CAP PRODUCT

For $n \geq k$ define
$$C_n(X;R) \times C^k(X;R) \to C_{n-k}(X;R)$$
$$(c, \alpha) \mapsto c \cap \alpha$$

by $\sigma \cap \alpha = \alpha(\sigma \circ F[e_0, \ldots, e_k]) \cdot (\sigma \circ F[e_k, \ldots, e_n])$ for any n-simplex σ

This ensures

- $c \cap (\alpha \cup \beta) = (c \cap \alpha) \cap \beta$
- if $k = n$ then the tautological pairing $\alpha(c) \in R$ equals the image of $c \cap \alpha \in C_0(X;R)$ under counting map $C_0(X;R) \to R$

Also $\partial(c \cap \alpha) = (-1)^k (\partial c \cap \alpha - c \cap d\alpha)$

$\rightsquigarrow$ well-defined product $H_n(X;R) \times H^k(X;R) \to H_{n-k}(X;R)$

For $f: X \to Y$, $c \in H_n(X;R)$ and $\alpha \in H^k(Y;R)$

$$(f_* c) \cap \alpha = f_*(c \cap f^\# \alpha) \in H_{n-k}(Y;R)$$

GEOMETRIC INTERPRETATION OF CUP AND CAP

If M is an n-dimensional manifold then we could pretend

$$C_k(M) = \{\text{compact oriented dim } k \text{ submfds } X \subset M \text{ with boundary}\}$$

If $X \subset M$ is closed (ie compact and $\partial X = \emptyset$) then $\rightsquigarrow [X] \in H_k(M;\mathbb{Z})$

If M is smooth then submfd $X, Y \subset M$ intersect <u>transversely</u> if
$T_pX + T_pY = T_pM$ for all $p \in X \cap Y$.

If $\dim X + \dim Y = \dim M$ that implies $X \cap Y$ discrete,
So finite if ~~discrete~~ compact.

For M^n smooth and oriented and $Y \subset M$ compact oriented $(n-k)$-submfd
can pretend to define $\varphi_Y : C_k(M) \to \mathbb{Z}$ by counting
transverse intersection points (with signs)

"$\partial Y = \emptyset \Rightarrow d\varphi_Y = 0$" $\rightsquigarrow$ $PD_M(Y) \in H^k(M;\mathbb{Z})$

make precise with Thm isomorphism

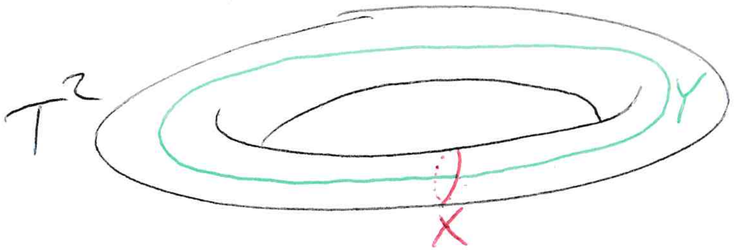
If $Z \subset M$ is a closed oriented $(n-\ell)$-submfd intersecting Y transversely then

$$[Z] \cap PD_M(Y) = [Y \cap Z] \in H_{n-k-\ell}(M)$$

$$PD_M(Y) \cup PD_M(Z) = PD_M(Y \cap Z) \in H^{k+\ell}(M;\mathbb{Z})$$

<u>Example</u>

$H_0 \cong H_2 \cong H^0 \cong H^2 \cong \mathbb{Z}$, $H_1 \cong H^1 \cong \mathbb{Z}^2$



	0	1	2
Generators for H_k	$[pt]$	$[X]$, $[Y]$	$[T^2]$
for H^k	$PD[T^2]$ (= consts points)	$PD[X]$, $PD[Y]$	$PD[pt]$

Ring structure of $H^*(T^2;\mathbb{Z})$ determined by

$$PD[X] \cup PD[Y] = PD[pt]$$

$$PD[X] \cup PD[X] = PD[Y] \cup PD[Y] = 0$$

$\rightsquigarrow$ "intersection form" on $H^1(T^2;\mathbb{Z})$ represented by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

1.4 POINCARÉ DUALITY

ORIENTATIONS

Let M be an n-dimensional topological manifold.
Then every $x \in M$ has a neighbourhood $U \subseteq M$ st $U \cong \mathbb{R}^n$.

By excision

$$H_n(M, M\setminus\{x\}) \cong H_n(U, U\setminus\{x\}) \cong H_n(\mathbb{R}^n, \mathbb{R}^n\setminus\{0\}) \cong \mathbb{Z}$$

has two generators. Those can be interpreted as two choices of orientation "at" x.
Given the nbhd U, there is a notion of whether an orientation at another $y \in U$ agrees with or is opposite to one at x since

$$H_n(M, M\setminus\{x\}) \cong H_n(U, U\setminus B) \cong H_n(M, M\setminus\{y\})$$

for any compact ball B containing both x and y.

