundle.

Since ~~~~ $\exists$ nowhere-vanishing section of E
$\iff$ $\exists$ section of ~~~~ corresponding $U(1)$-bundle
$\iff$ bundle is trivial
the above claims thus say cx line bundle E over CW complex is trivial iff $c_1(E)=0$

CHARACTERISATION OF EULER CLASS

Theorem

If for each n, ε_n is a $H^n(\,\cdot\,;\mathbb{Z})$-valued characteristic class of oriented real rank n bundles st

$$\varepsilon_{m+n}(E\oplus E') = \varepsilon_m(E)\cup\varepsilon_m(E')$$

then $\varepsilon_n(E) = ke(E)$ for some integer k

(i.e. Euler class is unique multiplicative class up to scale).

Proof uses a version of splitting principle to reduce to question of uniqueness up to scale of char classes for rank 2 bundles.

For any oriented real vector bundle $E\to X$, there is an $f: Y\to X$ st f^*E is a direct sum of oriented real bundles of rank ≤ 2.

Remark

For a non-oriented rank n bundle $E\to X$, could use the Thom class in $H^n(E,\dot{E};\mathbb{Z}/2)$ to define a characteristic class in $H^n(X;\mathbb{Z}/2)$. That turns out to be the n^{th} Stiefel-Whitney class w_n, which is defined for bundles of any rank.