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Deformations of asymptotically cylindrical G_2-manifolds

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Abstract

We prove that for a 7-dimensional manifold M with cylindrical ends the moduli space of exponentially asymptotically cylindrical torsion-free G_2-structures is a smooth manifold (if non-empty), and study some of its local properties. We also show that the holonomy of the induced metric of an exponentially asymptotically cylindrical G_2-manifold is exactly G_2 if and only if the fundamental group $\pi_1(M)$ is finite and neither M nor any double cover of M is homeomorphic to a cylinder.

1. *Introduction*

The holonomy group G_2 appears as an exceptional case in Berger's classification of the Riemannian holonomy groups [**2**]. A metric with holonomy contained in G_2 can be defined in terms of a parallel non-degenerate differential 3-form, often called a torsion-free G_2-structure. A manifold M has cylindrical ends if a complement of a compact subset of M is identified with $X \times \mathbb{R}^+$, for some compact manifold X called the cross-section of M. An important class of metrics on such manifolds are the exponentially asymptotically cylindrical (EAC) ones, i.e. metrics which are exponentially asymptotic to a product metric on the cylindrical part. An EAC G_2-manifold is a 7-dimensional EAC manifold M with holonomy contained in G_2, and its metric is defined by a torsion-free EAC G_2-structure. Section 2 covers this background in more detail.

On an EAC G_2-manifold M^7 the group of EAC diffeomorphisms isotopic to the identity acts on the space of torsion-free EAC G_2-structures by pull-backs. We define the moduli space of torsion-free EAC G_2-structures to be the resulting quotient $\mathcal{M}_+$. The precise definition involves a normalisation, which can also be interpreted as dividing by the rescaling action of $\mathbb{R}^+$ (see remark 3·1). The main result of the paper is theorem 3·2, which states that on an EAC G_2-manifold M^7 the moduli space $\mathcal{M}_+$ is a smooth manifold. The proof of theorem 3·2 is a generalisation of an argument for the compact case outlined by Hitchin in [**11**]. If X^6 is the cross-section of M then the dimension of the moduli space is given by the formula

$$\dim \mathcal{M}_+ = b^4(M) + \tfrac{1}{2}b^3(X) - b^1(M) - 1.$$

The cross-section X of a manifold M with cylindrical ends can be regarded as the 'boundary at infinity' of M, in the sense that M can be identified with the interior of a compact manifold with boundary X. The asymptotic limit of an EAC torsion-free G_2-structure on M induces a Calabi-Yau structure on X, and the proof of theorem 3·2 requires understanding of the deformations of this structure on the boundary. Theorem 3·3 states that the moduli space $\mathcal{N}$ of Calabi-Yau structures on a compact connected oriented manifold X^6 is a smooth manifold of dimension $b^3(X) + b^2(X) - b^1(X) - 1$. This is a special case of a more general result due to Tian [**26**] and Todorov [**27**]. The argument given here is based on an elementary application of the implicit function theorem, and is helpful for proving theorem 3·2.

There is natural 'boundary map' $B : \mathcal{M}_+ \to \mathcal{N}$ which sends a class of torsion-free G_2-structures on M to the class of Calabi-Yau structures that their asymptotic limit defines on X. Theorem 3·6 states that this map is a submersion onto its image, which is a submanifold of $\mathcal{N}$, and an open subset of a subspace $\mathcal{N}_A$ determined by the topology of the pair (M, X).

The final result of the paper is a topological criterion for when the holonomy of the metric associated to a torsion-free EAC G_2-structure is exactly G_2, rather than a subgroup. Theorem 3·8 states that $Hol(M) = G_2$ for an EAC G_2-manifold M if and only if the fundamental group $\pi_1(M)$ is finite and neither M nor any double cover of M is homeomorphic to a cylinder.

Precise statements for all the main results are given in section 3. Section 4 provides the needed deformation theory for compact Calabi-Yau 3-folds, extending a construction of the moduli space of torsion-free $SL(\mathbb{C}^3)$-structures by Hitchin in [**11**]. In section 5 we review Hodge theory for EAC manifolds, which is an important tool in the proofs of the main results. The main theorem 3·2 is proved in section 6 along with results about the local properties of the moduli space, and theorem 3·8 is proved in section 7.

It is complicated to produce examples of G_2-manifolds with holonomy exactly G_2. The first compact examples were constructed by Joyce in [**12**]. Also, in [**14**] Kovalev constructs non-trivial EAC G_2-manifolds (with holonomy $SU(3)$), and uses them in a gluing construction to produce compact manifolds with holonomy exactly G_2. There are no known examples of EAC manifolds with holonomy exactly G_2, but the author hopes to construct such manifolds in a future paper, adapting the methods of [**12**] to the EAC context.

2. *Background material*

In this section we review definitions and notation for Riemannian holonomy, G_2-structures on 7-manifolds, Calabi-Yau structures on 6-manifolds, and manifolds with cylindrical ends.

2·1. *Holonomy*

We define the holonomy group of a Riemannian manifold. For a fuller discussion of holonomy see e.g. Joyce [**13**, Chapter 2].

Definition 2·1. Let M^n be a manifold with a Riemannian metric g. If $x \in M$ and γ is a closed piecewise C^1 loop in M based at x then the parallel transport around γ (with respect to the Levi-Civita connection of the metric) defines an orthogonal linear map $P_\gamma : T_xM \to T_xM$. The *holonomy group* $Hol(g, x) \subseteq O(T_xM)$ at x is the group generated by $\{P_\gamma : \gamma \text{ is a closed loop based at } x\}$.

If $x, y \in M$ and τ is a path from x to y we can define a group isomorphism $Hol(g, x) \to Hol(g, y)$ by $P_\gamma \mapsto P_\tau \circ P_\gamma \circ P_\tau^{-1}$. Provided that M is connected we can therefore identify $Hol(g, x)$ with a subgroup of $O(n)$, independently of x up to conjugacy, and talk simply of the holonomy group of g.

There is a correspondence between tensors fixed by the holonomy group and parallel tensor fields on the manifold.

PROPOSITION 2·2 ([**13**, Proposition 2.5.2]). *Let (M^n, g) be a Riemannian manifold, $x \in M$ and E a vector bundle on M associated to TM. If s is a parallel section of E then $s(x)$ is fixed by $Hol(g, x)$. Conversely if $s_0 \in E_x$ is fixed by $Hol(g, x)$ then there is a parallel section s of E such that $s(x) = s_0$.*

2·2. *G_2-structures*

An effective approach to G_2-structures is to define them in terms of *stable* 3*-forms*. Here we outline the properties of G_2-structures, and explain their relation to metrics with holonomy G_2. For a more complete explanation see e.g. [**13**, Chapter 10].

Recall that G_2 can be defined as the automorphism group of the normed algebra of octonions. Equivalently, G_2 is the stabiliser in $GL(\mathbb{R}^7)$ of

$$\varphi_0 = dx^{123} + dx^{145} + dx^{167} + dx^{246} - dx^{257} - dx^{347} - dx^{356} \in \Lambda^3(\mathbb{R}^7)^*. \tag{2·1}$$

If V is a dimension 7 real vector space and $\varphi \in \Lambda^3 V^*$ then we call φ *stable* if it is equivalent to φ_0 under some isomorphism $V \cong \mathbb{R}^7$. We denote the set of stable 3-forms by $\Lambda^3_+ V^*$. Since the action of $GL(V)$ on $\Lambda^3 V^*$ has stabiliser G_2 at a stable form φ, and $\dim G_2 = 14$, it follows by dimension-counting that $\Lambda^3_+ V^*$ is open in $\Lambda^3 V^*$. Since $G_2 \subset SO(7)$ each $\varphi \in \Lambda^3_+ V^*$ naturally defines an inner product g_φ and an orientation.

Definition 2·3. Let M^7 be an oriented manifold. A *G_2-structure* on M is a section φ of $\Lambda^3_+ T^*M$ which defines the given orientation on M.

Since a G_2-structure φ on M induces a Riemannian metric g_φ it also defines a Levi-Civita connection ∇_φ, a Hodge star $*_\varphi$ and a codifferential d^*_φ.

Definition 2·4. A G_2-structure φ on an oriented manifold M^7 is *torsion-free* if $\nabla_\varphi \varphi = 0$. A *$G_2$-manifold* is an oriented manifold M^7 equipped with a torsion-free G_2-structure φ and the associated Riemannian metric g_φ.

Gray observed that a G_2-structure is torsion-free if and only if it is closed and coclosed.

THEOREM 2·5 ([**24**, Lemma 11.5]). *A G_2-structure φ on M^7 is torsion-free if and only if $d\varphi = 0$ and $d^*_\varphi \varphi = 0$.*

As an immediate application of proposition 2·2 we have that metrics with holonomy contained in G_2 correspond to torsion-free G_2-structures.

COROLLARY 2·6. *Let M^7 be a manifold with Riemannian metric g. Then $Hol(g)$ is a subgroup of $G_2 \subset O(7)$ if and only if there is a torsion-free G_2-structure φ on M such that $g = g_\varphi$.*

The condition that $Hol(g) \subseteq G_2$ imposes algebraic constraints on the curvature of g. In particular

THEOREM 2·7 ([**24**, Proposition 11.8]). *The metric of a torsion-free G_2-structure φ is Ricci-flat.*

The deformation problem for torsion-free G_2-structures on a compact oriented manifold M^7 was solved by Joyce in [**12**]. Let $\mathcal{X}$ be the space of smooth torsion-free G_2-structures on M, and $\mathcal{D}$ the group of diffeomorphisms of M isotopic to the identity. $\mathcal{D}$ acts on $\mathcal{X}$ by pull-backs, and the *moduli space of torsion-free G_2-structures* on M is the space of orbits

$\mathcal{M} = \mathcal{X}/\mathcal{D}$. Elements of $\mathcal{X}$ are closed 3-forms, so define cohomology classes. This gives a well-defined map to de Rham cohomology $\pi_H : \mathcal{M} \to H^3(M)$. Joyce proved that

THEOREM 2·8 ([**13**, Theorem 10.4.4]). *Let M^7 be a compact G_2-manifold. Then $\mathcal{M}$ is a smooth manifold of dimension $b^3(M)$, and $\pi_H : \mathcal{M} \to H^3(M)$ is a local diffeomorphism.*

The main result of this paper generalises theorem 2·8 to the case when M is an EAC G_2-manifold.

2·3. *Calabi-Yau* 3*-folds*

One common definition of a Calabi-Yau manifold is that it is a Riemannian manifold X^{2n} with holonomy contained in $SU(n)$. The stabiliser in G_2 of a vector in $\mathbb{R}^7$ is isomorphic to $SU(3)$, and Calabi-Yau 3-folds will appear naturally as the cross-sections of EAC G_2-manifolds.

For our purposes it is convenient to define a *Calabi-Yau structure* on a 6-dimensional manifold X in terms of a pair of closed differential forms (Ω, ω). This will make the relation to G_2-structures clear. The role of the 'stable' 3-form Ω is discussed by Hitchin in [**11**]. Let

$$\Omega_0 = dx^{135} - dx^{146} - dx^{236} - dx^{245} \in \Lambda^3(\mathbb{R}^6)^*, \tag{2·2a}$$

$$\omega_0 = dx^{12} + dx^{34} + dx^{56} \in \Lambda^2(\mathbb{R}^6)^*. \tag{2·2b}$$

For an oriented real vector space V of dimension 6 let $\Lambda^3_+ V^*$ be the set of $\Omega \in \Lambda^3 V^*$ such that Ω is equivalent to Ω_0 under some linear isomorphism $V \cong \mathbb{R}^6$. We call such Ω *stable*. If we identify $\mathbb{R}^6$ with $\mathbb{C}^3$ by taking $z^1 = x^1 + ix^2$, $z^2 = x^3 + ix^4$, $z^3 = x^5 + ix^6$ then

$$\Omega_0 = \mathrm{re}\ dz^1 \wedge dz^2 \wedge dz^3,$$
$$\omega_0 = \tfrac{i}{2}(dz^1 \wedge d\bar{z}^1 + dz^2 \wedge d\bar{z}^2 + dz^3 \wedge d\bar{z}^3).$$

The stabiliser of Ω_0 in $GL_+(\mathbb{R}^6)$ is $SL(\mathbb{C}^3)$. Therefore each $\Omega \in \Lambda^3_+ V^*$ defines a complex structure J on V, and there is a unique $\hat{\Omega} \in \Lambda^3_+ V^*$ such that $\Omega + i\hat{\Omega}$ is a $(3,0)$-form. We say that Ω defines an $SL(\mathbb{C}^3)$-structure on V. (Reversing the orientation of V changes the sign of both J and $\hat{\Omega}$.) By dimension-counting $\Lambda^3_+ V^*$ is an open subset of $\Lambda^3 V^*$.