$\rightsquigarrow$ local orientations form a topological space, which is a double cover of M

Call M <u>orientable</u> if that double cover is trivial.
Then an orientation is a section.

More generally, for any ring R one can say that an "R-orientation" is an analogously consistent assignment

point $x \in M$ $\mapsto$ generator of $H_n(M, M \setminus \{x\}; R) \cong R$.

R-orientation exists $\Leftrightarrow$ M orientable or $2=0$ in R

(If M is connected and R-orientable, then R-orientations $\Leftrightarrow$ units in R)

In particular, there is always a unique $\mathbb{Z}/2$-orientation!

Call $\mu \in H_n(M, R)$ a <u>fundamental class</u> if for any $x \in M$, the image of μ in $H_n(M, M \setminus \{x\}; R)$ is a generator.

<u>Theorem</u>

For a closed (compact without boundary) n-dimensional M,

fundamental classes in $H_n(M; R)$ $\Longleftrightarrow$ R-orientations

and if M is R-orientable and connected then $H_n(M; R) \cong R$.

Given an R-orientation on M, denote the corresponding fundamental class by $[M]$.

More generally, even if M is not compact (but has no boundary) one can say that for any compact $A \subset M$

R-orientation of $M \rightsquigarrow$ "local" fundamental class $[M|A] \in H_n(M, M \setminus A; R)$

POINCARÉ DUALITY THEOREM

Closed version

For a closed topological n-mfd M with fundamental class $[M] \in H_n(M;R)$

$$D_M : H^k(M;R) \to H_{n-k}(M;R), \quad \alpha \mapsto [M] \cap \alpha$$

is an isomorphism.

In particular, if M is closed and orientable then the Betti numbers satisfy

$$b_k(M) = b_{n-k}(M)$$

(ie $\operatorname{rank} H_k(M) = \operatorname{rank} H_{n-k}(M)$)

Also $\dim_{\mathbb{Z}/2} H_k(M;\mathbb{Z}/2) = \dim_{\mathbb{Z}/2} H_{n-k}(M;\mathbb{Z}/2)$ for any closed M. (orientable or not)

<u>Non-compact version</u> (needed in proof of closed version)

Let M be a manifold (without boundary).

Let $C^*_{cpt}(M;R) \subset C^*(M;R)$ be the subcomplex of cochains φ st
$\exists$ cpt $K \subset M$ st $\varphi(\sigma)=0$ for any $\sigma : \Delta_k \to M \setminus K$

$\rightsquigarrow$ <u>compactly supported cohomology</u> $H^*_{cpt}(M;R)$

If $[\varphi] \in H^k_{cpt}(M;R)$ then φ also represents a class in
$H^k(M, M\setminus K;R)$ for some compact K.

$\rightsquigarrow$ well-defined $[M \setminus K] \cap [\varphi] \in H_{n-k}(M ~~\quad~~ ;R)$ ~~\quad~~

~~in\quad~~

$\rightsquigarrow$ well-defined $D_M : H^k_{cpt}(M;R) \to H_{n-k}(M;R)$

That is an isomorphism.

<u>Remark</u>

For a <u>compact</u> manifold M <u>with</u> boundary

$H_k(M \setminus \partial M;R) \cong H_k(M;R)$, $\quad H^k_{cpt}(M\setminus\partial M;R) \cong H^k(M,\partial M;R)$

1.5 PERFECT PAIRINGS

INTERSECTION FORM

For a closed oriented n-dimensional M, consider the pairing

$$I_M : H^k(M;\mathbb{Z}) \times H^{n-k}(M;\mathbb{Z}) \to \mathbb{Z}$$
$$(\alpha, \beta) \mapsto (\alpha \cup \beta)[M]$$

If α or β has finite order then so does $I_M(\alpha,\beta) \in \mathbb{Z}$,
ie $I_M(\alpha,\beta) = 0$, so I_M descends to free quotient

$$I_M : FH^k(M;\mathbb{Z}) \times FH^{n-k}(M;\mathbb{Z}) \to \mathbb{Z}$$

<u>Corollary</u>

I_M is <u>perfect</u>, ie induces an isomorphism

$$FH^k(M;\mathbb{Z}) \to \mathrm{Hom}(FH^{n-k}(M;\mathbb{Z}), \mathbb{Z})$$

Proof:
Since $(\alpha\cup\beta)[M] = \beta([M]\cap\alpha)$, the map is composition of
$FH^k(M;\mathbb{Z}) \to FH_{n-k}(M;\mathbb{Z})$, $\alpha \mapsto [M]\cap\alpha$ isom. by Poincaré
and $FH_{n-k}(M;\mathbb{Z}) \to \mathrm{Hom}(FH^{n-k}(M;\mathbb{Z}),\mathbb{Z})$, $c \mapsto (\beta \mapsto \beta(c))$ isom. by U.C.

