2.3 CW COMPLEXES

For a space Y & cts $\varphi : S^{n-1} \to Y$, call

$$Y \cup_\varphi e^n := (Y \sqcup D^n) / (\varphi(s) \sim s \text{ for } s \in S^{n-1})$$

the result of attaching an n-cell to Y by φ
(the "n-cell" e^n is the image in $Y \cup_\varphi e^n$ of interior of D^n).

A **CW complex** X is a union of n-skeleta $X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots$ st

- X_n is obtained from X_{n-1} by attaching a (possibly infinite) set of n-cells
- X has the weak topology of the union, ie $U \subseteq X$ is open $\Leftrightarrow$ $U \cap X_n$ is open in X_n, $\forall n$

Result is "closure-finite": closure of any n-cell meets only finitely many lower cells.

$A \subseteq X$ is a **subcomplex** if $A \cap X_n$ is a union of cells for each n.

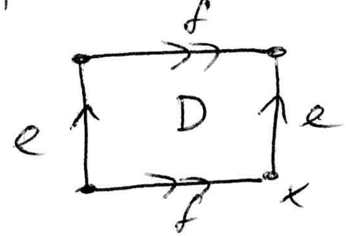
$f : X \to Y$ is a **cellular map** if $f(X_n) \subseteq Y_n$ for each n.

USEFUL PROPERTIES

- Any ~~[illegible]~~ closed smooth manifold (and every closed manifold of dim $\neq 4$) is homeomorphic to a CW complex
- Homology and cohomology can be computed from cellular chain complex.

Example

$T^2 = S^1 \times S^1 \cong$ CW complex with a single 0-cell x, two 1-cells e, f, one 2-cell D



" $\partial D = e + f - e - f = 0$

$\partial e = \partial f = x - x = 0$ "

$$\mathbb{Z} \xleftarrow{0} \mathbb{Z}^2 \xleftarrow{0} \mathbb{Z}$$

$\rightsquigarrow H_0(T^2) \cong H_2(T^2) \cong \mathbb{Z}$

$H_1(T^2) \cong \mathbb{Z}^2$

- CW complexes are well controlled by their homotopy groups thanks to

<u>Whitehead's theorem</u>

If X, Y are CW complexes and $f : X \to Y$ is a weak homotopy equivalence (ie $f_* : \pi_n X \xrightarrow{\cong} \pi_n Y$) then f is a homotopy equivalence

- Many results can be proved for CW complexes by induction and then generalised to all spaces using

<u>CW approximation theorem</u>

For any space Y, $\exists$ CW complex X with weak homotopy equivalence $f : X \to Y$.
Moreover, X is unique up to homotopy equivalence.

TWO BASIC INDUCTION RESULTS

Homotopy extension property of CW pairs

ie $A \subseteq X$ is a subcomplex

Let (X,A) a CW pair, $f : X \to Y$ and $F : A \times [0,1] \to Y$ a homotopy from $f|_A$ (ie $F(a,0) = f(a)$, $\forall a \in A$).

Then $\exists\, G : X \times [0,1] \to Y$ st

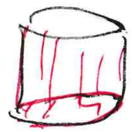
$$G(x,0) = f(x), \quad \forall x \in X$$
$$G(a,t) = F(a,t), \quad \forall a \in A,\ t \in [0,1]$$

Proof:

Suppose $\exists$ extension $G_n : (X_n \cup A) \times [0,1] \to Y$.

To extend G_n to $(X_{n+1} \cup A) \times [0,1]$ amounts to extending to each $(n+1)$-cell e^{n+1} not in A.

That means extending a map to $e^{n+1} \times [0,1]$ from $(\partial e^{n+1}) \times [0,1] \cup e^{n+1} \times \{0\}$



Compression lemma

Let f be a map from a CW pair (X,A) to a pair of top spaces (Y,B) $(B \neq \emptyset)$. (i.e. $f(A) \subseteq B$)
For each n st $X \setminus A$ has some n-cells, suppose that $\pi_n(Y,B,y_0)=0$ for all $y_0 \in B$.
Then f is homotopic relative to A to a map $X \to B$.

