THOM ISOMORPHISM THEOREM

A real rank n vector bundle can be described as an $\mathbb{R}^n$-fibre bundle $\pi : E \to X$ together with choices of local trivialisations $\pi^{-1}(U) \cong U \times \mathbb{R}^n$ st transition functions are linear on each fibre.
It is oriented if these linear maps have positive determinant.
In that case the identification of a fibre $F = \pi^{-1}(x)$ with $\mathbb{R}^n$ picks out a well-defined choice of generator of

$$H^n(F, F \setminus \{0\}; \mathbb{Z}) \cong H^n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}; \mathbb{Z}) \cong \mathbb{Z}$$

Theorem

- Let $\pi : E \to X$ an oriented real rank n bundle, and $\dot{E}$ complement to zero section. There is a unique "Thom class"

$$u(E) \in H^n(E, \dot{E}; \mathbb{Z})$$

such that for each fibre $F = \pi^{-1}(x)$, the restriction of $u(E)$ to F defines the oriented generator in $H^n(F, F \setminus \{0\}; \mathbb{Z})$.

- If we drop "oriented", same claim holds with $\mathbb{Z}$ replaced by $\mathbb{Z}/2$. (Then $H^n(F, F\setminus\{0\}; \mathbb{Z}/2) \cong \mathbb{Z}/2$ has a unique non-zero element.)

<u>Proof sketch</u>

For a trivial bundle, the claim + $H^k(E, \dot{E}) = 0$ for $k<n$ holds by Künneth formula (or by induction on n).

If that is true for U, V and $U \cap V$ then it also holds for $U \cup V$ by Mayer–Vietoris.

If there is a finite cover of trivialising nbhds we are done by induction.
More generally, can reduce to case of CW complexes, where one can take limits. $\square$

Combining with Leray-Hirsch yields:
for each k

$$H^k(X;\mathbb{Z}) \to H^{n+k}(E,\dot{E},\mathbb{Z}), \quad \alpha \mapsto \pi^*\alpha \cup u(E)$$

is an isomorphism.

Uniqueness of the Thom class implies

- if $f: Y \to X$ and $F: f^*E \to E$ is the tautological map on the total space of pull-back of $\pi: E \to X$ then
$$F^*(u(E)) = u(f^*E) \in H^n(f^*E, f^*\dot{E}; \mathbb{Z})$$
- for a direct sum of a rank m bundle $E \to X$ and rank n bundle $E' \to X$ (i.e. fibre of $E \oplus E'$ over $x \in X$ is $E_x \oplus E'_x$) with projections $p: E \oplus E' \to E$, $p': E \oplus E' \to E'$ from the total space
$$p^*(u(E)) \cup (p')^*(u(E')) = u(E \oplus E') \in H^{m+n}(E \oplus E', \dot{\widehat{E \oplus E'}}; \mathbb{Z})$$

Remark
If X is compact then $H^*(E,\dot{E};G) \cong H^*_{cpt}(E;G)$

THOM CLASSES AND POINCARÉ DUALS OF SUBMANIFOLDS

Suppose X is smooth and $E \to X$ is a smooth oriented real rank n bundle. Then $u(E) \in H^n(E, \dot{E}; \mathbb{Z})$ can be interpreted as "Poincaré dual" of zero section.

If $Y \subset E$ is a closed oriented n-submfd intersecting zero section transversely (at finitely many points $x_1, \ldots, x_k \in X$) then evaluation of $u(E)$ on $[Y]$ depends only on nbhds $U_i \subset Y$ of the x_i: by excision $u(E)[Y]$ is a sum of contributions from $H^n(U_i, U_i \setminus \{x_i\}; \mathbb{Z})$. Each U_i is homotopic to inclusion of a fibre, so get ± 1 depending on orientations.

Thus $u(E)[Y]$ counts how many times Y intersects the zero section, with signs.

Proposition

If $X^{n-k}, Y^{n-l} \subset M$ are both closed as subsets of M and intersect transversely, then

$$PD_M(X) \cup PD_M(Y) = PD_M(X \cap Y) \in H^{k+l}(M;\mathbb{Z})$$

Proof:

$N_{X\cap Y}$ = direct sum of restriction to $X \cap Y$ of normal bundles of X and of Y, Claim follows from $u(E \oplus E') = u(E) \cup u(E')$. $\square$

If the embedded submfd $i: X^{n-k} \hookrightarrow M^n$ is compact (without boundary), then clearly we can also define a preimage $PD^c_M(X) \in H^k_{cpt}(M;\mathbb{Z})$ of $PD_M(X)$.

