ALGEBRAIC TOPOLOGY

aims to provide tools to answer questions such as

- are two topological spaces homeomorphic, homotopy equivalent, ... ?
- are two fibres bundles isomorphic?
- are two smooth manifolds with added structure (like an almost-complex structure) structure-preserving diffeomorphic?

The tools are algebraic invariants such as

- homotopy groups $\pi_n(X, x_0)$ and homology groups $H_n(X;R)$
- cohomology ring $H^*(X;R)$ (+ extra structure of cohomology operations)

of a topological space X, or characteristic classes $\in H^*(X;R)$ of G-bundle $E \to X$

1 HOMOLOGY AND COHOMOLOGY

Chain complex C_*: Sequence of abelian groups (or "better")

$$\cdots \to C_3 \xrightarrow{\partial} C_2 \xrightarrow{\partial} C_1 \xrightarrow{\partial_1} C_0$$

with homomorphisms $\partial_n : C_n \to C_{n-1}$ st $\partial_{n-1} \circ \partial_n = 0$

$\rightsquigarrow$ homology groups $H_n(C) = \dfrac{\ker \partial_n}{\operatorname{im} \partial_{n+1}}$ $\dfrac{\text{cycles}}{\text{boundaries}}$

Cochain complex C^*: Same thing, except homomorphisms $d_n : C^n \to C^{n+1}$ increase degree

$$C^0 \xrightarrow{d} C^1 \xrightarrow{d} C^2 \xrightarrow{d} C^3 \to \cdots$$

$\rightsquigarrow$ cohomology groups $H^n(C) = \dfrac{\ker d_n}{\operatorname{im} d_{n-1}}$ $\dfrac{\text{cocycles}}{\text{exact}}$

Can obtain cochain complex from chain complex by $C^n := \operatorname{Hom}(C_n, G)$ (and vice versa)

One can cook up many different kinds of (co)chain complexes whose (co)homologies are related for sufficiently nice~~s~~ spaces.

- Simplicial complex = finite union of simplices, any two of which intersect along subsimplex

  $\rightsquigarrow$ simplicial chain complex
  C_* = free group generated by simplices

!

- CW complex X = union of n-skeleta X_n, where X_n is obtained by attaching a collection of n-discs to X_{n-1} by attaching maps $\varphi : S^{n-1} \to X_{n-1}$.

  $\rightsquigarrow$ cellular chain complex has C_n generated by n-cells, with $\partial : C_n \to C_{n-1}$ determined by the attaching maps.

- For any topological space X, the singular chain complex has C_n generated by cts maps $\{$(standard n-simplex) $\to X\}$

- For a smooth manifold M, the differential forms $\Omega^*(M)$ with exterior derivative d form a cochain complex $\rightsquigarrow$ de Rham cohomology ($\Omega^*(M)$ is a differential graded algebra)

- For an open cover $\mathcal{U}$ of X, can define a cochain complex whose n-cochains are functions $\{n$-tuples of elements in $\mathcal{U}$ with non-trivial intersection$\} \to G$
  The direct limit under refining the covers $\rightsquigarrow$ Čech cohomology $\check{H}^*(X; G)$ (also works for sheaves)

1.1 SINGULAR HOMOLOGY AND COHOMOLOGY

Singular chains and homology

k simplex $[v_0, \dots, v_k] \subseteq \mathbb{R}^n$:
convex hull of $k+1$ points $v_0, \dots, v_k$ not contained in any affine dim $k-1$ subspace

Standard n-simplex $\Delta_n = [e_0, \dots, e_n] = \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1}$ st $t_i \geq 0$ and $\sum t_i = 1\}$

std basis

"Framing" of $[v_0, \dots, v_n]$: $F[v_0, \dots, v_n] : \Delta_n \to [v_0, \dots, v_n]$

$(t_0, \dots, t_n) \mapsto \sum t_i v_i$

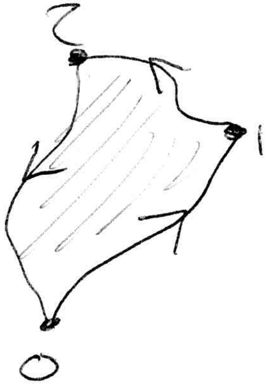
For a topological space X, a <u>singular simplex</u> in X is a cts map

$$\sigma : \Delta_n \to X$$

Its ith face is

$$\sigma^{(i)} := \sigma \circ F[v_0, \dots, \underbrace{\hat{v}_i}_{\text{omit}}, \dots, v_n] : \Delta_{n-1} \to X$$