Proof:
Suppose f maps $X_{n-1} \to B$. If $X \setminus A$ has no n-cells then also $f(X_n) \subseteq B$.
Otherwise, for each n-cell e^n in $X \setminus A$ (and arbitrary $p \in \partial e^n$)
the restriction of f to e^n defines an element of $\pi_n(Y,B,f(p))$.
So hypothesis allows you to homotope f on e^n to maps into B
$\Rightarrow$ $f|X_n$ is homotopic relative to $A \cap X_n$ to a map $X_n \to B$
By homotopy extension, f is homotopic relative to $X_{n-1} \cup A$
to a map that sends $X_n \to B$.
Can concatenate all these homotopies to a well-defined homotopy
(because for each fixed n, X_n is moved only by finitely many)

PROOF OF WHITEHEAD'S THEOREM

$f: X \to Y$ map of CW complexes st $f_*: \pi_n X \cong \pi_n Y$.
Want to show f is homotopy equivalence.

1) Suppose f is inclusion of a subcomplex.
   Then $\pi_n(Y,X) = 0$, $\forall n$
   Applying compression lemma to identity $(Y,X) \to (Y,X)$
   $\Rightarrow$ id_Y is homotopic relative to X to a map $Y \to X$
   So X is a deformation retract of Y.
   In particular, $X \hookrightarrow Y$ is a homotopy equivalence.

2) Suppose f is cellular. Then the mapping cylinder $M_f = X \times [0,1] \cup Y/\sim$
   is a CW complex, and $\simeq Y$. Apply (1) to the inclusion $X \hookrightarrow M_f$

3) In general, reduce to (2) by

<u>Cellular approximation theorem</u>

Any map of CW complexes is homotopic to a cellular map

CW APPROXIMATION

Proposition

For any space Y, $\exists$ CW complex X and weak homotopy equiv. $f : X \to Y$.

Proof:
WLOG Y path-connected. Let $X_0 = \{x_0\}$.
Inductively define n-skeleton X_n with n-equivalence $f_n : X_n \to Y$
(i.e. $(f_n)_* : \pi_k X_n \to \pi_k Y$ isomorphisms for $k<n$, surjective for $k=n$)

- for generators of $\ker\left((f_n)_* : \pi_n X_n \to \pi_n Y\right)$ pick representatives $\varphi : S^n \to X_n$ and attach $(n+1)$-cells along φ and define f_{n+1} on that $(n+1)$-cell by a null-homotopy $f_n \circ \varphi$.
- for generators of $\pi_{n+1} Y$, pick representatives $h : S^{n+1} \to Y$, attach $(n+1)$-cell to X_n by constant map to x_0, define f_{n+1} on that $(n+1)$-cell by h. $\square$

Clear from proof that if Y is n-connected one needs no k-cells for $0<k\leq n$.

Exercise If $f : X \to Y$ and $f' : X' \to Y$ are weak homotopy equivalences and X is CW then there is also a weak homotopy equivalence $g : X \to X'$.

This CW approximation is unique up to homotopy equivalence.

<u>Remark</u>

So if a CW complex is n-connected it is homotopy equivalent to one without k-cells for $0 < k \leq n$.

Converse holds by cellular approximation.

E.g. that shows S^n is $(n-1)$-connected.
(Already knew that by Hurewicz, and also that $\pi_n S^n \cong \mathbb{Z}$)

but higher homotopy groups are hard to understand.

2.4 FIBRATIONS

Definition

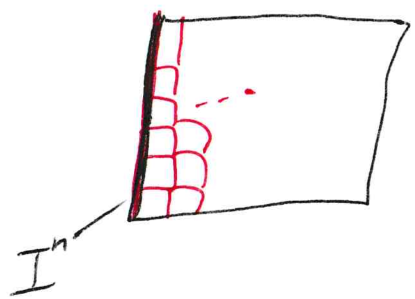
$p : E \to X$ is a <u>Serre fibration</u> if it has the homotopy lifting property for discs D^n, $n > 0$ (or equivalently for cubes I^n)

ie for any $g : D^n \to E$ and $G : D^n \times [0,1] \to X$ st

$G(x,0) = \pi(g(x))$, $\forall x \in D^n$

$$\begin{tikzcd} D^n \arrow[r, "g"] \arrow[d, hook] & E \arrow[d, "p"] \\ D^n \times [0,1] \arrow[r, "G"'] \arrow[ur, "\tilde{G}"] & X \end{tikzcd}$$

there is $\tilde{G} : D^n \times [0,1] \to E$ st $p \circ \tilde{F} = F$ and $\tilde{G}|_{D^n \times \{0\}} = g$



Lemma

Any fibre bundle $p: E \to X$ is a fibration

Proof:

Given $g: I^n \to E$ and $G: I^n \times I \to X$ homotopy from $p \circ g$, we can divide I^{n+1} into small cubes st G-image of each is contained in a trivialising nbhd. For each small cube, we can always find a lift of G while prescribing the lift on a connected set of faces, as long as we don't prescribe it on <u>all</u> faces. $\square$

Remark

- If
$$\begin{tikzcd} E \arrow[rr, "g"] \arrow[dr, "p"'] & & E' \arrow[dl, "p'"] \\ & X & \end{tikzcd}$$
commutes and g is fibre-preserving homotopy equivalence, then p is fibration $\Leftrightarrow$ p' is.