If X is also oriented, we may also consider $[X] := i_*[X] \in H_{n-k}(M)$

Proposition

If $X^{n-k} \subset M^n$ is embedded and M and X are both oriented, then the Poincaré duality isomorphism $D_M : H^k_{cpt}(M;\mathbb{Z}) \to H_{n-k}(M)$ maps $PD^c_M(X)$ to $[X]$.

For an embedded submfd $i : X^{n-k} \hookrightarrow M^n$ (without boundary) that is closed as a subset of M, a tubular nbhd $U \subset M$ of X is diffeomorphic to the total space of its normal bundle N_X (which has rank k)

Given an orientation on N_X (eg coming from orientations on M and X) let $PD_M(X) \in H^k(M;\mathbb{Z})$ be the image of $u(N_X)$ under

$$H^k(N_X, \dot{N}_X; \mathbb{Z}) \cong H^k(U, U \setminus X; \mathbb{Z}) \underset{\substack{\uparrow \\ \text{excision}}}{\cong} H^k(M, M \setminus X; \mathbb{Z}) \underset{\substack{\uparrow \\ \text{restriction}}}{\longrightarrow} H^k(M;\mathbb{Z})$$

<u>Proposition</u>

If $f : N \to M$ is any smooth map transverse to $X \subset M$ (ie for any $y \in N$ st $f(y) = x \in X$, $(\operatorname{im} Df_y) + T_x X = T_x M$) then $f^*(PD_M(X)) = PD_N(f^{-1}(X))$.

<u>Proof</u>

f maps a tubular nbhd $V \subseteq N$ of $Y := f^{-1}(X)$ to U.
V is diffeomorphic to total space of N_Y, which is $\cong f^* N_X$.
Claim follows as $u(f^* N_X) = f^* u(N_X)$. $\square$

Proof

Let $U \subset M$ be the tubular nbhd of X as before.
We have a commutative diagram

$$\begin{tikzcd} u(N_X) \in H^k_{cpt}(U;\mathbb{Z}) \arrow[r] \arrow[d, "D_U"'] & H^k_{cpt}(M;\mathbb{Z}) \ni PD^c_M(X) \arrow[d, "D_M"] \\ H_{n-k}(U) \arrow[r] & H_{n-k}(M) \end{tikzcd}$$

$H_{n-k}(U) \cong H_{n-k}(X)$ generated by $[X]$

Suffices to say D_U (equivalently, Poincaré duality isom. on total space of N_X) maps generator $u(N_X)$ to the generator $[X]$.

Corollary

If $X^{n-k}, Y^{n-l} \subset M^n$ are embedded oriented and intersect transversely, and X is compact and Y is closed as a subset of M, then

$$[X] \cap PD_M(Y) = [X \cap Y] \in H_{n-k-l}(M)$$

Proof: Let $i: X \hookrightarrow M$ inclusion
$i_*[X] \cap PD_M(Y) = i_*([X] \cap i^* PD_M(Y)) = i_*(D_X(PD_X(X \cap Y)) = i_*[X\cap Y]$. $\square$

DEGREE

As a special case, consider a proper map $f: M \to N$ of oriented connected mfds of dim n.

Proper = pre-image of any compact set is compact
$\Rightarrow$ induced $f^*: H^*_{cpt}(N;\mathbb{Z}) \to H^*_{cpt}(M;\mathbb{Z})$.

By Poincaré duality
$H^n_{cpt}(M;\mathbb{Z}) \cong H^n_{cpt}(N;\mathbb{Z}) \cong \mathbb{Z}$ generated by PD(point)

So f^* is identified with multiplication by an integer.
Call that integer the <u>degree</u> of f.

f transverse to $x \in N \Leftrightarrow Df_y : T_yM \to T_xN$ surjective for any $y \in f^{-1}(x)$
$\Leftrightarrow x$ is a regular value

<u>Sard's theorem</u>
Critical values of f (ie non-regular ones) have measure 0 in N.