$\square$

Any perfect pairing between two different free groups looks the same as any other one, but perfect pairing of a group with itself is interesting.
So focus on middle cohomology of an even-dim mfd.

Corollary
Any closed oriented $(4n+2)$-mfd M has $b_{2n+1}(M)$ even.

Proof:
I_M on $FH^{2n+1}(M;\mathbb{Z})$ is perfect and antisymmetric,
so represented by $\left(\begin{array}{c|c} 0 & I_k \\ \hline I_k & 0 \end{array}\right)$ for some k, then $b_{2n+1}(M) = 2k$. □

For a closed oriented $4n$-mfd M, I_M on $FH^{2n}(M;\mathbb{Z})$ is symmetric, so makes it a unimodular lattice.
Apart from rank $= b_{2n}(M)$, key invariants of a lattice are

- signature $\sigma \in \mathbb{Z}$
- type (even if $x.x$ is even for all $x \in$
  odd if $x.x$ is odd for some x)

Example

Intersection forms of $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ and $S^2 \times S^2$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

have rank $= 2$ and signature $= 0$, but are distinguished by type.

Indefinite lattices are classified by rank, signature and type, but classification of definite lattices is more complicated.

Freedman's Theorem

Simply-connected closed 4-manifolds are classified up to homeomorphism by their intersection form and Kirby-Siebenman invariant $\in \mathbb{Z}/2$.

Donaldson's Diagonalisation Theorem

If the intersection form of a smooth closed simply-connected 4-manifold is definite then it is diagonalisable.

Exercise

The signature

- of ~~[scribbled out]~~ the boundary of any compact oriented (4n+1)-manifold is zero
- is additive under connected sums
- is multiplicative under products

$\leadsto$ signature defines a ring homomorphism

$$\Omega_*^{SO} \to \mathbb{Z}$$

$\uparrow$

ring of oriented manifolds modulo bordism

TORSION LINKING FORM

For a closed oriented n-dimensional M, the torsion linking form

$$\begin{aligned} TH^{k+1}(M) \times TH^{n-k}(M) &\to \mathbb{Q}/\mathbb{Z} \\ (x, y) &\mapsto \ell(x,y) \end{aligned}$$

is defined by

$$\ell(x,y) = (z \cup y)[M]$$

for z any pre-image of x under Bockstein map (of $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$)

$$\beta : H^k(M; \mathbb{Q}/\mathbb{Z}) \to H^{k+1}(M; \mathbb{Z})$$

This makes sense because

x torsion $\Rightarrow$ $x \in \ker H^{k+1}(M; \mathbb{Z}) \to H^{k+1}(M; \mathbb{Q})$

y torsion $\Rightarrow$ $w \cup y = 0$ for any $w \in H^k(M; \mathbb{Q})$, so changing z by adding w does not affect $\ell(x,y)$.

Lemma

a) $\ell(x,y) = (-1)^{(k+1)(n-k)} \ell(y,x)$

b) ℓ is <u>perfect</u> in the sense that the induced map

$$TH^{n-k}(M) \to \mathrm{Hom}(TH^{k+1}(M), \mathbb{Q}/\mathbb{Z}), \quad y \mapsto (x \mapsto \ell(x,y))$$

is an isomorphism

Proof

a) If $x = [\varphi] \in H^{k+1}(M;\mathbb{Z})$ is m-torsion, say $m\varphi = d\rho$, $\rho \in C^k(M;\mathbb{Z})$

$y = [\psi] \in H^{n-k}(M;\mathbb{Z})$ is n-torsion, say $n\psi = d\sigma$, $\sigma \in C^{n-k-1}(M;\mathbb{Z})$

then we can expand definition more explicitly as

$$\ell(x,y) = \frac{1}{m}(\rho \cup \psi)[M] \in \left(\tfrac{1}{m}\mathbb{Z}\right)/\mathbb{Z} < \mathbb{Q}/\mathbb{Z}$$

$$\ell(y,x) = \frac{1}{n}(\sigma \cup \varphi)[M] \in \left(\tfrac{1}{n}\mathbb{Z}\right)/\mathbb{Z}$$

so

$$\ell(x,y) - (-1)^{(k+1)(n-k)}\ell(y,x) = \frac{1}{mn}(d(\rho \cup \sigma))[M] = 0.$$

b) Factor the map as composition of

$$TH^{n-k}(M) \to TH_k(M), \quad y \mapsto [M] \cap y \qquad \text{isom. by Poincaré}$$

$$TH_k(M) \to \mathrm{Hom}(TH^{k+1}(M), \mathbb{Q}/\mathbb{Z}), \quad c \mapsto (x \mapsto (\beta^{-1}(x))c)$$

isom. by universal coefficients. □

Geometric interpretation:

Suppose M is smooth, and $x = PD_M(X)$, $y = PD_M(Y)$ for closed oriented smooth submfds $X, Y \subset M$.