Similarly the stabiliser in $GL(\mathbb{R}^6)$ of the pair (Ω_0, ω_0) is $SU(3)$, so we can define an $SU(3)$-structure on V to be a pair $(\Omega, \omega) \in \Lambda^3 V^* \times \Lambda^2 V^*$ which is equivalent to (Ω_0, ω_0) under some oriented linear isomorphism $V \cong \mathbb{R}^6$. An $SU(3)$-structure naturally defines a complex structure J as above, and also an inner product $g(\cdot,\cdot) = \omega(\cdot, J\cdot)$. With respect to the Hodge star defined by this metric $\hat{\Omega} = *\Omega$.

Definition 2·9. An $SU(3)$*-structure* on an oriented manifold X^6 is a pair of forms $(\Omega, \omega) \in \Omega^3(X) \times \Omega^2(X)$ which defines an $SU(3)$-structure on each tangent space.

(Ω, ω) is said to be a *Calabi-Yau structure* if $\nabla\Omega = 0$, $\nabla\omega = 0$ with respect to the metric induced by (Ω, ω). X equipped with the structure (Ω, ω) and the associated Riemannian metric is called a *Calabi-Yau 3-fold.*

If X is a Calabi-Yau 3-fold in this sense then by proposition 2·2 the holonomy of the induced metric is contained in $SU(3)$, and conversely any metric with holonomy contained in $SU(3)$ is induced by some Calabi-Yau structure. The almost complex structure J defined by a Calabi-Yau structure is integrable, and the metric is Kähler. Moreover $\Omega + i\hat{\Omega}$ is a global holomorphic $(3,0)$-form, and the metric is Ricci-flat.

Calabi-Yau structures on X^6 are equivalent to torsion-free cylindrical G_2-structures on $X \times \mathbb{R}$ in the following sense.

Definition 2·10. Let X^6 a compact oriented manifold, and denote by t the $\mathbb{R}$-coordinate on the cylinder $X \times \mathbb{R}$. A G_2-structure φ on $X \times \mathbb{R}$ is *cylindrical* if it is translation-invariant and the associated metric is a product metric $g_\varphi = g_X + dt^2$, for some metric g_X on X.

Comparing the point-wise models (2·1) and (2·2) it is easy to see that (Ω, ω) is an $SU(3)$-structure on X with metric g_X if and only if the translation-invariant G_2-structure $\varphi = \Omega + dt \wedge \omega$ on $X \times \mathbb{R}$ defines the product metric $g_X + dt^2$. $Hol(g_X + dt^2) \subseteq G_2$ if and only if $Hol(g_X) \subseteq SU(3)$, so

PROPOSITION 2·11. *(Ω, ω) is a Calabi-Yau structure on X^6 if and only if $\Omega + dt \wedge \omega$ is a torsion-free cylindrical G_2-structure on $X \times \mathbb{R}$.*

Remark 2·12. If $\varphi = \Omega + dt \wedge \omega$ is a cylindrical G_2-structure then

$$*_\varphi \varphi = \tfrac{1}{2}\omega^2 - dt \wedge \hat{\Omega}.$$

2·4. *Manifolds with cylindrical ends*

We define manifolds with cylindrical ends and their long exact sequence for cohomology relative to the boundary. We also define exponentially asymptotically cylindrical (EAC) metrics and G_2-structures.

Definition 2·13. A manifold M is said to have *cylindrical ends* if M is written as union of two pieces M_0 and M_∞ with common boundary X, where M_0 is compact, and M_∞ is identified with $X \times \mathbb{R}^+$ by a diffeomorphism (identifying ∂M_∞ with $X \times \{0\}$). X is called the *cross-section* of M.

A *cylindrical coordinate* on M is a smooth function $t : M \to \mathbb{R}$ which is equal to the $\mathbb{R}^+$-coordinate on M_∞ and is negative in the interior of M_0.

The interior of any compact manifold with boundary can be considered as a manifold with cylindrical ends by the collar neighbourhood theorem. Conversely if M has cylindrical ends then we can compactify M by including it in $\overline{M} = M_0 \cup (X \times [0, \infty])$, i.e. by 'adding a copy of X at infinity'. The cohomology of $\overline{M}$ relative to its boundary can be identified with $H^*_{cpt}(M)$, the cohomology of the complex $\Omega^*_{cpt}(M)$ of compactly supported forms. The long exact sequence for relative cohomology of $\overline{M}$ can be written as

$$\cdots \longrightarrow H^{m-1}(X) \xrightarrow{\partial} H^m_{cpt}(M) \xrightarrow{e} H^m(M) \xrightarrow{j^*} H^m(X) \longrightarrow \cdots \tag{2·4}$$

$e : H^m_{cpt}(M) \to H^m(M)$ is induced by the inclusion $\Omega^*_{cpt}(M) \hookrightarrow \Omega^*(M)$. The image of e is the subspace of cohomology classes with compact representatives.

Definition 2·14. Let $H^m_0(M) = \mathrm{im}\,(e : H^m_{cpt}(M) \to H^m(M))$.

Cylindrical ends allow us to define a notion of asymptotic translation-invariance.

Definition 2·15. A tensor field on $X \times \mathbb{R}$ is called *translation-invariant* if it is invariant under the obvious $\mathbb{R}$-action on $X \times \mathbb{R}$.

Definition 2·16. Let M be a manifold with cylindrical ends. Call a smooth function $\rho : M \to \mathbb{R}$ a *cut-off function for the cylinder* if it is 0 on the compact piece M_0 and 1 outside a compact subset of M.

If s_∞ is a section of a vector bundle associated to the tangent bundle on the cylinder $X \times \mathbb{R}$ and ρ is a cut-off function for the cylinder on M then ρs_∞ can be considered to be a section of the corresponding vector bundle over M.

Definition 2·17. Let M be a manifold with cylindrical ends and cross-section X. Pick an arbitrary product metric $g_X + dt^2$ on $X \times \mathbb{R}$, and a cut-off function ρ for the cylinder. A section s of a vector bundle associated to TM is said to be *decaying* if $\|\nabla^k s\| \to 0$ uniformly on X as $t \to \infty$ for all $k \geqslant 0$. s is said to be *asymptotic* to a translation-invariant section s_∞ of the corresponding bundle on $X \times \mathbb{R}$ if $s - \rho s_\infty$ decays.

Similarly s is said to be *exponentially decaying* with rate $\delta > 0$ if $e^{\delta t}\|\nabla^k s\|$ is bounded on M_∞ for all $k \geqslant 0$, and *exponentially asymptotic* to a translation-invariant section s_∞ if $s - \rho s_\infty$ decays exponentially. Denote by $C^\infty_\delta(E)$ the space of sections of E which decay exponentially with rate δ.

The natural choice of topology on $C^\infty_\delta(E)$ is to require the linear isomorphism $C^\infty_\delta(E) \to C^\infty(E)$, $s \mapsto e^{\delta t} s$ to be a homeomorphism.

Definition 2·18. A metric g on a manifold M with cylindrical ends is said to be *EAC* if it is exponentially asymptotic to a product metric $g_X + dt^2$ on $X \times \mathbb{R}^+$. An EAC manifold is a manifold with cylindrical ends equipped with an EAC metric.

Definition 2·19. Let M be a manifold with cylindrical ends and cross-section X. A diffeomorphism Ψ_∞ of the cylinder $X \times \mathbb{R}$ is said to be cylindrical if it is of the form

$$\Psi_\infty(x, t) = (\Xi(x), t + h),$$

where Ξ is a diffeomorphism of X and $h \in \mathbb{R}$. A diffeomorphism Ψ of M is said to be *EAC* with rate $\delta > 0$ if there is a cylindrical diffeomorphism Ψ_∞ of $X \times \mathbb{R}$, a real $T > 0$ and an exponentially decaying vector field V on M such that on $X \times (T, \infty)$

$$\Psi = (\exp V) \circ \Psi_\infty.$$

Definition 2·20. Let M^7 be a connected oriented manifold with cylindrical ends and cross-section X^6. A G_2-structure φ on M is said to be EAC if it is exponentially asymptotic to a cylindrical G_2-structure on $X \times \mathbb{R}$ (cf. definition 2·10). M equipped with a torsion-free EAC G_2-structure and the associated metric is called an *EAC G_2-manifold*.

If φ is a torsion-free EAC G_2-structure then note that the associated metric g_φ is EAC, and that by proposition 2·11 the asymptotic limit defines a Calabi-Yau structure on the cross-section X.

The next theorem implies that an EAC G_2-manifold is not interesting unless it has a single end. The theorem can be proved using the Cheeger-Gromoll splitting theorem, and is also proved using more elementary methods by Salur [**25**].

THEOREM 2·21. *Let M be an orientable connected asymptotically cylindrical Ricci-flat manifold. Then either M has a single end, i.e. its cross-section X is connected, or M is a cylinder $X \times \mathbb{R}$ with a product metric.*

3. *Statement of results*

Let M^7 be a connected oriented manifold with cylindrical ends and cross-section X^6. For $\delta > 0$ let $\mathcal{X}_\delta$ be the space of torsion-free exponentially asymptotically cylindrical (EAC)

G_2-structures with rate δ on M (see definition 2·20). $\mathcal{X}_\delta$ has the topology of a subspace of the space of exponentially asymptotically translation-invariant 3-forms.

Let $\mathcal{X}_+ = \bigcup_{\delta>0} \mathcal{X}_\delta$. If $\delta_1 > \delta_2 > 0$ then the inclusion $\mathcal{X}_{\delta_1} \hookrightarrow \mathcal{X}_{\delta_2}$ is continuous, so we can give $\mathcal{X}_+$ the direct limit topology, i.e. $U \subseteq \mathcal{X}_+$ is open if and only if $U \cap \mathcal{X}_\delta$ is open in $\mathcal{X}_\delta$ for all $\delta > 0$. Similarly let $\mathcal{D}_+$ be the group of EAC diffeomorphisms of M with any positive rate (in the sense of definition 2·19) that are isotopic to the identity. $\mathcal{D}_+$ acts on $\mathcal{X}_+$ by pull-backs, and the *moduli space of torsion-free EAC G_2-structures* on M is the quotient $\mathcal{M}_+ = \mathcal{X}_+/\mathcal{D}_+$.

Remark 3·1. The definition of an EAC G_2-structure φ that is used involves a normalisation – if t is the cylindrical coordinate on M then $\|\frac{\partial}{\partial t}\| \to 1$ uniformly on X as $t \to \infty$ (in the metric defined by φ), so a scalar multiple $\lambda\varphi$ is not an EAC G_2-structure. This normalisation is the most convenient to work with, but a different choice of normalisation (e.g. that $Vol(X) = 1$ in the induced metric on the boundary) would of course give the same results. Another interpretation is that $\mathbb{R}^+$ acts on the moduli space of unnormalised EAC G_2-structures by rescaling, and that $\mathcal{M}_+$ is the resulting quotient.

In the compact case theorem 2·8 gives a description of the moduli space of torsion-free G_2-structures using the natural projection map to the de Rham cohomology. In the EAC case, however, it is not enough to consider

$$\pi_H : \mathcal{M}_+ \to H^3(M), \quad \varphi\mathcal{D}_+ \mapsto [\varphi].$$

We also need to consider the boundary values of φ to get an adequate description. Any $\varphi \in \mathcal{X}_+$ is asymptotic to some $\Omega + dt \wedge \omega$ with $(\Omega, \omega) \in \Omega^3(X) \times \Omega^2(X)$. Let

$$\pi_{\mathcal{M}} : \mathcal{M}_+ \to H^3(M) \times H^2(X), \quad \varphi\mathcal{D}_+ \mapsto ([\varphi], [\omega]). \tag{3·1}$$

The main theorem we shall prove is

THEOREM 3·2. *$\mathcal{M}_+$ is a smooth manifold, and $\pi_{\mathcal{M}} : \mathcal{M}_+ \to H^3(M) \times H^2(X)$ is an immersion.*

In order to prove theorem 3·2 we will need to understand the deformations of the 'boundary' of an EAC G_2-manifold, i.e. the deformations of torsion-free cylindrical G_2-structures. By proposition 2·11 this corresponds to deformations of Calabi-Yau structures. Let $\mathcal{Y}$ be the set of Calabi-Yau structures (Ω, ω) on X, and $\mathcal{D}_X$ the group of diffeomorphisms of X isotopic to the identity. The *moduli space of Calabi-Yau structures on X* is $\mathcal{N} = \mathcal{Y}/\mathcal{D}_X$, and there is a natural projection to the de Rham cohomology

$$\pi_{\mathcal{N}} : \mathcal{N} \to H^3(X) \times H^2(X), \quad (\Omega, \omega)\mathcal{D}_X \mapsto ([\Omega], [\omega]). \tag{3·2}$$

THEOREM 3·3. *Let X^6 be a compact connected Calabi-Yau 3-fold. The moduli space $\mathcal{N}$ of Calabi-Yau structures on X is a manifold,*

$$\dim \mathcal{N} = b^3(X) + b^2(X) - b^1(X) - 1, \tag{3·3}$$

and $\pi_{\mathcal{N}} : \mathcal{N} \to H^3(X) \times H^2(X)$ is an immersion.