Let $C_n(X) :=$ free abelian group generated by singular n-simplices



Then defining $\partial : C_n(X) \to C_{n-1}(X)$

$$\partial \sigma = \sum_{i=0}^{n} (-1)^i \sigma^{(i)}$$

makes $C_*(X)$ a chain complex

$\rightsquigarrow$ singular homology groups $H_n(X)$

Example

$X = \{\text{point}\}$ has $C_n \cong \mathbb{Z}$ for each $n \geq 0$

$\partial : C_n \to C_{n-1}$ is isomorphism for even $n > 0$ and zero for odd n, so $H_n(X) \cong 0$ for $n > 0$
$\cong \mathbb{Z}$ for $n = 0$

C_n infinitely generated for anything but finite spaces, but H_* finitely generated eg for compact manifolds, so lots of redundancy.

Simple cases

For any path-connected space X

- $C_0(X) \to \mathbb{Z}$, $\sigma \mapsto 1$ induces $H_0(X) \cong \mathbb{Z}$
- Homomorphism $\pi_1(X) \to H_1(X)$ induces $\pi_1(X)/\text{derived} \cong H_1(X)$ (Hurewicz theorem)

COEFFICIENTS AND COCHAINS

For any abelian group G, we get from singular chain complex of X

- another chain complex $C_*(X;G) = C_*(X) \otimes_{\mathbb{Z}} G$
  $\rightsquigarrow$ Singular homology $H_*(X;G)$ with coefficients in G
- a cochain complex $C^*(X;G) = \mathrm{Hom}(C_*(X), G)$
  with $d: C^n \to C^{n+1}$ the dual to ∂, ie
  $(d\alpha)c = \alpha(\partial c)$ $\rightsquigarrow$ Singular cohomology $H^*(X;G)$ with coefficients in G.

<u>Remarks</u>

- Always have a pairing $H^n(X;G) \times H_n(X) \to G$
- If G is a ring, then $H^*(X;G)$ has a natural ring structure (but $H_*(X;G)$ does not!)

FUNCTORIALITY

For any cts map $f: X \to Y$ and singular simplex $\sigma: \Delta_n \to X$ in X

$$f_\# \sigma := f \circ \sigma : \Delta_n \to Y$$

is a singular simplex in Y

$\rightsquigarrow f_\# : C_*(X) \to C_*(Y)$, which is a "chain map", ie $\partial f_\# = f_\# \partial$

$\rightsquigarrow$ induced homomorphism $f_* : H_n(X) \to H_n(Y)$

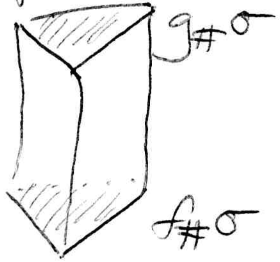
__Proposition__

If $f \simeq g : X \to Y$ (ie $\exists F: X \times [0,1] \to Y$ st $F(x,0) = f(x)$ and $F(x,1) = g(x)$)

then $f_* = g_* : H_n(X) \to H_n(Y)$

Proof:

Intuitively, given any singular simplex $\sigma : \Delta_n \to X$ we would like to think of the "singular prism"



$$\Delta_n \times [0,1] \to Y, \quad (d,t) \mapsto F(\sigma(d),t)$$

as a chain $P(\sigma) \in C_{n+1}(Y)$

$$\partial P(\sigma) = \underbrace{f_\#\sigma}_{\text{bottom}} - \underbrace{g_\#\sigma}_{\text{top}} + \underbrace{P(\partial\sigma)}_{\text{sides}}$$

Formalise by dividing $\Delta_n \times [0,1]$ into $\overset{n+1}{\vee}$ $(n+1)$-simplices

$\rightsquigarrow$ "prism operator" $P : C_n(X) \to C_{n+1}(Y)$ which is a "chain homotopy" between $f_\#$ and $g_\#$, meaning

$$\partial P - P\partial = f_\# - g_\#$$

$\Rightarrow [f_\# c] = [g_\# c] \in H_n(Y)$ whenever $\partial c = 0$. $\square$

Similarly, $f : X \to Y$ induces pull-backs $f^* : H^n(Y;G) \to H^n(X;G)$.

Corollary

If $f : X \to Y$ is a homotopy equivalence (ie $\exists g : Y \to X$ st $f \circ g \simeq \mathrm{Id}_Y$ and $g \circ f \simeq \mathrm{Id}_X$) then f_* and f^* are isomorphisms.