- Fibres of a fibration need not be homotopy equivalent, but have homotopy equivalent CW approximations.

Theorem

Let $p : E \to X$ a fibration, with X path-connected.
Let $e_0 \in E$, $x_0 := p(e_0)$, $F := \pi^{-1}(x_0)$.
Then the homomorphisms $p_* : \pi_n(E, F, e_0) \to \pi_n(X, x_0)$
are isomorphism for all n.
Hence $\exists$ LES

$$\pi_k F \to \pi_k E \to \pi_k X \xrightarrow{\partial} \pi_{k-1} F \to \cdots \qquad \to \pi_0 F \to \pi_0 E \to 0$$

Note

$\partial : \pi_n X \to \pi_{n-1} F$ can be described as:
for $g : D^n \to X$ st g maps ∂D^n to x_0,
thinking of g as homotopy from constant map
$\leadsto$ lift $G : D^n \to E$.
Restricting G to ∂D^n defines map $S^{n-1} \to F$.
Take that to represent $\partial [g] \in \pi_{n-1} F$.

Examples

- For a covering map $p : E \to X$ with X connected
  $\pi_n F = 0$ for $n \geq 1$, $\pi_0 F \leftrightarrow F$.
  So $\pi_n E \cong \pi_n X$ for $n \geq 2$ (also easily proved from lifting properties of p)
  $0 \to \pi_1 E \to \pi_1 X \to \pi_0 F \to 0 \Rightarrow \pi_1 E$-cosets in $\pi_1 X$ biject with F

- Hopf fibration $p : S^{2n+1} \to \mathbb{C}P^n$ has fibre S^1, so
  $\pi_2 \mathbb{C}P^n \cong \mathbb{Z}$ (also from Hurewicz)
  $\pi_k \mathbb{C}P^n \cong \pi_k S^{2n+1}$ for $k \geq 3$

  If we let $S^\infty = \bigcup_n S^n$, $\mathbb{C}P^\infty = \bigcup_n \mathbb{C}P^n$ there is also
  S^1-fibration $S^\infty \xrightarrow{p} \mathbb{C}P^\infty$.
  S^∞ is contractible, so $\pi_k \mathbb{C}P^\infty = 0$ for all $k \geq 3$.

- $SO(n+1) \to S^n$ has fibre $SO(n)$
  $A \mapsto Ae_0$

  $\Rightarrow \pi_k SO(n+1) \cong \pi_k SO(n)$ for $k \leq n-2$

  $\Rightarrow \pi_k SO(m) \cong \pi_k SO(n)$ for $m, n \geq k+2$.

  $\rightsquigarrow$ stable homotopy groups $\pi_k SO := \pi_k SO(n)$ for all $n \geq k+2$.

- Similarly $SU(n)$ is a fibre of $SU(n+1) \to S^{2n+1}$

  $\Rightarrow \pi_k SU(n+1) \cong \pi_k SU(n)$ for $k < 2n$

  $\Rightarrow \pi_k SU(m) \cong \pi_k SU(n)$ for $2m, 2n > k$

Eg $\pi_3 SU(n) \cong \pi_3 SU(2) \cong \pi_3 S^3 \cong \mathbb{Z}$ for any $n \geq 2$

WEAK HOMOTOPY EQUIVALENCES AND FIBRE ~~[illegible]~~ BUNDLES

Proposition

Let $p: E \to X$ a fibre bundle and $g: Y \to X$ a weak homotopy equivalence. Let $G: g^*E \to E$ be the tautological map from the total space of the pull-back bundle $g^*E \to Y$. Then G is a weak homotopy equivalence.

Proof:

Fibre over y_0 is homeomorphic to the fibre F over $x_0 := g(y_0)$.

The LESes of $E \to X$ and $g^*E \to Y$ fit into a commutative diagram

$$\begin{tikzcd}
\pi_k F \arrow[r] \arrow[d, "\cong"] & \pi_k(g^*E) \arrow[r] \arrow[d, "G_*"] & \pi_k Y \arrow[r] \arrow[d, "g_*"] & \pi_{k-1} F \arrow[d, "\cong"] \\
\pi_k F \arrow[r] & \pi_k E \arrow[r] & \pi_k X \arrow[r] & \pi_{k-1} F
\end{tikzcd}$$

Then g_* isomorphism $\Rightarrow$ G_* isomorphism by 5-lemma. $\square$

Combining with CW approximation theorem, this allows claims about cohomology of fibre bundles to be reduced to case where base is CW complex.