Remark 3·4. The definition of a Calabi-Yau 3-fold X^6 used here allows $Hol(X)$ to be a proper subgroup of $SU(3)$. If $Hol(X)$ is exactly $SU(3)$ (so X is irreducible as a Riemannian manifold) then $b^1(X) = 0$, and the formula for the dimension simplifies to $b^3(X)+b^2(X)-1$.

If X is an irreducible Calabi-Yau manifold then for any Calabi-Yau structure (Ω, ω) on X and $\lambda \in \mathbb{R}^+$ we can define a torsion-free product G_2-structure $\varphi = \Omega + \lambda d\theta \wedge \omega$ on $X \times S^1$. The metric defined by φ is the product of the Calabi-Yau metric on X and the metric on S^1 with radius λ (cf. proposition 2·11). The moduli space of such torsion-free product G_2-structures has dimension

$$\dim \mathcal{N} + 1 = b^3(X) + b^2(X) = b^3(X \times S^1),$$

which equals the dimension of the moduli space of torsion-free G_2-structures on $X \times S^1$ by theorem 2·8.

Theorem 3·3 is actually a special case of a known result. The deformation theory for complex manifolds was developed by Kodaira and Spencer. Tian [**26**] and Todorov [**27**] showed independently that on a compact connected Calabi-Yau manifold X^{2n} with holonomy exactly $SU(n)$ the deformations of the complex structure are 'unobstructed'. This implies that the moduli space of complex structures on X is a manifold of dimension $2h^{1,n-1}(X)$ ($h^{p,q}(X)$ denote the Hodge numbers of X, i.e. the dimension of the Dolbeault cohomology $H^{p,q}(X)$). It is easy to deduce from this and Yau's solution of the Calabi conjecture [**28**] that the moduli space of Calabi-Yau structures on a complex n-fold (in the sense of an integrable complex structure with a Kähler metric and a holomorphic $(n, 0)$-form of norm 1) is a manifold of dimension

$$2h^{1,n-1}(X) + h^{1,1}(X) + h^{n,0}(X) \tag{3·4}$$

(cf. [**13**, Section 6.8]). By [**13**, Proposition 6.2.6] $h^{m,0}(X) = 0$ for $0 < m < n$ and $h^{n,0}(X) = 1$ when X^{2n} is compact connected and $Hol(X) = SU(n)$, so if $n = 3$ then $b^3(X) = 2h^{1,2}(X) + 2$ and $b^2(X) = h^{1,1}(X)$. Thus the expression (3·4) for the dimension of the moduli space can be rewritten as $b^3(X) + b^2(X) - 1$ when $n = 3$, which agrees with the formula stated in theorem 3·3.

We will give a different proof of theorem 3·3 in section 4. We produce pre-moduli spaces by an elementary application of the implicit function theorem, extending arguments of Hitchin in [**11**]. These pre-moduli spaces are also used in the proof of theorem 3·2.

In subsection 6·8 we look at some local properties of $\mathcal{M}_+$. Its dimension is given by

PROPOSITION 3·5. $\dim \mathcal{M}_+ = b^4(M) + \frac{1}{2}b^3(X) - b^1(M) - 1.$

We also study the properties of the boundary map $B : \mathcal{M}_+ \to \mathcal{N}$, which sends a G_2-structure on M to the Calabi-Yau structure on X defined by its asymptotic limit. Denote by A^m the image of the pull-back map $j^* : H^m(M) \to H^m(X)$ in the long exact sequence for relative cohomology (2·4). If φ is asymptotic to $\Omega + dt \wedge \omega$ then

$$[\Omega] = j^*([\varphi]) \in A^3,$$
$$\tfrac{1}{2}[\omega^2] = j^*([*_\varphi \varphi]) \in A^4,$$

so the image of $B : \mathcal{M}_+ \to \mathcal{N}$ is contained in

$$\mathcal{N}_A = \{(\Omega, \omega)\mathcal{D}_X \in \mathcal{N} : [\Omega] \in A^3, [\omega^2] \in A^4\}. \tag{3·5}$$

It turns out that – locally at least – these necessary conditions for a point to be in the image are also sufficient.

THEOREM 3·6. *The image of*

$$B : \mathcal{M}_+ \to \mathcal{N}_A \tag{3·6}$$

is open in $\mathcal{N}_A$ and a submanifold of $\mathcal{N}$. The map is a submersion onto its image.

Since the methods used are entirely local they do not tell us anything about the global properties of $\mathcal{M}_+$ or the image of (3·6).

We will show that the fibres of the submersion (3·6) are locally diffeomorphic to the compactly supported subspace $H^3_0(M) \subseteq H^3(M)$. The fibre over (Ω, ω) corresponds to the moduli space of torsion-free G_2-structures asymptotic to $\Omega + dt \wedge \omega$. Thus

COROLLARY 3·7. *The moduli space of torsion-free G_2-structures on M exponentially asymptotic to a fixed cylindrical G_2-structure on $X \times \mathbb{R}$ is a manifold locally diffeomorphic to $H^3_0(M)$.*

In the proof of proposition 3·5 we find that $\dim H^3_0(M) = b^3(M) - \frac{1}{2}b^3(X)$, so

$$\dim \mathcal{N}_A = b^4(M) - b^3(M) + b^3(X) - b^1(M) - 1.$$

Theorem 2·21 implies that if M is a G_2-manifold then either M is a cylinder $X \times \mathbb{R}$ (with a product metric) or M has a single end. If M is a cylinder $X \times \mathbb{R}$ then the only possible torsion-free G_2-structure asymptotic to a given cylindrical G_2-structure φ_∞ is φ_∞ itself, so the moduli space of asymptotically cylindrical torsion-free G_2-structures on M is equivalent to the moduli space of Calabi-Yau structures on X (we can compute that $H^m_0(X \times \mathbb{R}) = 0$ for all m, so this agrees with corollary 3·7). The moduli space will therefore only be interesting when M has a single end. We will not need to assume this in the proof of theorem 3·2 though.

Finally, in section 7 we find a topological condition for when the holonomy group of an EAC G_2-manifold is exactly G_2, as opposed to a proper subgroup. For compact G_2-manifolds it is well-known that the holonomy is exactly G_2 if and only if the fundamental group is finite. In the EAC case we need to take into account that a product cylinder $X^6 \times \mathbb{R}$ may have finite fundamental group, but cannot have holonomy G_2. The correct statement is

THEOREM 3·8. *Let M^7 be an EAC G_2-manifold. Then $Hol(M) = G_2$ if and only if the fundamental group $\pi_1(M)$ is finite and neither M nor any double cover of M is homeomorphic to a cylinder.*

4. *Deformations of compact Calabi-Yau* 3*-folds*

In [**11**] Hitchin uses elementary methods to construct the moduli space of torsion-free $SL(\mathbb{C}^3)$-structures on a compact manifold X^6. In a sense this provides 'half' the deformation theory for compact Calabi-Yau 3-folds. In this section we show how to extend Hitchin's arguments to construct pre-moduli spaces of Calabi-Yau structures, which we require for the proof of the main theorem 3·2. This also gives an elementary proof of theorem 3·3, stating that the moduli space of Calabi-Yau structures on X is a manifold.

4·1. *Harmonic forms and holonomy*

We first review how a holonomy constraint on a Riemannian manifold gives rise to decompositions of harmonic forms, similar to the Hodge decomposition on a Kähler manifold. This is explained in more detail in [**13**, Section 3.5].

Let H be a closed subgroup of $SO(n)$, and M^n an oriented Riemannian manifold with $Hol(M) \subseteq H$, equipped with a corresponding H-structure. Suppose that $\Lambda^m\mathbb{R}^n$ decomposes as an orthogonal direct sum of subrepresentations $\Lambda^m\mathbb{R}^n = \bigoplus \Lambda^m_d\mathbb{R}^n$ under the action of H (we will indicate the rank of the subrepresentations by the index d). Then there is a corresponding H-invariant decomposition of the exterior product bundle

$$\Lambda^m T^*M = \bigoplus \Lambda^m_d T^*M. \tag{4·1}$$

We write $\Omega^m_d(M)$ for the space sections of $\Lambda^m_d T^*M$, the 'forms of type d'. We define projections $\pi_d : \Lambda^m T^*M \to \Lambda^m_d T^*M$. These induce maps $\pi_d : \Omega^m(M) \to \Omega^m_d(M)$ on the sections, and allow us to decompose forms into type components. As observed by Chern in [**6**], the condition that $Hol(M) \subseteq H$ ensures that the Hodge Laplacian respects these type decompositions.

PROPOSITION 4·1 ([**13**, Theorem 3.5.3]). *Let M^n be a Riemannian manifold with $Hol(M) \subseteq H$. If $\Lambda^m_d T^*M$ is an H-invariant subbundle of $\Lambda^m T^*M$ then $\triangle$ commutes with π_d on $\Omega^m(M)$, and maps $\Omega^m_d(M)$ to itself. Moreover, if $\Lambda^k_e T^*M$ is an H-invariant subbundle of $\Lambda^k T^*M$ and $\phi : \Lambda^m_d T^*M \to \Lambda^k_e T^*M$ is H-equivariant then the diagram below commutes.*

$$\begin{tikzcd}
\Omega^m_d(M) \arrow[r, "\phi"] \arrow[d, "\triangle"'] & \Omega^k_e(M) \arrow[d, "\triangle"] \\
\Omega^m_d(M) \arrow[r, "\phi"] & \Omega^k_e(M)
\end{tikzcd}$$

It follows that given an H-invariant decomposition (4·1) of $\Lambda^m T^*M$ into subbundles there is a corresponding decomposition of the harmonic forms

$$\mathcal{H}^m = \bigoplus \mathcal{H}^m_d.$$

If M is compact then by Hodge theory the natural map $\mathcal{H}^m \to H^m(M)$ is an isomorphism. If we let $H^m_d(M)$ be the image of $\mathcal{H}^m_d$ then we obtain a decomposition of the de Rham cohomology

$$H^m(M) = \bigoplus H^m_d(M). \tag{4·2}$$

In this section we consider a Calabi-Yau 3-fold X^6. The standard representation of $SU(3)$ on $\mathbb{R}^6$ is irreducible, and $\Lambda^m\mathbb{R}^6$ decomposes as

$$\begin{aligned}
\Lambda^2\mathbb{R}^6 &= \Lambda^2_1\mathbb{R}^6 \oplus \Lambda^2_6\mathbb{R}^6 \oplus \Lambda^2_8\mathbb{R}^6,\\
\Lambda^3\mathbb{R}^6 &= \Lambda^3_{1\oplus1}\mathbb{R}^6 \oplus \Lambda^3_6\mathbb{R}^6 \oplus \Lambda^3_{12}\mathbb{R}^6,\\
\Lambda^4\mathbb{R}^6 &= \Lambda^4_1\mathbb{R}^6 \oplus \Lambda^4_6\mathbb{R}^6 \oplus \Lambda^4_8\mathbb{R}^6.
\end{aligned}$$

Each of the subrepresentations $\Lambda^m_d\mathbb{R}^6$ is irreducible, but $\Lambda^3_{1\oplus1}\mathbb{R}^6$ is trivial of rank 2. The corresponding decompositions of the exterior cotangent bundles of X are related to the Hodge decomposition, e.g. $\Lambda^2_1 T^*X \oplus \Lambda^2_8 T^*X$ is the bundle of real (1, 1)-forms, while $\Lambda^2_6 T^*X$ consists of the real and imaginary parts of forms of type (2, 0).

4·2. *Pre-moduli space of Calabi-Yau structures*

Let X^6 a compact connected oriented manifold. Recall that in subsection 2·3 we defined a Calabi-Yau structure on X in terms of a pair of forms $(\Omega, \omega) \in \Omega^3(X) \times \Omega^2(X)$. As defined in section 3 the *moduli space of Calabi-Yau structures* on X is $\mathcal{N} = \mathcal{Y}/\mathcal{D}_X$, where $\mathcal{Y}$ is the set of Calabi-Yau structures (Ω, ω) on X^6, and $\mathcal{D}_X$ is the group of diffeomorphisms of

X isotopic to the identity. To prove theorem 3·3 we find pre-moduli spaces of Calabi-Yau structures, i.e. manifolds $\mathcal{Q} \subseteq \mathcal{Y}$ that are homeomorphic to open sets in $\mathcal{N}$, and can therefore be used as charts.