If $mX = \partial Z$ for smooth $Z \subset M$ intersecting Y transversely, then

$$\ell(x,y) = \frac{1}{m}(\#(Z \cap Y) \text{ counted with signs})$$

<u>Exercise</u>

For q coprime to p, the lens space $L_{p,q}$ is the quotient of $S^3 = \{(z,w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\}$ by $\mathbb{Z}/p$ action generated by $(z,w) \mapsto (e^{\frac{2\pi i}{p}} z, e^{\frac{2\pi q i}{p}} w)$. Then

$$H^2(M;\mathbb{Z}) \cong H_1(M) \cong \pi_1 M \cong \mathbb{Z}/p$$

has a generator x st $\ell(x,x) = \frac{q}{p} \in \mathbb{Q}/\mathbb{Z}$

Theorem (Barden)

Smooth closed simply-connected 5-manifolds M are classified up to diffeomorphism by $H^3(M;\mathbb{Z})$ together with torsion-linking form on $TH^3(M)$ and homomorphism $H^3(M;\mathbb{Z}) \to \mathbb{Z}/2$ defined by second Stiefel-Whitney class $w_2 \in H^2(M;\mathbb{Z}/2)$

1.6 COHOMOLOGY OF FIBRE BUNDLES

Recall: $\pi : E \to X$ is a fibre bundle with typical fibre F if for every $x \in X$ there is a trivialising nbhd $U \subset X$, ie

$$\pi^{-1}(U) \xrightarrow{\simeq} U \times F \quad \text{commuting with projection to } U.$$

For $f : Y \to X$, there is a pull-back bundle

$$f^* E = \{ (y, e) \in Y \times E \text{ st } f(y) = \pi(e) \}$$

"fibre of f^*E over $y \in Y$ is a copy of the fibre of E over $f(y) \in X$"

LERAY-HIRSCH AND THE KÜNNETH FORMULA

Künneth formula describes cohomology of a product.
One version is

Theorem Let R be a ring, and X and F some top spaces, and suppose $H^*(F;R)$ is a free finitely-generated R-module.
Let $\pi : X \times F \to X$ and $\psi : X \times F \to F$ be the projections.
Then

$$\begin{aligned} H^*(X;R) \times H^*(F;R) &\longrightarrow H^*(X \times F;R) \\ (\alpha, \beta) &\longmapsto \pi^*\alpha \cup \psi^*\beta \end{aligned}$$

induces an isomorphism

$$H^*(X;R) \otimes_R H^*(F;R) \cong H^*(X \times F;R)$$

(so ring structure of $H^*(X \times F;R)$ is completely determined by the rings $H^*(X;R)$ and $H^*(Y;R)$)

That version of the Künneth formula can be regarded as a special case of

<u>Leray–Hirsch theorem</u>

Let $\pi : E \to X$ a fibre bundle, R a ring, and suppose $\exists\, \alpha_1, \dots, \alpha_n \in H^*(E; R)$ st for each fibre F, the restrictions of $\alpha_1, \dots, \alpha_n$ to F form an R-module basis of $H^*(F; R)$.
Then $\alpha_1, \dots, \alpha_n$ is a basis for $H^*(E; R)$ regarded as an $H^*(X; R)$-module (with "scalar multiplication"

$$\begin{aligned} H^*(X;R) \times H^*(E;R) &\to H^*(E;R) \\ (\alpha, \beta) &\mapsto \pi^*\alpha \cup \beta \end{aligned}$$

)

Hypothesis satisfied eg for

- projectivisation of any complex vector bundle
- projectivisation of any real vector bundle, with $R = \mathbb{Z}/2$

Proof sketch

First suppose X and F are CW complexes.
Then prove Künneth formula by induction on skeleta of X.

Thus L-H holds for $\pi^{-1}(U)$ for any trivialising nbhd U.
Use Mayer-Vietoris to argue that if L-H holds for $\pi^{-1}(U)$, $\pi^{-1}(V)$ and $\pi^{-1}(U \cap V)$ then it also holds for $\pi^{-1}(U \cup V)$.

Done by induction if X has finite cover by trivialising nbhds.
Otherwise limiting argument works if X is CW complex.
In general, reduce to CW case by "CW approximation". $\square$