2.5 CELLULAR (CO)HOMOLOGY

For a CW complex X

excision $\Rightarrow$ $$H_k(X_n, X_{n-1}; G) \cong H^k(X_n, X_{n-1}; G) \cong \begin{cases} \text{free } G \text{ module generated by } n\text{-cells} & \text{if } k=n \\ 0 & \text{otherwise} \end{cases},$$

Making $C_n^{CW}(X) := H_n(X_n, X_{n-1})$ into a chain complex that computes (co)homology of X is therefore useful in practice.

The boundary maps $\partial_n : H_n(X_n, X_{n-1}) \to H_{n-1}(X_{n-1}, X_{n-2})$ are defined using maps in the LESes for relative homology

$$\cancel{H_n(X_{n-1})} \to H_n(X_n) \xrightarrow{j_n} H_n(X_n, X_{n-1}) \xrightarrow{\delta_n} H_{n-1}(X_{n-1}) \to$$

Setting $\partial_n := j_{n-1} \circ \delta_n$ ensures

$$\partial_{n-1} \circ \partial_n = j_{n-2} \circ \cancel{(\delta_{n-1} \circ j_{n-1})} \circ \delta_n = 0$$

If $\alpha \in \ker \partial_n$ then also $\delta_n \alpha = 0$, so $\alpha = j_n \beta$ for some $\beta \in H_n(X_n)$.
The image of β in $H_n(X)$ is independent of choice β, and 0 iff $\alpha \in \operatorname{Im} \partial_{n+1}$

$\rightsquigarrow$ isomorphism $H_n^{CW}(X) = \dfrac{\ker \partial_n}{\operatorname{im} \partial_{n+1}} \to H_n(X)$

Similarly $C_n^{CW}(X;G) := C_n^{CW}(X)\otimes G \cong H_n(X_n, X_{n-1}; G)$

and $C^n_{CW}(X;G) := \mathrm{Hom}(C_n^{CW}(X), G) \cong H^n(X_n, X_{n-1}; G)$

compute $H_*(X;G)$ and $H^*(X;G)$

(but does not say much about cup products on $H^*(X;R)$)

<u>Example</u>

$\mathbb{C}P^n$ is a ~~...~~ CW complex with $2k$-skeleton $\mathbb{C}P^k$ and a single $2k$-cell for each $0\le k\le n$

$$\Rightarrow H_k(\mathbb{C}P^n; G) \cong H^k(\mathbb{C}P^n; G) \cong \begin{cases} G & \text{for } 0\le k\le 2n \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

(Poincaré duality $\Rightarrow$

if $\omega\in H^2(\mathbb{C}P^n;\mathbb{Z})$ is a generator then

$\omega^k\in H^{2k}(\mathbb{C}P^n;\mathbb{Z})$ is a generator for each $0\le k\le n$.

So $H^*(\mathbb{C}P^n;\mathbb{Z}) \cong \mathbb{Z}[\omega]/\omega^{k+1}$)

Geometric interpretation of ∂_n:

For an n-cell e^n in X with attaching map $\varphi: S^{n-1} \to X_{n-1}$, writing $\partial_n e^n = \sum c_i e_i^{n-1} \in C_{n-1}^{CW}(X)$ the coefficients are

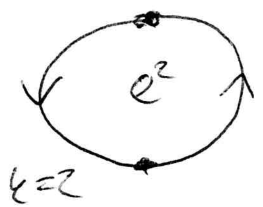
c_i = degree of map $S^{n-1} \to X_{n-1} / (X_{n-1} \setminus e_i^{n-1}) \cong S^{n-1}$

= degree of restriction of φ to $\varphi^{-1}(e_i^{n-1}) \to e_i^{n-1}$

<u>Example</u>

$\mathbb{R}P^n$ is a CW complex with k-skeleton $\mathbb{R}P^k$, and a single k-cell for each $0 \le k \le n$.

Attaching map for the k-cell is quotient map $\varphi: S^{k-1} \to \mathbb{R}P^{k-1}$.



For k even, $\mathbb{R}P^{k-1}$ is orientable and $\deg \varphi = 2$

For k odd the $(k-1)$-cell is still covered twice but with opposite orientations, so relevant degree $= 0$.