On 6-dimensional manifolds we have the following convenient characterisation of Calabi-Yau structures (cf. Hitchin [**10**, Section 2])

LEMMA 4·2. *Let X^6 an oriented manifold, and $(\Omega, \omega) \in \Omega^3(X) \times \Omega^2(X)$. Suppose that Ω is stable, so that it defines an almost complex structure J and a 3-form $\hat{\Omega}$. Then the following conditions are sufficient to ensure that (Ω, ω) is a Calabi-Yau structure:*

(i) $\frac{1}{4}\Omega \wedge \hat{\Omega} = \frac{1}{6}\omega^3$,

(ii) $\Omega \wedge \omega = 0$,

(iii) $g(\cdot, \cdot) = \omega(\cdot, J\cdot)$ *is positive-definite,*

(iv) $d\Omega = d\hat{\Omega} = 0$,

(v) $d\omega = 0$.

Recall that $\hat{\Omega}$ is the unique 3-form such that $\Omega + i\hat{\Omega}$ has type $(3, 0)$ with respect to the complex structure defined by Ω. $\hat{\Omega}$ depends smoothly on the stable form Ω. The derivative of $\Omega \mapsto \Omega \wedge \hat{\Omega}$ must be proportional to $\cdot \wedge \hat{\Omega}$ since it is $SU(3)$-equivariant. The proportionality constant is 2 as $\hat{\Omega}$ is homogeneous of degree 1 in Ω (cf. [**11**, page 10]).

Now consider a fixed Calabi-Yau structure (Ω, ω). Any tangent (σ, τ) to a path of $SU(3)$-structures through (Ω, ω) must satisfy the linearisation of the point-wise algebraic conditions (i) and (ii) in lemma 4·2, i.e.

$$L_1(\sigma, \tau) = \sigma \wedge \hat{\Omega} - \tau \wedge \omega^2 = 0, \qquad (4\cdot3a)$$

$$L_2(\sigma, \tau) = \sigma \wedge \omega + \Omega \wedge \tau = 0. \qquad (4\cdot3b)$$

Definition 4·3. Let

$$\mathcal{H}_{SU} = \{(\sigma, \tau) \in \mathcal{H}^3 \times \mathcal{H}^2 : L_1(\sigma, \tau) = L_2(\sigma, \tau) = 0\}.$$

The natural map $\pi_{\mathcal{N}} : \mathcal{H}_{SU} \to H^3(X) \times H^2(X)$, $(\sigma, \tau) \mapsto ([\sigma], [\tau])$ is injective by Hodge theory for compact manifolds. Proposition 4·1 implies that $L_1 : \mathcal{H}^3 \times \mathcal{H}^2 \to \mathcal{H}^6$ and $L_2 : \mathcal{H}^3 \times \mathcal{H}^2 \to \mathcal{H}^5$ are surjective, so

$$\dim \mathcal{H}_{SU} = b^3(X) + b^2(X) - b^1(X) - 1. \qquad (4\cdot4)$$

Theorem 3·3 will follow from the existence of a pre-moduli space near each (Ω, ω) with tangent space $\mathcal{H}_{SU}$.

PROPOSITION 4·4. *For any $(\Omega, \omega) \in \mathcal{Y}$ there is a manifold $\mathcal{Q} \subseteq \mathcal{Y}$ near (Ω, ω) such that the natural map $\mathcal{Q} \to \mathcal{N}$ is a homeomorphism onto an open subset. The tangent space of $\mathcal{Q}$ at (Ω, ω) is $\mathcal{H}_{SU}$.*

We find $\mathcal{Q}$ with the desired properties using a slice construction. Pick some $k \geqslant 1$, $\alpha \in (0, 1)$ and let $\mathcal{Z}^3_k \times \mathcal{Z}^2_k$ be the space of pairs of closed Hölder $C^{k,\alpha}$ 3- and 2-forms. $\mathcal{Y} \hookrightarrow \mathcal{Z}^3_k \times \mathcal{Z}^2_k$ continuously. We find a direct complement K in $\mathcal{Z}^3_k \times \mathcal{Z}^2_k$ to the tangent space of the $\mathcal{D}_X$-orbit at (Ω, ω), and use a small neighbourhood $\mathcal{S}$ of (Ω, ω) in the affine space $(\Omega, \omega) + K$ as a *slice* for the $\mathcal{D}_X$-action.

Let $\mathcal{Q} \subseteq \mathcal{S}$ be the subspace of elements which define Calabi-Yau structures. Subsection 4·3 summarises Hitchin's construction of the moduli space of torsion-free $SL(\mathbb{C}^3)$-structures (stable forms Ω satisfying condition (iv) in lemma 4·2). In subsection 4·4 we extend those arguments in order to prove that $\mathcal{Q} \subseteq \mathcal{S}$ is a submanifold by an application of the implicit function theorem.

By elliptic regularity the elements of $\mathcal{Q}$ are smooth, and slice arguments show that $\mathcal{Q}$ is homeomorphic to a neighbourhood of $(\Omega, \omega)\mathcal{D}_X$ in $\mathcal{N}$. Such arguments will be explained in detail in the more complicated EAC case (cf. propositions 6·21 and 6·23). Thus $\mathcal{Q}$ satisfies the conditions of proposition 4·4.

We will use the deformation theory of Calabi-Yau structures in the proof of the main theorem 3·2 on the moduli space of torsion-free EAC G_2-structures. We will also require the following property of the pre-moduli spaces of Calabi-Yau structures.

PROPOSITION 4·5. *Let $\mathcal{Q}$ be the pre-moduli space of Calabi-Yau structures near* (Ω, ω). *If $\mathcal{Q}$ is taken sufficiently small then all elements of $\mathcal{Q}$ have the same stabiliser in $\mathcal{D}_X$.*

Proof. Let $\mathcal{I} \subseteq \mathcal{D}_X$ be the stabiliser of (Ω, ω). $\mathcal{I}$ is contained in the isometry group of a Riemannian metric, so it is a compact Lie group. By shrinking $\mathcal{Q}$ we may assume that it is mapped to itself by $\mathcal{I}$. Since $\pi_{\mathcal{N}} : \mathcal{Q} \to H^3(X) \times H^2(X)$ is injective and $\mathcal{D}_X$-invariant $\mathcal{I}$ acts trivially on $\mathcal{Q}$.

Conversely, it follows from [**7**, Theorem 7.1(2)] that if $\phi \in \mathcal{D}_X$ fixes an element of $\mathcal{Q}$ sufficiently close to (Ω, ω) then $\phi \in \mathcal{I}$. □

4·3. *Torsion-free $SL(\mathbb{C}^3)$-structures*

Recall from page 314 that an $SL(\mathbb{C}^3)$-structure on an oriented dimension 6 vector space V is defined by a stable 3-form $\Omega \in \Lambda^3_+ V^*$.

Definition 4·6. An $SL(\mathbb{C}^3)$*-structure* on an oriented manifold X^6 is a section Ω of $\Lambda^3_+ T^*X$. Ω is *torsion-free* if $d\Omega = d\hat{\Omega} = 0$.

If Ω is torsion-free then so is the almost complex structure J it defines, and $\Omega + i\hat{\Omega}$ is a global holomorphic $(3, 0)$-form. A torsion-free $SL(\mathbb{C}^3)$-structure is therefore equivalent to a complex structure with trivial canonical bundle, together with a choice of trivialisation.

The next two propositions are a summary of Sections 6.1 and 6.2 in [**11**]. Let X^6 be a compact oriented manifold. Fix a torsion-free $SL(\mathbb{C}^3)$-structure Ω on X, and pick an arbitrary Riemannian metric that is Hermitian with respect to the complex structure defined by Ω. Take $k \geqslant 1$, $\alpha \in (0, 1)$, and let $\mathcal{Z}^3_k$ be the space of closed $C^{k,\alpha}$ 3-forms. Abbreviate $\Lambda^m T^*X$ to Λ^m. The complex structure J defines a vector bundle splitting

$$\Lambda^2 = \Lambda^2_6 \oplus \Lambda^2_9,$$

where Λ^2_9 denotes the bundle of real $(1, 1)$-forms, and Λ^2_6 has type $(2, 0) + (0, 2)$.

PROPOSITION 4·7. *There is an L^2-orthogonal direct sum decomposition*

$$\mathcal{Z}^3_k = \mathcal{H}^3 \oplus W_1 \oplus dC^{k+1,\alpha}\big(\Lambda^2_6\big),$$

and the projections are bounded in the Hölder $C^{k,\alpha}$-norm.

Let $P_1 : C^{k,\alpha}(\Lambda^3) \to W_1$ be the L^2-orthogonal projection. If β is sufficiently close to Ω then β is stable, and $\hat{\beta}$ is well-defined. On a neighbourhood of Ω in $\mathcal{Z}^3_k$ we define

$$F_1(\beta) = P_1(*\hat{\beta}).$$

PROPOSITION 4·8. *The derivative* $(DF_1)_\Omega : \mathcal{Z}^3_k \to W_1$ *is* 0 *on* $\mathcal{H}^3 \oplus dC^{k+1,\alpha}(\Lambda^2_6)$ *and bijective on* W_1. *Furthermore for* $\beta \in \mathcal{Z}^3_k$ *sufficiently close to* Ω

$$F_1(\beta) = 0 \Leftrightarrow d\hat{\beta} = 0.$$

In [**11**] the content of the above two propositions is used with a slice argument to construct a moduli space of torsion-free $SL(\mathbb{C}^3)$-structures. In the next subsection we extend the argument to prove proposition 4·4.

4·4. *Proof of proposition* 4·4

We now explain how to define a slice for the $\mathcal{D}_X$-action at $(\Omega, \omega) \in \mathcal{Y}$. We find a function with surjective derivative whose zero set in the slice is precisely the subspace of Calabi-Yau structures $\mathcal{Q}$. Together with elliptic regularity and slice arguments this proves proposition 4·4, and hence theorem 3·3.

Abbreviate $\Lambda^m T^* X$ to Λ^m. As described in subsection 4·1 the Calabi-Yau structure (Ω, ω) induces decompositions $\Lambda^2 = \Lambda^2_1 \oplus \Lambda^2_6 \oplus \Lambda^2_8$ etc.

The tangent space to the $\mathcal{D}_X$-orbit at (Ω, ω) in $\mathcal{Z}^3_k \times \mathcal{Z}^2_k$ is

$$T = \{(d(V \lrcorner \Omega), d(V \lrcorner \omega)) : V \in C^{k+1,\alpha}(TX)\}.$$

The sections of Λ^2_6 are precisely $V \lrcorner \Omega$ for vector fields V, so by proposition 4·7 we may take $K = (\mathcal{H}^3 \oplus W_1) \times \mathcal{Z}^2_k$ as a complement of T in $\mathcal{Z}^3_k \times \mathcal{Z}^2_k$. It is not clear *a priori* that $K \cap T = 0$, but it will turn out to be so (corollary 4·16). We use a small neighbourhood $\mathcal{S}$ of (Ω, ω) in the affine space $(\Omega, \omega) + K$ as a slice for the $\mathcal{D}_X$-action.

The pre-moduli space $\mathcal{Q} \subseteq \mathcal{S}$ is the subspace of elements which define Calabi-Yau structures. By lemma 4·2 $\mathcal{Q}$ is the zero set of

$$\mathcal{S} \to C^{k-1,\alpha}(\Lambda^4) \times C^{k,\alpha}(\Lambda^5) \times C^{k,\alpha}(\Lambda^6), \quad (\beta, \gamma) \mapsto \left(d\hat{\beta}, \beta \wedge \gamma, \tfrac{1}{4}\beta \wedge \hat{\beta} - \tfrac{1}{6}\gamma^3\right),$$

but this function does not have surjective derivative. Part of the work of obtaining a more appropriate function is already done – we can replace the first component by F_1 defined in (4·3). We need to find a more suitable second component.

(Ω, ω) defines a complex structure on X, and in particular gives us the conjugate differential $d^c = i(\bar{\partial} - \partial)$. Note that $dd^c = -d^c d = 2i\partial\bar{\partial}$. Since X is a Kähler manifold the dd^c-lemma holds.