RELATIVE VERSIONS

Let $A \subset X$ a subspace. Then $C_*(A) \subset C_*(X)$ is closed under ∂

$\rightsquigarrow$ $C_*(X,A) := C_*(X)/C_*(A)$ is a chain complex too

$\rightsquigarrow$ relative homology $H_*(X,A)$ (often this is the same as $\tilde{H}_*(X/\sim)$, where $\sim$ crushes A to a point)

"Classes in $H_n(X,A)$ are represented by chains that can have boundary, as long as the boundary is contained in A"

Dually, let $C^*(X,A;G) \subseteq C^*(X;G)$ the subcomplex of cochains that vanish on any singular simplex wholly contained in A (ie kernel of restriction $C^*(X;G) \to C^*(A;G)$)

$\rightsquigarrow$ relative cohomology $H^*(X,A;G)$

These behave functorially for "maps of pairs"
$(X, A) \to (Y, B)$, ie $f: X \to Y$ sd $f(A) \subseteq B$
induces $f_*: H_n(X, A) \to H_n(Y, B)$ and $f^*: H^n(Y, B) \to H^n(X, A)$
(In particular, if $X = Y$ and $A = \emptyset$ to get homomorphism $H_n(X) \to H_n(X, B)$)

Excision theorem

If $Z \subset A \subset X$ and the closure of Z is contained in the interior of A then

$$H_n(X \setminus Z, A \setminus Z) \cong H_n(X, A)$$

Proof:
Using barycentric subdivision, any singular simplex in X is equal modulo boundaries to a sum of simplices wholly contained in either A or $X \setminus Z$. $\square$

...

1.2 HOMOLOGICAL ALGEBRA

LONG EXACT SEQUENCES

A sequence of homomorphisms $A \xrightarrow{f} B \xrightarrow{g} C$ is <u>exact</u> at B if $\operatorname{im} f = \ker g$

chain maps, ie $\partial f = f \partial$

<u>Snake lemma</u>

Let $0 \to A_* \xrightarrow{f} B_* \xrightarrow{g} C_* \to 0$ be a short exact sequence of chain complexes. Define a homomorphism $\delta : H_n(C) \to H_{n-1}(A)$ as follows: For a cycle in $c \in C_n$, pick $b \in g^{-1}(c) \subseteq B_n$, and let $a = f^{-1}(\partial b)$, and set $\delta([c]) = [a]$.

Then the sequence

$$\dots \to H_n(A) \xrightarrow{f_*} H_n(B) \xrightarrow{g_*} H_n(C) \xrightarrow{\delta} H_{n-1}(A) \to \dots$$

is exact.

- For a pair (X,A) by definition have SESes
$$0 \to C_\#(A) \to C_\#(X) \to C_\#(X,A) \to 0$$
and
$$0 \to C^\#(X,A;G) \to C^\#(X;G) \to C^\#(A;G) \to 0$$
$\rightsquigarrow$ LESes for relative homology and cohomology

- If $U, V \subseteq X$ are open and $X = U \cup V$ then inclusion $C_\#(U) + C_\#(V) \hookrightarrow C_\#(X)$ induces isomorphism on homology (by same argument as in excision). Therefore the SES
$$0 \to C_\#(U \cap V) \to C_\#(U) \oplus C_\#(V) \to \underset{\substack{\text{\rotatebox{90}{$\subseteq$}}\\ C_\#(X)}}{C_\#(U) + C_\#(V)} \to 0$$
induces <u>Mayer-Victoris sequence</u>
$$\cdots \to H_n(U \cap V) \to H_n(U) \oplus H_n(V) \to H_n(X) \to H_{n-1}(U \cap V) \to \cdots$$
Either can be used to compute
$$H^n(S^n) = \begin{cases} \mathbb{Z} & \text{for } k = 0, n \\ 0 & \text{otherwise} \end{cases}$$
by induction

UNIVERSAL COEFFICIENTS

Let R be a ring and M an R-module (eg $R=\mathbb{Z}$ and M any abelian group)

For any chain complex C_* there are well-defined homomorphisms

$$(1) \quad H_n(C;R) \otimes_R M \to H_n(C;M)$$
$$[c]m \mapsto [cm]$$

$$(2) \quad H^n(C;M) \longrightarrow \mathrm{Hom}_R(H_n(C;R), M)$$
$$[\varphi] \longmapsto ([c] \mapsto \varphi(c))$$

Theorem

a) (1) is injective and (2) is surjective

b) If M is a free R-module or $R=\mathbb{Z}$ and $M=\mathbb{Q}$ then both are isomorphisms.

c) coker (1) and ker (2) can be described in terms of H_{n-1} and Ext and Tor functors.