So $C_*^{CW}(\mathbb{R}P^n; G)$ is $G \xleftarrow{0} G \xleftarrow{2} G \xleftarrow{0} G \xleftarrow{2} \cdots$

$$H_k(\mathbb{R}P^n; G) \cong \begin{cases} G & \text{for } k=0 \text{ and } k=n \text{ if } n \text{ is odd} \\ G/2G & \text{for } k \text{ odd}, \ 0<k<n \\ T_2 G & \text{for } k \text{ even}, \ 0<k\le n \end{cases}$$

In particular, $H_k(\mathbb{R}P^n; \mathbb{Z}/2) \cong H^k(\mathbb{R}P^n; \mathbb{Z}/2) \cong \mathbb{Z}/2$ for each $0 \le k \le n$.

<u>Remark</u>

A cellular map $f: X \to Y$ induces a chain map $f_\#: C_n^{CW}(X) \to C_n^{CW}(Y)$, and hence homomorphisms on homology and cohomology.

2.6 EILENBERG-MACLANE SPACES AND COHOMOLOGY OPS

Definition

X is called an Eilenberg-MacLane space if $\pi_n X \neq 0$ for a single $n > 0$ and X is path-connected.

For such a space with a choice of isomorphism $\pi_n X \cong G$ call X a $K(G,n)$

Theorem

For any $n > 0$ and group G (abelian if $n \neq 1$) there exists a CW complex $K(G,n)$.
Moreover, that is unique up to homotopy equivalence, and. can be taken to have no k-cells for $0 < k < n$.

Proof:

Existence is CW approximation: Take union of n-cells corresponding to generators of G, add $(n+1)$-cells to impose relations, and k-cells with $k > n+1$ to kill higher homotopy groups.

Uniqueness can be seen as consequence of next theorem. □

COHOMOLOGY OPERATIONS

Definition

A cohomology operation Θ defines (for some fixed m, n and groups G, K) for every topological space X

- a function

$$\Theta : H^n(X; G) \to H^m(X; K)$$

st for any $g : X \to Y$ and $\alpha \in H^n(Y; G)$

$$g^* \Theta(\alpha) = \Theta(g^* \alpha)$$

Examples

- Powers of cup product $H^n(X; R) \to H^{kn}(X; R)$, $\alpha \mapsto \alpha^k$
- $H^n(X; G) \to H^n(X; K)$ induced by group homomorphism $G \to K$
- Bockstein map $H^n(X; G) \to H^{n+1}(X; K)$ from SES of coefficient groups $0 \to K \to H \to G \to 0$.

If Y is a $K(G,n)$ then by universal coefficients + Hurewicz

$$H^n(Y;G) \cong \mathrm{Hom}(H_n(Y);G) \cong \mathrm{Hom}(G,G)$$

Thus $H^n(Y;G)$ has a canonical element u_G corresponding to identity $G \to G$ (depends on the choice of isomorphism $\pi_n Y \cong G$)

Theorem

If X is a CW complex and Y is a $K(G,n)$ then

$$\begin{aligned} [X,Y] &\longrightarrow H^n(X;G) \\ f &\longmapsto f^* u_G \end{aligned}$$

is a bijection.

Theorem

There is a bijection

$$\left\{\begin{matrix}\text{cohomology operations} \\ \text{with given } m, n, G, K\end{matrix}\right\} \longrightarrow H^m(K(G,n);K)$$

$$\Theta \longmapsto \Theta(u_G)$$

Proof:

For any $\beta \in H^m(K(G,n);K)$ define Θ by

- If X is a CW complex and $\alpha \in H^n(X;G)$, consider the unique homotopy class of maps $f: X \to K(G,n)$ s.t. $f^* u_G = \alpha$, and set $\Theta(\alpha) = f^* \beta \in H^m(X;K)$.
- For arbitrary X, consider a CW approximation. $\square$

Example

$K(\mathbb{Z}/2, 1) = \mathbb{R}P^\infty = S^\infty/\pm 1$

$H^*(\mathbb{R}P^\infty; \mathbb{Z}/2) \cong \mathbb{Z}/2[\omega]$ for $\omega \in H^1(\mathbb{R}P^\infty; \mathbb{Z}/2)$ non-zero.

$\rightsquigarrow$ no cohomology operations $\Theta : H^1(\ ; \mathbb{Z}/2) \to H^k(- ; \mathbb{Z}/2)$ other than cup powers.

Interesting consequence:
For any top space X, the Bockstein map
$$H^1(X; \mathbb{Z}/2) \to H^2(X; \mathbb{Z}/2)$$
of the SES $0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$
must agree with the cup squaring map.