THEOREM 4·9. *Any real exact form of type* $(1, 1)$ *on a compact Kähler manifold* X *is* dd^c*-exact.*

PROPOSITION 4·10. *There is an* L^2*-orthogonal direct sum decomposition*

$$\mathcal{Z}^5_k = \mathcal{H}^5 \oplus dC^{k+1,\alpha}\left(\Lambda^4_6\right) \oplus W_2, \tag{4·5}$$

where $W_2 = \{d\eta \in dC^{k+1,\alpha}(\Lambda^4) : \pi_6 d^* d\eta = 0\}$. *The projection*

$$P_2 : C^{k,\alpha}(\Lambda^5) \to \mathcal{H}^5 \oplus dC^{k+1,\alpha}\left(\Lambda^4_6\right) \tag{4·6}$$

is bounded in Hölder $C^{k,\alpha}$*-norm.*

Proof. The operator $\pi_6 d^* \oplus d \oplus d^c : \Gamma(\Lambda^5) \to \Gamma(\Lambda^4_6 \oplus \Lambda^6 \oplus \Lambda^6)$ is overdetermined elliptic, and its formal adjoint is $d + d^* + d^{c*}$. Hence the operator $d\pi_6 d^* + d^* d + d^{c*} d^c$ is elliptic and formally self-adjoint on sections of Λ^5, so

$$\begin{aligned} C^{k,\alpha}(\Lambda^5) &= (d\pi_6 d^* + d^*d + d^{c*}d^c)C^{k+2,\alpha}(\Lambda^5) \oplus \ker(d\pi_6 d^* + d^*d + d^{c*}d^c) \\ &\subseteq dC^{k+1,\alpha}\big(\Lambda^4_6\big) + d^*C^{k+1,\alpha}(\Lambda^6) + d^{c*}C^{k+1,\alpha}(\Lambda^6) + \ker \pi_6 d^* \\ &\subseteq dC^{k+1,\alpha}\big(\Lambda^4_6\big) + \ker \pi_6 d^*, \end{aligned}$$

and the last sum is clearly direct. $\ker \pi_6 d^*$ contains $\mathcal{H}^5$ and $d^*C^{k+1,\alpha}(\Lambda^6)$, so splits as $\mathcal{H}^5 \oplus W_2 \oplus d^*C^{k+1,\alpha}(\Lambda^6)$. □

d^c gives a convenient characterisation of the splitting (4·5).

PROPOSITION 4·11. $\Omega \wedge \mathcal{Z}^2_k = \mathcal{H}^5 \oplus dC^{k+1,\alpha}\big(\Lambda^4_6\big) = \mathcal{Z}^5_k \bigcap \ker d^c$.

Proof. $\Omega \wedge \mathcal{H}^2 = \mathcal{H}^5$ by proposition 4·1, while

$$\Omega \wedge dC^{k+1,\alpha}(\Lambda^1) = d(\Omega \wedge C^{k+1,\alpha}(\Lambda^1)) = dC^{k+1,\alpha}\big(\Lambda^4_6\big).$$

A real 4-form has type (2, 2) if and only if its Λ^4_6 part vanishes. For $d\eta \in dC^{k+1,\alpha}(\Lambda^4)$ applying the dd^c-lemma therefore gives

$$\pi_6 d^*d\eta = 0 \Leftrightarrow d^*d\eta \in d^*d^{c*}C^{k+1,\alpha}(\Lambda^6) \Leftrightarrow d\eta \in P_E d^{c*}C^{k+1,\alpha}(\Lambda^6),$$

where $P_E : C^{k,\alpha}(\Lambda^5) \to dC^{k+1,\alpha}(\Lambda^4)$ is the L^2-orthogonal projection to the exact forms. Thus $W_2 = P_E d^{c*}C^{k+1,\alpha}(\Lambda^6)$, which is the L^2-orthogonal complement in $dC^{k+1,\alpha}(\Lambda^4)$ to $dC^{k+1,\alpha}(\Lambda^4) \bigcap \ker d^c$. Hence $\mathcal{H}^5 \oplus dC^{k+1,\alpha}\big(\Lambda^4_6\big)$ and $\mathcal{Z}^5_k \bigcap \ker d^c$ are both the L^2-orthogonal complement to W_2 in $\mathcal{Z}^5_k$, so they must be equal. □

Definition 4·12. Let $U \subseteq \mathcal{Z}^3_k \times \mathcal{Z}^2_k$ be a small neighbourhood of (Ω, ω), and

$$F : U \to W_1 \times \big(\mathcal{H}^5 \oplus dC^{k+1,\alpha}\big(\Lambda^4_6\big)\big) \times C^{k,\alpha}(\Lambda^6), \quad (\beta, \gamma) \mapsto (F_1(\beta), F_2(\beta, \gamma), F_3(\beta, \gamma)),$$

where $F_1(\beta)$ is defined by (4·3), $F_2(\beta, \gamma) = P_2(\beta \wedge \gamma)$ with P_2 defined by (4·6), and $F_3(\beta, \gamma) = \frac{1}{4}\beta \wedge \hat{\beta} - \frac{1}{6}\gamma^3$.

PROPOSITION 4·13. *Let (β, γ) be a zero of F sufficiently close to (Ω, ω). Then (β, γ) is a Calabi-Yau structure.*

Proof. By proposition 4·8 $F_1(\beta) = 0$ implies that $d\hat{\beta} = 0$. Using lemma 4·2 it suffices to show that $\beta \wedge \gamma = 0$. Since β is a torsion-free $SL(\mathbb{C}^3)$-structure it defines an integrable complex structure J_β. Let d^c_β be the conjugate differential with respect to J_β. Since $d\gamma = 0$ the $(3, 0) + (0, 3)$-part of $d^c_\beta\gamma$ with respect to J_β vanishes. Thus

$$d^c_\beta(\beta \wedge \gamma) = -\beta \wedge d^c_\beta\gamma = 0. \tag{4·7}$$

Let $\eta = \beta \wedge \gamma$. $F_2(\beta, \gamma) = 0$ implies that $\eta \in W_2$. $d^c : L^2_1(\Lambda^5) \to L^2(\Lambda^6)$ is bounded below transverse to its kernel. $W_2 \bigcap \ker d^c = 0$ by proposition 4·11, so there is a constant A independent of $\eta \in W_2$ such that

$$\|d^c\eta\|_{L^2} \geqslant A\|\eta\|_{L^2_1}.$$

The map $C^1(\Lambda^3) \times L^2_1(\Lambda^2) \to L^2(\Lambda^5)$, $(\beta, \eta) \mapsto d^c_\beta\eta$ is differentiable in β and bounded linear in η, so there is a constant B such that for β sufficiently close to Ω

$$\|(d^c - d^c_\beta)\eta\|_{L^2} \leqslant B\|\beta - \Omega\|_{C^1}\|\eta\|_{L^2_1}.$$

Hence

$$\left\|d^c_\beta\eta\right\|_{L^2} \geqslant \|d^c\eta\|_{L^2} - \left\|(d^c - d^c_\beta)\eta\right\|_{L^2} \geqslant (A - B\|\beta - \Omega\|_{C^1})\|\eta\|_{L^2_1}. \qquad (4\cdot8)$$

Combining (4·7) and (4·8) gives that if $\|\beta - \Omega\|_{C^1} < A/B$ then $\beta \wedge \gamma = 0$. □

PROPOSITION 4·14. $(DF)_{(\Omega,\omega)} : \mathcal{Z}^3_k \times \mathcal{Z}^2_k \to W_1 \times \left(\mathcal{H}^5 \oplus dC^{k+1,\alpha}\left(\Lambda^4_6\right)\right) \times C^{k,\alpha}(\Lambda^6)$ *is surjective.*

Proof. Suppose $(\chi_1, \chi_2, \chi_3) \in W_1 \times (\mathcal{H}^5 \oplus dC^{k+1,\alpha}\left(\Lambda^4_6\right)) \times C^{k,\alpha}(\Lambda^6)$. By proposition 4·8 there is $\sigma \in \mathcal{Z}^3_k$ such that $(DF_1)_\Omega\sigma = \chi_1$. By proposition 4·11 there is $\tau \in \mathcal{Z}^2_k$ such that $\Omega \wedge \tau = \chi_2 - P_2(\sigma \wedge \omega)$. For $f \in C^{k+2,\alpha}(\mathbb{R})$, $\lambda \in \mathbb{R}$

$$(DF_3)_{(\Omega,\omega)}(0, dd^c f + \lambda\omega) = -\tfrac{1}{2}\omega^2 \wedge (dd^c f + \lambda\omega) = -\tfrac{1}{2}(\triangle f + \lambda)\omega^3.$$

Since $C^{k,\alpha}(\Lambda^6) = (\mathbb{R} \oplus \triangle C^{k+2,\alpha}(\mathbb{R}))\omega^3$ we can therefore find f, λ such that

$$(DF_3)_{(\Omega,\omega)}(\sigma, \tau + dd^c f + \lambda\omega) = \chi_3.$$

Then $(DF)_{(\Omega,\omega)}(\sigma, \tau + dd^c f + \lambda\omega) = (\chi_1, \chi_2, \chi_3)$. □

Recall that the tangent space to the slice is $K = (\mathcal{H}^3 \oplus W_1) \times \mathcal{Z}^2_k$, and that in definition 4·3 we took $\mathcal{H}_{SU}$ to be the harmonic tangents at (Ω, ω) to the space of $SU(3)$-structures.

PROPOSITION 4·15. *The kernel of* $(DF)_{(\Omega,\omega)}$ *in* K *is* $\mathcal{H}_{SU}$.

Proof. It is obvious that $\mathcal{H}_{SU}$ is contained in the kernel. The projection from the kernel to the $\mathcal{Z}^3_k$-component has image contained in $\mathcal{H}^3 \subseteq \mathcal{Z}^3_k$ and kernel contained in $0 \times \mathcal{H}^2_8 \subseteq K$, so by dimension-counting $\mathcal{H}_{SU}$ is all of the kernel. □

As a consequence we have that K really is transverse to T, as claimed earlier.

COROLLARY 4·16. $K \cap T = 0$.

Proof. If $K \cap T$ were non-trivial the kernel of $(DF)_{(\Omega,\omega)}$ in K would contain some non-zero exact forms. □

Now we can apply the implicit function theorem to F to show that the $\mathcal{Q}$ is a manifold with tangent space $\mathcal{H}_{SU}$ at (Ω, ω). Proposition 4·4 follows by elliptic regularity and slice arguments.

5. *Hodge theory*

We wish to study the moduli space of torsion-free EAC G_2-structures on a manifold with cylindrical ends in terms of the projection (3·1) to the de Rham cohomology. In order to do this we require results about Hodge theory on EAC manifolds. We also explain how the type decompositions of de Rham cohomology discussed in subsection 4·1 behave on EAC G_2-manifolds.

5·1. *Analysis of the Laplacian*

We review some results that we shall need about analysis of elliptic asymptotically translation-invariant operators on manifolds with cylindrical ends, and explain how they can be applied to the Laplacian of an EAC metric. For more detail about the analysis see e.g. Lockhart [**17**], Lockhart and McOwen [**18**], and Maz'ya and Plamenevskiĭ [**19**].

Definition 5·1. Let M^n be a manifold with cylindrical ends. If g is an asymptotically translation-invariant metric on M, E is a vector bundle associated to TM, $\delta \in \mathbb{R}$ and s is a section of E define the *Hölder norm with weight* δ (or $C^{k,\alpha}_{\delta}$-norm) of s in terms of the Hölder norm associated to g by

$$\|s\|_{C^{k,\alpha}_{\delta}(g)} = \|e^{\delta t}s\|_{C^{k,\alpha}(g)},$$

where t is the cylindrical coordinate on M. Denote the space of sections of E with finite $C^{k,\alpha}_{\delta}$-norm by $C^{k,\alpha}_{\delta}(E)$.

Up to equivalence, the weighted norms are independent of the choice of EAC metric g, and of the choice of t on the compact piece M_0. In particular, as topological vector spaces $C^{k,\alpha}_{\delta}(E)$ are independent of these choices.

We will want to use that d and d^* are formal adjoints in integration by parts arguments. On a manifold with cylindrical ends this is only justified if the rate of decay of the product is positive.

LEMMA 5·2. *Let M^n be a manifold with cylindrical ends equipped with an asymptotically translation-invariant metric. Suppose that $\beta \in C^{k,\alpha}_{\delta_1}(\Lambda^m T^*M)$, $\gamma \in C^{k,\alpha}_{\delta_2}(\Lambda^{m+1}T^*M)$ with $k \geqslant 1$ and $\delta_1 + \delta_2 > 0$. Then*

$$<d\beta, \gamma>_{L^2} = <\beta, d^*\gamma>_{L^2}.$$

Let M be a manifold with cylindrical ends, E, F vector bundles associated to TM and A a linear smooth order r differential operator $\Gamma(E) \to \Gamma(F)$. The restriction of A to the cylindrical end M_∞ can be written in terms of the Levi-Civita connection of an arbitrary product metric on $X \times \mathbb{R}$ as

$$A = \sum_{i=0}^{r} a_i \nabla^i \tag{5·1}$$

with coefficients $a_i \in C^\infty((TM)^i \otimes E^* \otimes F)$. A is said to be *asymptotically translation-invariant* if the coefficients a_i are. Then A induces bounded linear maps

$$A : C^{k+r,\alpha}_{\delta}(E) \to C^{k,\alpha}_{\delta}(F)$$

for any $\delta \in \mathbb{R}$. One of the main results of [**18**] is Theorem 6.2, which states that if A is elliptic then these maps are Fredholm for all but a discrete set of values of δ, and also relates the index for different values of δ. [**18**, Theorem 7.4] is a corollary, which computes the index of self-adjoint asymptotically translation-invariant elliptic operators for small weights. This can be applied in particular to the Hodge Laplacian of an asymptotically cylindrical metric, as in [**17**, Section 3].