$\rightsquigarrow$ almost all points in N have $\deg(f)$ preimages (counted with sign, depending on whether Df_y preserves orientation or not).

2 HOMOTOPY GROUPS AND CW COMPLEXES

2.1 HOMOTOPY GROUPS AND THEIR LONG EXACT SEQUENCE

Notation: For $U \subset A \subset X$, $V \subset B \subset Y$ let

$[X, Y]$ = homotopy classes of maps $f: X \to Y$ $= [(X, \emptyset), (Y, \emptyset)]$

$[(X,A), (Y,B)]$ — " — s.t. $f(A) \subseteq B$

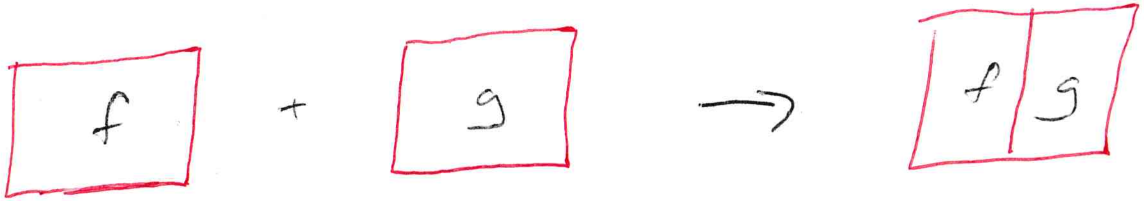
$[(X,A,U), (Y,B,V)]$ = ...

Given a base point $x_0 \in X$, the homotopy group $\pi_n(X, x_0)$ can be described as a set as eg

$$[(S^n, s_0), (X, x_0)], \quad \underset{\text{ball}}{[(D^n, S^{n-1}), (X, x_0)]} \quad \text{or} \quad \underset{\text{cube } (I = [0,1])}{[(I^n, \partial I^n), (X, x_0)]}$$

$\pi_0(X, x_0)$ = {path-components of X}, a "pointed set" (distinguished element is component containing x_0)

For $n \geq 1$, $\pi_n(X, x_0)$ has a group structure, easiest to describe in cube picture.

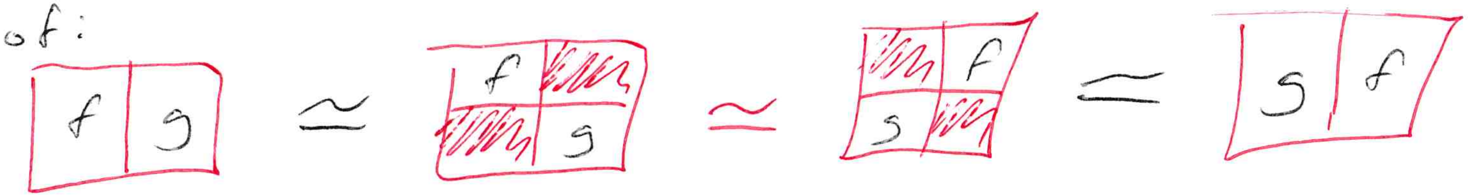


red $\mapsto x_0$

Lemma

$\pi_n(X, x_0)$ is abelian for $n \geq 2$

Proof:



□

~~For~~ Any path γ from x_0 to x_1 $\rightsquigarrow$ isomorphism $\pi_n(X, x_0) \to \pi_n(X, x_1)$, depending only on homotopy class $[\gamma]$. So

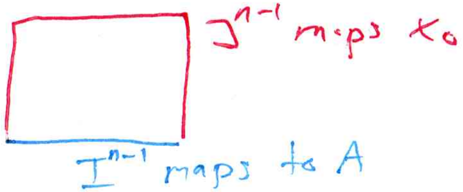
- base point "doesn't matter" provided X is path-connected
- $\pi_1(X, x_0)$ acts on $\pi_n(X, x_0)$ (if $n=1$ this is just conjugation action)

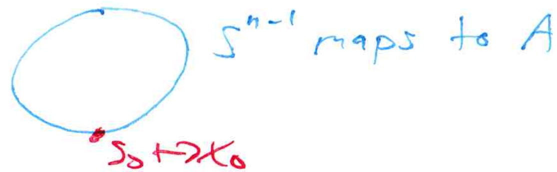
RELATIVE HOMOTOPY GROUPS

Consider I^{n-1} as one face $I^{n-1} \times \{0\} \subset \partial I^n$, and let $J^{n-1} \subset \partial I^n$ be the closure of the complement of I^{n-1}.