Corollary

If $H_n(X)$ is finitely-generated then

$$\operatorname{rank} H_n(X) = \operatorname{rank} H^n(X;\mathbb{Z}) = \dim_{\mathbb{Q}} H_n(X;\mathbb{Q}) = \dim_{\mathbb{Q}} H^n(X;\mathbb{Q})$$

Call this the Betti number $b_n(X)$.

So in principle the group isomorphism class of homology and cohomology with any coefficients can be recovered from knowing just $H_*(C)$

but
- non-natural, so eg doesn't tell everything about behaviour of $f_*: H_n(X;M) \to H_n(Y;M)$
- doesn't say much about cup product structure on $H^*(X;R)$

Also often interesting to see relation to SES of coefficient groups $0 \to E \to F \to G \to 0$

$\rightsquigarrow$ LES $H^n(C;E) \to H^n(C;F) \to H^n(C;G) \xrightarrow{\beta} H^{n+1}(C,E)$

where the snake map β is called the <u>Bockstein map</u>

For $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$, the kernel $K \subseteq H^n(C;\mathbb{Z})$ of map to $H^n(C;\mathbb{Q})$ is the same as for map to ~~H^n~~ $\mathrm{Hom}(H_n(C),\mathbb{Z})$, so $K =$ ~~ker~~ $\mathrm{im}\,\beta$.

One can define a pairing $K \times TH_{n-1}(C) \to \mathbb{Q}/\mathbb{Z}$

$(\phi, c) \mapsto T(\phi, c)$

by $T(\phi,c) = \psi(c)$ for any $\psi \in \beta^{-1}(\phi) \subset H^{n-1}(C;\mathbb{Q}/\mathbb{Z})$

<u>Proposition</u>

T is perfect, ie induces $K \cong \mathrm{Hom}(TH_{n-1}(C), \mathbb{Q}/\mathbb{Z})$

If H_* is finitely generated then RHS $\cong TH_{n-1}$, so

$H^n(C;\mathbb{Z}) \cong$ free part of $H_n(C) \oplus$ torsion part of $H_{n-1}(C)$

non-natural!

1.3 PRODUCT STRUCTURE

THE CUP PRODUCT

Let R be a ring. For $\varphi \in C^k(X;R)$ and $\psi \in C^\ell(X;R)$ define $\varphi \cup \psi \in C^{k+\ell}(X;R)$ by

$$(\varphi \cup \psi)\sigma = \varphi(\underbrace{\sigma \circ F[e_0, \dots, e_k]}_{\text{"front } k\text{-face"}})\, \psi(\underbrace{\sigma \circ F[e_k, \dots, e_{k+\ell}]}_{\text{"back } \ell\text{-face"}}) \in R$$

for any singular simplex $\sigma : \Delta_{k+\ell} \to X$.

Basic features

- associative
- Leibniz rule $d(\varphi \cup \psi) = d\varphi \cup \psi + (-1)^k \varphi \cup d\psi$
  $\Rightarrow$ induces well-defined product $H^k(X;R) \times H^\ell(X;R) \to H^{k+\ell}(X;R)$
  $([\varphi], [\psi]) \mapsto [\varphi \cup \psi]$
- functorial: if $f : X \to Y$, then $f^* : H^*(Y;R) \to H^*(X;R)$ has $f^*(\alpha \cup \beta) = f^*\alpha \cup f^*\beta$

This makes $H^*(X;R) = \bigoplus_n H^n(X;R)$ a graded ring.
It is graded commutative, even though the product on cochains is not!

Proposition

$$[\varphi] \cup [\psi] = (-1)^{kl} [\psi] \cup [\varphi]$$

for $[\varphi] \in H^k(X;R)$, $[\psi] \in H^l(X;R)$

Proof:

For $k, l \geq 1$ define another product

$$C^k(X;R) \times C^l(X;R) \to C^{k+l-1}(X;R)$$
$$(\alpha, \beta) \mapsto \alpha \cup_1 \beta$$

by

$$(\alpha \cup_1 \beta)\sigma = \sum_{i=0}^{l-1} \alpha(\sigma \circ F[e_i, \dots, e_{i+k}])\, \beta(\sigma \circ F[e_0, \dots, e_i, e_{i+k}, \dots, e_{k+l-1}])$$

Then

$$\alpha \cup \beta - (-1)^{kl} \beta \cup \alpha = (-1)^{k+l} \left(d\alpha \cup_1 \beta + \alpha \cup_1 d\beta + (-1)^k d(\alpha \cup_1 \beta) \right) \qquad \square$$

So if $d\alpha = d\beta = 0$, then RHS is exact.

While cohomology with rational or real coefficients can also be obtained from a graded commutative cochain algebra, integral cohomology cannot!

$\rightsquigarrow$ Steenrod operations