PROPOSITION 5·3. *Let M be an asymptotically cylindrical manifold, and ϵ_1 the largest real such that*

$$\triangle : C^{k+2,\alpha}_{\pm\delta}(\Lambda^m T^*M) \to C^{k,\alpha}_{\pm\delta}(\Lambda^m T^*M) \tag{5·2}$$

is Fredholm for all m and $0 < \delta < \epsilon_1$. Then the index of (5·2) is $\mp(b^{m-1}(X) + b^m(X))$ for all $0 < \delta < \epsilon_1$.

Remark 5·4. Strictly speaking, the results in [**18**] use weighted Sobolev spaces rather than weighted Hölder spaces, but the arguments are the same in both cases. See also [**19**, Theorem 6.4].

LEMMA 5·5. *ϵ_1 depends only on the asymptotic model $g_X + dt^2$ for the metric on M. Furthermore, ϵ_1 is a lower semi-continuous function of g_X with respect to the C^1-norm.*

Proof. ϵ_1^2 is in fact the smallest positive eigenvalue λ_1 of the Hodge Laplacian $\triangle_X$ defined by g_X on $\Omega^*(X)$. To prove the proposition it therefore suffices to show that λ_1 is lower semi-continuous in g_X.

Let g, g' be smooth Riemannian metrics on X, $\triangle$, $\triangle'$ their Laplacians and λ_1, λ_1' the smallest positive eigenvalues of the Laplacians. Let T be the $L^2(g)$-orthogonal complement to $\ker \triangle$ in $C^{2,\alpha}(\Lambda^* T^* X)$. Then for any $\beta \in T$ with unit $L^2(g)$-norm

$$\lambda_1 \leqslant \langle \triangle\beta, \beta\rangle_{L^2(g)} = \|d\beta\|^2_{L^2(g)} + \|d^*\beta\|^2_{L^2(g)}.$$

Since $d + d^*$ is an elliptic operator it gives a Fredholm map $L^2_1(\Lambda^* T^* X) \to L^2(\Lambda^* T^* X)$, so it is bounded below transverse to its kernel. In other words, there is a constant C_1 such that $\|\beta\|^2_{L^2_1(g)} \leqslant C_1(\|d\beta\|^2_{L^2(g)} + \|d^*\beta\|^2_{L^2(g)})$ for any $\beta \in T \cap L^2_1(\Lambda^* T^* X)$.

Let e_1 be an eigenvector of $\triangle'$ with eigenvalue λ_1'. By Hodge theory for compact manifolds $\ker \triangle$ and $\ker \triangle'$ have the same dimension, so $(\ker \triangle' \oplus \mathbb{R}e_1) \cap T$ is non-trivial. Hence

$$\lambda_1' \geqslant \frac{\langle \triangle'\beta, \beta\rangle_{L^2(g')}}{\langle \beta, \beta\rangle_{L^2(g')}} = \frac{\|d\beta\|^2_{L^2(g')} + \|d^{*'}\beta\|^2_{L^2(g')}}{\|\beta\|^2_{L^2(g')}}$$

for some $\beta \in T$ with unit $L^2(g)$-norm. The RHS depends differentiably on g' (with respect to the $C^1(g)$-norm) and the derivative at $g' = g$ can be estimated in terms of $\|\beta\|^2_{L^2_1(g)}$. Therefore there is a constant C_2 (independent of β) such that for any g' close to g

$$\begin{aligned}\lambda_1' &\geqslant \frac{\|d\beta\|^2_{L^2(g')} + \|d^{*'}\beta\|^2_{L^2(g')}}{\|\beta\|^2_{L^2(g')}} \\ &\geqslant \|d\beta\|^2_{L^2(g)} + \|d^*\beta\|^2_{L^2(g)} - C_2\|g' - g\|_{C^1(g)}\|\beta\|^2_{L^2_1(g)} \geqslant \big(1 - \|g' - g\|_{C^1(g)} C_1 C_2\big)\,\lambda_1.\end{aligned}$$

Now let M be an EAC manifold with rate δ_0 and cross-section X, and assume that $0 < \delta < \min\{\epsilon_1, \delta_0\}$. We fix some notation for various spaces of harmonic forms.

Definition 5·6. Denote by

(i) $\mathcal{H}^m_\pm$ the space of harmonic m-forms in $C^{k,\alpha}_{\pm\delta}(\Lambda^m T^* M)$,

(ii) $\mathcal{H}^m_0$ the space of bounded harmonic m-forms on M,

(iii) $\mathcal{H}^m_\infty$ the space of translation-invariant harmonic m-forms on $X \times \mathbb{R}$,

(iv) $\mathcal{H}^m_X$ the space of harmonic m-forms on X.

By elliptic regularity $\mathcal{H}^m_\pm$ consists of smooth forms, and is independent of k for $k \geqslant 2$. The computation of the index in proposition 5·3 involves proving that

$$\mathcal{H}^m_\infty = \big\{\psi + dt \wedge \tau : \psi \in \mathcal{H}^m_X, \tau \in \mathcal{H}^{m-1}_X\big\}. \tag{5·3}$$

In particular the index of

$$\triangle : C^{k+2,\alpha}_\delta(\Lambda^m T^* M) \to C^{k,\alpha}_\delta(\Lambda^m T^* M) \tag{5·4}$$

is $-\dim \mathcal{H}^m_\infty$ for small positive δ. Knowing the index of the Laplacian on weighted Hölder spaces allows us to use an index-counting argument to deduce results about the kernel, and a Hodge decomposition statement.

Let $i = b^m(X) + b^{m-1}(X)$. $\mathcal{H}^m_\infty$ has dimension i, and the index of (5·4) is $-i$. $\triangle(\rho\mathcal{H}^m_\infty)$ and $\triangle(\rho t\mathcal{H}^m_\infty)$ consist of exponentially decaying forms. Therefore

$$\triangle : C^{k+2,\alpha}_\delta(\Lambda^m T^*M) \oplus \rho\mathcal{H}^m_\infty \oplus \rho t\mathcal{H}^m_\infty \to C^{k,\alpha}_\delta(\Lambda^m T^*M) \qquad (5{\cdot}5)$$

is well-defined, and its index is $+i$. (5·4) has kernel $\mathcal{H}^m_+$, and by integration by parts the image contained in the orthogonal complement of $\mathcal{H}^m_-$. (5·5) has kernel contained in $\mathcal{H}^m_-$ and image contained in the orthogonal complement of $\mathcal{H}^m_+$. Hence

$$i \leqslant \dim \mathcal{H}^m_- - \dim \mathcal{H}^m_+,$$
$$-i \leqslant \dim \mathcal{H}^m_+ - \dim \mathcal{H}^m_-.$$

Since equality holds the kernel of (5·5) is exactly $\mathcal{H}^m_-$.

PROPOSITION 5·7. *Let M be an exponentially asymptotically cylindrical manifold with rate δ_0, $k \geqslant 0$ and $0 < \delta < \min\{\epsilon_1, \delta_0\}$. Then*

$$\mathcal{H}^m_- \subseteq C^{k,\alpha}_\delta(\Lambda^m T^*M) \oplus \rho\mathcal{H}^m_\infty \oplus \rho t\mathcal{H}^m_\infty,$$
$$\mathcal{H}^m_0 \subseteq C^{k,\alpha}_\delta(\Lambda^m T^*M) \oplus \rho\mathcal{H}^m_\infty,$$

and $\mathcal{H}^m_+$ is precisely the space of L^2-integrable harmonic forms on M.

Also the image of (5·5) is exactly the L^2-orthogonal complement of $\mathcal{H}^m_+$ in $C^{k,\alpha}_\delta(\Lambda^m T^*M)$. We can use this to prove an EAC analogue of the Hodge decomposition for compact manifolds:

$$C^{k,\alpha}_\delta(\Lambda^m T^*M) = \mathcal{H}^m_+ \oplus C^{k,\alpha}_\delta[d\Lambda^{m-1}T^*M] \oplus C^{k,\alpha}_\delta[d^*\Lambda^{m+1}T^*M], \qquad (5{\cdot}6)$$

where we let $C^{k,\alpha}_\delta[d\Lambda^{m-1}T^*M]$ and $C^{k,\alpha}_\delta[d^*\Lambda^{m+1}T^*M]$ denote the subspaces of $C^{k,\alpha}_\delta(\Lambda^m T^*M)$ consisting of exact and coexact forms, respectively.

5·2. *Hodge theory on EAC manifolds*

In this subsection we outline the correspondence between bounded harmonic forms and the de Rham cohomology on an oriented EAC Riemannian manifold M^n with cross-section X^{n-1}. For 'exact b-manifolds', which can be considered as a subclass of EAC manifolds, the type of results we need can be found in Section 6.4 of Melrose [**20**]. The arguments there carry over unchanged to the EAC case. The Hodge theory of EAC manifolds is also dealt with by Kovalev [**16**].

Definition 5·8. Let $\mathcal{H}^m_E, \mathcal{H}^m_{E^*} \subseteq \mathcal{H}^m_0$ denote the spaces of bounded exact and coexact harmonic forms respectively on M, and

$$\mathcal{H}^m_{abs} = \mathcal{H}^m_+ \oplus \mathcal{H}^m_{E^*}, \quad \mathcal{H}^m_{rel} = \mathcal{H}^m_+ \oplus \mathcal{H}^m_E.$$

The next theorem is part of [**20**, Theorem 6.18].

THEOREM 5·9. *Let M be an oriented EAC manifold. The natural projection map $\pi_H : \mathcal{H}^m_{abs} \to H^m(M)$ is an isomorphism.*

Since the kernel of $\pi_H : \mathcal{H}^m_0 \to H^3(M)$ is $\mathcal{H}^m_E$ the theorem implies that

$$\mathcal{H}^m_0 = \mathcal{H}^m_+ \oplus \mathcal{H}^m_{E^*} \oplus \mathcal{H}^m_E. \qquad (5{\cdot}7)$$

By proposition 5·7 any $\alpha \in \mathcal{H}^m_0$ is exponentially asymptotic to some $B(\alpha) \in \mathcal{H}^m_\infty$. By (5·3) $\mathcal{H}^m_\infty \cong \mathcal{H}^m_X \oplus dt \wedge \mathcal{H}^{m-1}_X$, so we get a boundary map

$$B : \mathcal{H}^m_0 \to \mathcal{H}^m_X \oplus dt \wedge \mathcal{H}^{m-1}_X, \quad \alpha \mapsto B_a(\alpha) + dt \wedge B_e(\alpha).$$

It is easy to see that for $\alpha \in \mathcal{H}^m_0$ the pull-back map j^* in the long exact sequence for relative cohomology (2·4) acts as

$$j^*([\alpha]) = [B_a(\alpha)] \in H^m(X), \tag{5·8}$$

and it follows that $\mathcal{H}^m_{rel} \subseteq \ker B_a$. Applying the Hodge star shows that also $\mathcal{H}^m_{abs} \subseteq \ker B_e$. Therefore B_a is injective on $\mathcal{H}^m_{E*}$ and 0 on $\mathcal{H}^m_{rel}$, while B_e is injective on $\mathcal{H}^m_E$ and 0 on $\mathcal{H}^m_{abs}$.

As a corollary of theorem 5·9 we can determine that the image of the space $\mathcal{H}^m_+$ of L^2 harmonic forms in the de Rham cohomology $H^m(M)$ is precisely the subspace $H^m_0(M)$ of compactly supported classes. This result appears as e.g. [**1**, Proposition 4.9], [**17**, Theorems 7.6 and 7.9], and [**20**, Proposition 6.14].

THEOREM 5·10. *Let M be an oriented EAC manifold. Then $\pi_H : \mathcal{H}^m_+ \to H^m(M)$ is an isomorphism onto $H^m_0(M)$.*

Proof. $\mathcal{H}^m_+$ is $\ker B_a$ in $\mathcal{H}^m_{abs}$, and it follows from theorem 5·9 that it is mapped isomorphically to $H^m_0(M) = \ker j^* \subseteq H^m(M)$. □

Definition 5·11. Let $\mathcal{A}^m = B_a(\mathcal{H}^m_0) \subseteq \mathcal{H}^m_X$, $\mathcal{E}^m = B_e(\mathcal{H}^{m+1}_0) \subseteq \mathcal{H}^m_X$, and let A^m, E^m be the subspaces of $H^m(X)$ that they represent.