For $x_0 \in A \subseteq X$ and $n \geq 1$, the relative homotopy group $\pi_n(X, A, x_0)$ can be described as a set as

$$[(I^n, \partial I^n, J^{n-1}), (X, A, x_0)] \quad \text{or} \quad [(D^n, S^{n-1}, s_0), (X, A, x_0)]$$





Can define a group structure on $\pi_n(X, A, x_0)$ by same picture as before if $n \geq 2$

Identity is represented by constant map with image x_0 (or any map with image in A).

Similarly, $\pi_n(X, A, x_0)$ is abelian if $n \geq 3$

THE LES

If $g: X \to Y$ maps $x_0 \mapsto y_0$ then composition induces homomorphism $g_*: \pi_n(X, x_0) \to \pi_n(Y, y_0)$

If also $g(A) \subseteq B$ then also $g_*: \pi_n(X, A, x_0) \to \pi_n(Y, B, x_0)$

In particular, for $x_0 \in A \subseteq X$ we have $\pi_n(A, x_0) \to \pi_n(X, x_0)$
$\pi_n(X, x_0) \to \pi_n(X, A, x_0)$

Also, for $[f] \in \pi_n(X, A, x_0)$ get a well-defined $\delta[f] \in \pi_{n-1}(A, x_0)$ by restricting a representative $f: I^n \to X$ to I^{n-1}!

Theorem

The sequence of homomorphisms / functions between pointed sets

$$\cdots \to \pi_2(X, A) \xrightarrow{\delta} \pi_1(A) \to \pi_1(X) \to \pi_1(X, A) \xrightarrow{\delta} \pi_0(A) \to \pi_0(X)$$

is exact

2.2 THE HUREWICZ THEOREM

The Hurewicz homomorphism $h : \pi_n(X, x_0) \to H_n(X)$ can be interpreted in several ways.

$f : (S^n, s_0) \to (X, x_0) \quad \mapsto \quad f_*[S^n]$

$f : (I^n, \partial I^n) \to (X, x_0) \quad \mapsto$ homology class represented by simplices obtained from triangulating I^n

$f : (\partial \Delta_{n+1}, e_0) \to (X, x_0) \quad \mapsto$ homology class represented by $\langle f \rangle$ = sum of restrictions of f to faces of Δ_{n+1}

Theorem

Let $n \geq 2$ and suppose X is $(n-1)$-connected, ie path-connected and $\pi_i(X) = 0$ for $1 \leq i < n$.

Then $h : \pi_n(X, x_0) \to H_n(X)$ is an isomorphism.

If $n = 1$ then instead h induces an isomorphism $\tilde{\pi}_1(X, x_0) \to H_1(X)$, where $\tilde{\pi}_1$ is the abelianisation, ie quotient of π_1 by the derived subgroup $[\pi_1, \pi_1]$

Clumsy but intuitive proof:

For each singular simplex $\tau : \Delta_k \to X$ with $k < n$, define a "cone" $C(\tau) : \Delta_{k+1} \to X$ st $e_0 \mapsto x_0$ and restriction to back face is given by τ, in such a way that

for any k-simplex $\sigma : \Delta_k \to X$ with $k \leq n$ one obtains a well-defined $g : (\partial\Delta_{k+1}, e_0) \to (X, x_0)$ by patching up σ on back face and cones over faces of σ.

Can do this recursively by taking $C(\tau)$ to be any extension of $g(\tau) : \partial\Delta_{k+1} \to X$ to interior, which is possible because X is $(n-1)$-connected

Then

a) $\langle g(\sigma) \rangle = \sigma + C(\partial\sigma) \in C_n(X)$ for any n-simplex σ

b) $\sum_j [g(j^{th} \text{ face of } f)] = [f] \in \pi_n(X, x_0)$ for any $f : (\partial\Delta_{n+1}, e_0) \to (X, x_0)$

a) means that for any class in $H_n(X)$ represented by a closed chain $\sum_i \sigma_i$ we can define a h-preimage by $\sum_i [g(\sigma_i)] \in \pi_n(X, x_0)$

On the other hand, suppose $f: (\partial\Delta_{n+1}, e_0) \to (X, x_0)$ represents an element in the kernel of h, say

$$\langle f \rangle = \partial\left(\sum \sigma_i\right) \in C_n(X) \qquad (*)$$

For each σ_i adding up $[g(\sigma_i^{(j)})] \in \pi_n(X, x_0)$ for the faces $j = 0, \dots, n+1$ gives identity.