A^m is of course just the image $j^*(H^m(M)) \subseteq H^m(X)$. The Hodge star on M identifies $\mathcal{H}^m_{abs}$ and $\mathcal{H}^{m-n}_{rel}$. If $\beta \in \mathcal{H}^m_0$ then $B_e(*\beta) = *B_a(\beta)$. Therefore the Hodge star on X identifies $\mathcal{A}^m$ with $\mathcal{E}^{n-m-1}$, and A^m with E^{n-m-1}. [**20**, Lemma 6.15] implies

PROPOSITION 5·12. *Let M^n be an oriented EAC manifold. Then*

$$\mathcal{H}^m_X = \mathcal{A}^m \oplus \mathcal{E}^m$$

is an orthogonal direct sum.

Finally, we observe

COROLLARY 5·13. *Let M^n be an oriented EAC manifold which has a single end (i.e. the cross-section X is connected). Then $e : H^1_{cpt}(M) \to H^1(M)$ is injective. $\mathcal{H}^1_E = 0$, and $\mathcal{H}^1_0 \to H^1(M)$ is an isomorphism.*

Proof. Consider the start of the long exact sequence for relative cohomology

$$H^0_{cpt}(M) \to H^0(M) \to H^0(X) \xrightarrow{\partial} H^1_{cpt}(M) \xrightarrow{e} H^1(M).$$

The dimensions of the first three terms are 0, 1, and 1, so $\partial = 0$, and thus e is injective.

$$\mathcal{H}^1_E \cong \mathcal{E}^0 \cong E^0 \cong \partial(H^0(X)) = \ker e \subseteq H^1_{cpt}(M),$$

so the result follows. □

5·3. *Hodge theory of EAC G_2-manifolds*

Let M^7 be an EAC G_2-manifold, with G_2-structure φ asymptotic to $\Omega + dt \wedge \omega$. (Ω, ω) is a Calabi-Yau structure on the cross-section X. Maps such as $\Omega^2(X) \to \Omega^4(X)$, $\sigma \mapsto \sigma \wedge \omega$ are $SU(3)$-equivariant, so by proposition 4·1 induce maps between type components of the spaces of harmonic forms. In this subsection we consider the relation between the type decompositions and the decomposition in proposition 5·12.

By remark 2·12 $*\varphi$ is asymptotic to $\frac{1}{2}\omega^2 - dt \wedge \hat{\Omega}$, where $\hat{\Omega}$ is the unique 3-form on X such that $\Omega + i\hat{\Omega}$ has type $(3, 0)$ as discussed in subsection 2·3.

LEMMA 5·14. *If* $\tau \in \mathcal{E}^m$ *then* $\tau \wedge \Omega \in \mathcal{E}^{m+3}$ *and* $\tau \wedge \frac{1}{2}\omega^2 \in \mathcal{E}^{m+4}$.
If $\sigma \in \mathcal{A}^m$ *then* $\sigma \wedge \Omega \in \mathcal{A}^{m+3}$, $\sigma \wedge \frac{1}{2}\omega^2 \in \mathcal{A}^{m+4}$, $\sigma \wedge \omega \in \mathcal{E}^{m+2}$ *and* $\sigma \wedge \hat{\Omega} \in \mathcal{E}^{m+3}$.

Proof. If $\chi \in \mathcal{H}_E^{m+1}$ with $B_e\chi = \tau$ then

$$\chi \wedge \varphi \in \mathcal{H}_0^{m+4} \Rightarrow dt \wedge \tau \wedge \Omega = B(\chi \wedge \varphi) \in dt \wedge \mathcal{E}^{m+3},$$
$$\chi \wedge *\varphi \in \mathcal{H}_0^{m+5} \Rightarrow dt \wedge \tau \wedge \tfrac{1}{2}\omega^2 = B(\chi \wedge *\varphi) \in dt \wedge \mathcal{E}^{m+4}.$$

If $\chi \in \mathcal{H}_0^m$ with $B_a\chi = \sigma$ then

$$\chi \wedge \varphi \in \mathcal{H}_0^{m+3} \Rightarrow \sigma \wedge \Omega + dt \wedge \sigma \wedge \omega = B(\chi \wedge \varphi) \in \mathcal{A}^{m+3} \oplus dt \wedge \mathcal{E}^{m+2},$$
$$\chi \wedge *\varphi \in \mathcal{H}_0^{m+4} \Rightarrow \sigma \wedge \tfrac{1}{2}\omega^2 + dt \wedge \sigma \wedge \hat{\Omega} = B(\chi \wedge *\varphi) \in \mathcal{A}^{m+4} \oplus dt \wedge \mathcal{E}^{m+3}.$$

Hodge theory for compact manifolds allows us to identify the de Rham cohomology of X with the harmonic m-forms on X. The L^2-inner product on $\mathcal{H}_X^m$ therefore gives an inner product on $H^m(X)$, and the Hodge star $* : \mathcal{H}_X^m \to \mathcal{H}_X^{6-m}$ gives isomorphisms

$$* : H^m(X) \to H^{6-m}(X).$$

This map is the composition of the metric isomorphism $H^m(X) \cong (H^m(X))^*$ with Poincaré duality $(H^m(X))^* \cong H^{6-m}(X)$. Proposition 5·12 implies that there is an orthogonal direct sum

$$H^m(X) = A^m \oplus E^m,$$

where $A^m = j^*(H^m(M))$ and $E^m = *A^{6-m}$. Let $A_6^2 = A^2 \cap H_6^2(X)$ etc.

PROPOSITION 5·15. *Let M be an EAC G_2-manifold with cross-section X. Then*

$$H_6^2(X) = A_6^2 \oplus E_6^2, \quad H_6^4(X) = A_6^4 \oplus E_6^4,$$

and the sums are orthogonal. Furthermore

(i) $H_6^2(X) \to H_6^4(X)$, $[\alpha] \mapsto *[\alpha]$ *maps* $A_6^2 \to E_6^4$, $E_6^2 \to A_6^4$,
(ii) $H^1(X) \to H_6^4(X)$, $[\alpha] \mapsto [\alpha] \cup [\Omega]$ *maps* $A^1 \to A_6^4$, $E^1 \to E_6^4$,
(iii) $H^1(X) \to H^5(X)$, $[\alpha] \mapsto [\alpha] \cup [\frac{1}{2}\omega^2]$ *maps* $A^1 \to A^5$, $E^1 \to E^5$.

Proof. (i) is obvious, since $*$ maps $A^m \leftrightarrow E^{6-m}$.
$[\alpha] \mapsto [\alpha] \cup [\Omega]$ is a bijection $H^1(X) \to H_6^4(X)$, and it maps $A^1 \hookrightarrow A^4$, $E^1 \hookrightarrow E^4$ by lemma 5·14. Thus (ii). It follows that that $A^1 \to A_6^4$, $E^1 \to E_6^4$ are both surjective, and that $H_6^4(X)$ splits as $A_6^4 \oplus E_6^4$. $H_6^2(X)$ splits too by (i).
(iii) follows from lemma 5·14 in the same way. □

When X is Kähler the complex structure J is parallel, and the Hodge Laplacian on forms commutes with the action of J. Thus $\mathcal{H}_X^1$ inherits a complex structure from X in the Kähler case, and the inner product on $\mathcal{H}_X^1$ is Hermitian. We identify $H^1(X) \leftrightarrow \mathcal{H}_X^1$ as usual to make $H^1(X)$ a Hermitian vector space.

PROPOSITION 5·16. *Let M be an EAC G_2-manifold with cross-section X. Then the pull-back $j^* : H^1(M) \hookrightarrow H^1(X)$ embeds $H^1(M)$ as a Lagrangian subspace of $H^1(X)$ with the symplectic form $< \cdot, J \cdot >$. In particular $b^1(M) = \frac{1}{2}b^1(X)$.*

Proof. By proposition 5·17 below j^* maps $H^1(M)$ isomorphically to its image A^1. The complex structure on $H^1(X)$ can be expressed as

$$J[\alpha] = *\big([\alpha] \cup \big[\tfrac{1}{2}\omega^2\big]\big).$$

Thus J maps A^1 to its orthogonal complement E^1 by proposition 5·15. □

On compact Ricci-flat manifolds harmonic 1-forms are parallel, and this can be generalised to the EAC case.

PROPOSITION 5·17. *If M is a Ricci-flat EAC manifold then $\mathcal{H}^1_0$ is the space of parallel 1-forms on M. In particular $\mathcal{H}^1_+ = 0$, and $j^* : H^1(M) \to H^1(X)$ is injective.*

Proof. This is proved by a standard 'Bochner argument'. For a 1-form ϕ the Weitzenböck formula [**3**, (1.155)] gives

$$\triangle\phi = \nabla^*\nabla\phi. \tag{5·9}$$

It follows that any parallel 1-form ϕ is harmonic, and parallel forms are of course bounded. To show that any bounded harmonic form is parallel we use (5·9) together with an integration by parts argument.

$\mathcal{H}^1_+$ consists of parallel decaying forms, so is 0. By theorem 5·10 the kernel $H^1_0(M)$ of $j^* : H^1(M) \to H^1(X)$ is represented by $\mathcal{H}^1_+$. □

6. *Deformation theory of EAC G_2-manifolds*

In this section we construct the moduli space of exponentially asymptotically cylindrical torsion-free G_2-structures on a manifold with a cylindrical end, and study its local properties.

Throughout the section M^7 is a connected oriented manifold with cylindrical ends, X^6 is its cross-section, t the cylindrical coordinate on M and ρ a cut-off function for the cylinder on M. We abbreviate $\Lambda^m T^*M$ to Λ^m.

6·1. *Proof outline*

We now set out to prove the main theorem 3·2. The argument is a generalisation of that used by Hitchin in [**11**] to construct the moduli space of torsion-free G_2-structures on a compact manifold. Hitchin shows that a G_2-structure is torsion-free if and only if it is a critical point of the volume functional $\varphi \mapsto Vol(\varphi)$ ($Vol(\varphi)$ is the total volume of the metric defined by φ). An appropriate modification of the map $\varphi \mapsto D_\varphi Vol$ has surjective derivative, and the implicit function theorem can be applied to find pre-moduli spaces of torsion-free G_2-structures which are manifolds.

In the EAC case it is not as natural to consider the volume functional, but we can nevertheless adapt the steps in Hitchin's proof to find pre-moduli spaces which are manifolds, and then apply slice arguments to show that the moduli space is a manifold. Like in e.g. Kovalev [**15**] (which studies deformations of EAC Calabi-Yau metrics) we can simplify the slice by understanding the deformations of the boundary – in this case we use the deformation theory of compact Calabi-Yau 3-folds described in section 5.

As an intermediate step in the proof of theorem 3·2 we construct moduli spaces of torsion-free exponentially cylindrical G_2-structures with some fixed rate $\delta > 0$. As before we let $\mathcal{X}_\delta$ be the space of torsion-free EAC G_2-structures with rate δ. Each $\varphi \in \mathcal{X}_\delta$ defines an EAC metric, and hence a parameter $\epsilon_1(\varphi)$ (cf. proposition 5·3). When we study a neighbourhood of φ we need to assume that $\delta < \epsilon_1(\varphi)$ in order to apply analysis results. We therefore let

$$\mathcal{X}'_\delta = \{\varphi \in \mathcal{X}_\delta : \epsilon_1(\varphi) > \delta\}.$$

ϵ_1 depends lower semi-continuously on the asymptotic model by lemma 5·5, so $\mathcal{X}'_\delta$ is an open subset of $\mathcal{X}_\delta$. Let $\mathcal{D}_\delta$ be the group of EAC diffeomorphisms of M with rate δ. The rate δ moduli space that we study is $\mathcal{M}_\delta = \mathcal{X}'_\delta/\mathcal{D}_\delta$.