On the other hand, $(*)$ means

$$\sum_{i,j} [g(\sigma_i^{(j)})] = \sum_j [g(j^{\text{th}} \text{ face of } f)] \underset{(5)}{=} [f]$$

So if $n \geq 2$ so that π_n is abelian then h is injective.

If $n = 1$, we just find elements in the kernel of h are equal to identity $\underline{\text{modulo commutators}}$, i.e. $\ker h = [\pi_1, \pi_1]$.

$\square$

Relative version of Hurewicz

If $A \subseteq X$ and A, X both path-connected, $n \geq 2$ and $\pi_i(X, A, x_0) = 0$ for $1 \leq i < n$, then $\pi_n(X, A, x_0) \to H_n(X, A)$ is surjective, with quotient generated by elements of the form

$$[\gamma][f] - [f] \qquad \text{for } [f] \in \pi_n(X, A, x_0) \text{ and } [\gamma] \in \pi_1(A, x_0)$$

In particular, if A is simply-connected then

$$\pi_n(X, A, x_0) \cong H_n(X, A)$$

WEAK HOMOTOPY EQUIVALENCES

If $f: X \to Y$ is a homotopy equivalence, then induced maps f_* on π_n and H_n and f^* on H^n are all isomorphisms.

Definition

Call $f: X \to Y$ a weak homotopy equivalence if $f_*: \pi_n(X, x_0) \to \pi_n(Y, f(x_0))$ is an isomorphism for every $x_0 \in X$ and n.

Proposition

Then $f_*: H_n(X; G) \to H_n(Y; G)$ and $f^*: H^n(X; G) \to H^n(Y; G)$ are isomorphisms for any n and coefficient group G.

Proof

WLOG X and Y are path-connected.

First suppose f in inclusion of a subspace.

Then $\pi_i(Y,X)=0$ for all i: by LES for relative homotopy groups

$H_i(Y,X)=0$ for all i: by relative version of Hurewicz

$H_i(Y,X;G)=0$ and $H^i(Y,X;G)=0$ for all i: by universal coefficients

Conclusion follows from LES for relative (co)homology.

In general, consider the <u>mapping cylinder</u> M_f of $f: X \to Y$, i.e.

$$M_f := X \times [0,1] \amalg Y/\sim, \text{ where } (x,1) \sim f(x)$$

Then $j: X \to M_f$, $x \mapsto (x,0)$ induces $j_*: H_n(X;G) \xrightarrow{\cong} H_n(M_f;G)$ etc

On the other hand, $p: M_f \to Y$, $p(x,t)=f(x)$, $p(y)=y$ is a homotopy equivalence, and

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "f"'] & M_f \arrow[d, "p"] \\ & Y \end{tikzcd}$$

commutes

So f_* is composition of isomorphisms j_* and p_*. $\square$

Partial converse

If X and Y are simply-connected and $f_*: H_n(X) \to H_n(Y)$ is an isomorphism for each n, then f is a weak homotopy equivalence.

Proof

Reduce to the case when f is an inclusion like before.

$H_n(Y,X) = 0$ for all $n \Rightarrow \pi_n(Y,X) = 0$ for all n by induction, using Hurewicz. $\square$

($\pi_1 X = 0 \Rightarrow \pi_n(Y,X) \to H_n(Y,X)$ is not just surjective but also injective).

Similarly, call $f: X \to Y$ an <u>n-equivalence</u> if

$f_*: \pi_i(X, x) \to \pi_i(Y, f(x))$ is an isomorphism for $i < n$
surjective for $i = n$

(If f is an inclusion, that is equivalent to $\pi_i(Y, X) = 0$
for $i \leq n$.)

That implies $f_*: H_i(X; G) \to H_i(Y; G)$ is an isomorphism for $i < n$
surjective for $i = n$

and $f^*: H^i(Y; G) \to H^i(X; G)$ is an isomorphism for $i < n$
injective for $i = n$.