Given $\varphi \in \mathcal{X}'_\delta$ let $\Omega + dt \wedge \omega = B(\varphi)$, i.e. the asymptotic limit of φ. If we identify $\Omega + dt \wedge \omega$ with the pair (Ω, ω) then by proposition 2·11 $\Omega + dt \wedge \omega$ defines a Calabi-Yau structure on X. By proposition 4·4 there is a pre-moduli space $\mathcal{Q}$ of Calabi-Yau structures near $\Omega + dt \wedge \omega$. Let

$$\mathcal{X}_\mathcal{Q} = \{\psi \in \mathcal{X}'_\delta : B(\psi) \in \mathcal{Q}\},$$

and let $\mathcal{D}_\mathcal{Q} \subseteq \mathcal{D}_\delta$ be the subgroup of diffeomorphisms asymptotic to automorphisms of the cylindrical G_2-structure $\Omega + dt \wedge \omega$. By proposition 4·5 $\mathcal{D}_\mathcal{Q}$ acts on $\mathcal{X}_\mathcal{Q}$, and we will see that $\mathcal{X}_\mathcal{Q}/\mathcal{D}_\mathcal{Q}$ maps homeomorphically to an open subset of $\mathcal{M}_\delta$.

We use slice arguments to study a neighbourhood of $\varphi\mathcal{D}_\mathcal{Q}$ in $\mathcal{X}_\mathcal{Q}/\mathcal{D}_\mathcal{Q}$. In order to be able to apply analysis results we need to use Banach spaces of forms, so we work with weighted Hölder $C^{k,\alpha}_\delta$ spaces, for some fixed $k \geqslant 1, \alpha \in (0, 1)$. Note that the boundary values of elements of $\mathcal{X}_\mathcal{Q}$ must lie in

$$\mathcal{Q}_A = \{\Omega' + dt \wedge \omega' \in \mathcal{Q} : [\Omega'] \in A^3, [\omega'^2] \in A^4\}, \tag{6·1}$$

where A^m is the image of $j^* : H^m(M) \to H^m(X)$ (cf. discussion before theorem 3·6). We use the cut-off function ρ to consider $\rho\mathcal{Q}_A$ as a subspace of smooth asymptotically translation-invariant 3-forms supported on the cylinder of M, and let

$$\mathcal{Z}^3_\mathcal{Q} \subseteq C^{k,\alpha}_\delta(\Lambda^3) + \rho\mathcal{Q}_A$$

be the subspace of closed forms. Then $\mathcal{X}_\mathcal{Q} \hookrightarrow \mathcal{Z}^3_\mathcal{Q}$ continuously. The main technical steps in the construction of the pre-moduli space near φ are to

(i) show that $\mathcal{Q}_A$ is a submanifold of $\mathcal{Q}$ (proposition 6·2), so that $\mathcal{Z}^3_\mathcal{Q}$ is a manifold,
(ii) find a complement K in $T_\varphi\mathcal{Z}^3_\mathcal{Q}$ for the tangent space to the $\mathcal{D}_\mathcal{Q}$-orbit at φ (proposition 6·11), and pick a submanifold $\mathcal{S} \subseteq \mathcal{Z}^3_\mathcal{Q}$ with $T_\varphi\mathcal{S} = K$,
(iii) show that the space of torsion-free G_2-structures $\mathcal{R}_\delta \subseteq \mathcal{S}$ is a submanifold (proposition 6·18),
(iv) show that the elements of $\mathcal{R}_\delta$ are smooth and exponentially asymptotically translation-invariant with rate δ (proposition 6·21).

In subsection 6·7 we then provide the slice arguments that show that $\mathcal{R}_\delta$ is homeomorphic to a neighbourhood in $\mathcal{X}_\mathcal{Q}/\mathcal{D}_\mathcal{Q}$, and therefore in $\mathcal{M}_\delta$. Hence $\mathcal{M}_\delta$ is a manifold for any $\delta > 0$. We then show that $\mathcal{M}_\delta$ is homeomorphic to an open subset of $\mathcal{M}_+$ for any $\delta > 0$, and deduce that $\mathcal{M}_+$ is a manifold.

Remark 6·1. The last step means that if $\varphi \in \mathcal{X}_+$ is EAC with rate $\delta_0(\varphi)$ then for any $0 < \delta < \min\{\delta_0(\varphi), \epsilon_1(\varphi)\}$ the pre-moduli space $\mathcal{R}_\delta$ gives a chart near φ not only in $\mathcal{M}_\delta$, but also in $\mathcal{M}_{\delta'}$ for any $\delta' > \delta$, and in $\mathcal{M}_+$. In other words, $\mathcal{R}_\delta$ is essentially independent of δ if δ is chosen sufficiently small.

6·2. *The boundary values*

As explained above we are restricting our attention to determining the space of torsion-free G_2-structures in $\mathcal{Z}^3_\mathcal{Q}$, whose boundary values lie in a space $\mathcal{Q}_A$. In order to make sense of this we first of all need to know that $\mathcal{Q}_A$ is a manifold. We show that here, and in the process essentially prove theorem 3·6.

Let X^6 be the cross-section of an EAC G_2-manifold M^7, and (Ω, ω) a Calabi-Yau structure on X defined by the limit of a torsion-free EAC G_2-structure on M. Let $\mathcal{Q}$ be the pre-moduli space of Calabi-Yau structures near (Ω, ω), and equivalently to (6·1) define

$$\mathcal{Q}_A = \{(\Omega', \omega') \in \mathcal{Q} : [\Omega'] \in A^3, [\omega'^2] \in A^4\}.$$

Since $\mathcal{Q}$ is diffeomorphic to a neighbourhood in the moduli space $\mathcal{N}$ of Calabi-Yau structures on X, $\mathcal{Q}_A$ is homeomorphic to a neighbourhood in the subspace $\mathcal{N}_A$ of classes which stand a chance of being the boundary values of EAC torsion-free G_2-structures, as discussed before theorem 3·6.

Recall that by proposition 4·4 the tangent space at (Ω, ω) to the pre-moduli space $\mathcal{Q}$ is the space $\mathcal{H}_{SU}$ of harmonic tangents to the $SU(3)$-structures. As before let $E^m \subseteq H^m(X)$ be the orthogonal complement of A^m, and let $\mathcal{A}^m, \mathcal{E}^m \subseteq \mathcal{H}^m_X$ denote the respective spaces of harmonic representatives. By lemma 5·14 $\tau \mapsto \omega \wedge \tau$ maps $\mathcal{A}^2 \to \mathcal{E}^4$ and $\mathcal{E}^2 \to \mathcal{A}^4$. Hence the linearisation of the condition $[\omega'^2] \in A^4$ is $[\tau] \in E^2$, and we would expect the tangent space to $\mathcal{Q}_A$ at (Ω, ω) to be

$$\mathcal{H}_{SU,A} = \{(\sigma, \tau) \in \mathcal{H}_{SU} : \sigma \in \mathcal{A}^3, \tau \in \mathcal{E}^2\}.$$

PROPOSITION 6·2. *Let (Ω, ω) be the Calabi-Yau structure induced on the cross-section X^6 of an EAC G_2-manifold M^7, and $\mathcal{Q}$ the pre-moduli space of Calabi-Yau structures near (Ω, ω). Then $\mathcal{Q}_A \subseteq \mathcal{Q}$ is a submanifold, and*

$$T_{(\Omega,\omega)}\mathcal{Q}_A = \mathcal{H}_{SU,A}.$$

Proof. The map $\mathcal{Q} \to H^3(X)$ is a submersion, so

$$\mathcal{Q}' = \{(\Omega', \omega') \in \mathcal{Q} : [\Omega'] \in A^3\}$$

is a submanifold of $\mathcal{Q}$. By proposition 5·15

$$H^4(X) = A^4_1 \oplus A^4_6 \oplus A^4_8 \oplus E^4_6 \oplus E^4_8,$$

where $A^4_6 = H^4_6(X) \cap A^4$ etc. Let

$$P_{E8} : H^4(X) \to E^4_8$$

be the orthogonal projection. For $(\Omega', \omega') \in \mathcal{Q}'$ let $E^{m\prime}$ be the orthogonal complement of A^m in $H^m(X)$ with respect to the metric defined by (Ω', ω'), and $P_{A'} : H^m(X) \to A^m$, $P_{E'} : H^m(X) \to E^{m\prime}$ the projections. Let

$$F : \mathcal{Q}' \to E^4_8, \quad (\Omega', \omega') \mapsto P_{E8}P_{E'}[\omega' \wedge \omega'].$$

We prove that $\mathcal{Q}_A$ is a submanifold of $\mathcal{Q}'$ by showing that it is the zero set of F, and that F has surjective derivative at (Ω, ω).

Suppose $F(\Omega', \omega') = 0$, and let $a = P_{E'}[\omega' \wedge \omega']$. Write $a = b + c$, with $b \in A^4$, $c \in E^4$. $P_{E8}a = 0 \Rightarrow \pi_8 c = 0$, so $c \in E^4_6$. Since $E^1 \to E^4_6$, $v \mapsto [\Omega] \cup v$ is an isomorphism $c = [\Omega] \cup v$ for some $v \in E^1$. In the inner product $<,>'$ on $H^*(X)$ defined by (Ω', ω')

$$<a, a>' = <a, [\Omega] \cup v>' = <a, [\Omega] \cup v - [\Omega'] \cup P_{E'}v>' \leqslant \|a\|'(\|[\Omega - \Omega']\|'\|v\|' + \|P_{A'}v\|').$$

The RHS can be estimated by $\|[\Omega - \Omega']\|(\|a\|')^2$ for (Ω', ω') close to (Ω, ω). Hence for (Ω', ω') sufficiently close to (Ω, ω)

$$F(\Omega', \omega') = 0 \Rightarrow P_{E'}[\omega'^2] = 0 \Rightarrow [\omega'^2] \in A^4.$$

So $\mathcal{Q}_A \subseteq \mathcal{Q}'$ is the zero set of F. It remains to show that F has surjective derivative. If $(\sigma, \tau) \in (\mathcal{A}^3 \times \mathcal{H}^2_X) \bigcap \mathcal{H}_{SU} = T_{(\Omega,\omega)}\mathcal{Q}'$ then since $[\omega^2] \in A^4$

$$DF_{(\Omega,\omega)}(\sigma, \tau) = P_{E8}P_E(2[\omega \wedge \tau]) = 2P_{E8}[\omega \wedge \tau].$$

Since $\mathcal{A}^2_8 \to \mathcal{E}^4_8,\ \tau \mapsto \omega \wedge \tau$ is a bijection the derivative maps $0 \times \mathcal{A}^2_8 \subseteq T_{(\Omega,\omega)}\mathcal{Q}'$ onto E^4_8.

By the implicit function theorem $\mathcal{Q}_A$ is a manifold, and the tangent space at (Ω, ω) is

$$\ker DF_{(\Omega,\omega)} = \mathcal{H}_{SU,A}. \qquad \square$$

COROLLARY 6·3. *$\mathcal{H}_{SU,A} \to \mathcal{A}^3,\ (\sigma, \tau) \mapsto \sigma$ is surjective, with kernel* $0 \times \mathcal{E}^2_8$.

Proof. The last part of the proof of the proposition actually shows that $\mathcal{Q}_A \to A^3$ is a submersion, so $\mathcal{H}_{SU,A} \to \mathcal{A}^3$ is surjective. This could also be deduced from lemma 5·14. By definition of $\mathcal{H}_{SU}$ the kernel consists of $(0, \tau) \in 0 \times \mathcal{E}^2$ satisfying the conditions (4·3), which reduce to $\pi_1\tau = \pi_6\tau = 0$. $\square$

Proposition 6·2 implies directly that a neighbourhood of the image of $B : \mathcal{M}_+ \to \mathcal{N}_A$ is a manifold. The rest of theorem 3·6 follows too, once we have proved the main result that $\mathcal{M}_+$ is a manifold. We will return to this in subsection 6·8.

6·3. *The Dirac operator*

We will use Fredholm properties of the Dirac operator associated to a G_2-structure φ to obtain a direct sum decomposition (proposition 6·11). The required properties of the Dirac operator can conveniently be deduced from the properties of the Laplacian discussed in subsection 5·1.

Since $G_2 \subseteq SO(7)$ is simply connected it can be regarded as a subgroup of $Spin(7)$, so a G_2-structure on a manifold M^7 induces a spin structure (in fact a converse also holds: an oriented manifold M^7 admits G_2-structures if and only if it admits a spin structure, cf. [**5**, Remark 3]). The spin structure defines a spinor bundle S, and the Dirac operator

$$\eth : \Gamma(S) \to \Gamma(S).$$

Recall that if $\mathbb{R}^7$ is identified with the imaginary part of the octonions $\mathbb{O}$ then G_2 corresponds to the group of normed-algebra automorphisms of $\mathbb{O}$. Indeed, the 3-form $\varphi_0 \in \Lambda^3(\mathbb{R}^7)^*$ stabilised by G_2 can be written in terms of the octonion multiplication $\cdot$ and the inner product $<,>$ as

$$\varphi_0(a, b, c) = <a \cdot b, c>.$$

The octonion multiplication by $\mathbb{R}^7$ on $\mathbb{O}$ induces a representation of the Clifford algebra $Cl(\mathbb{R}^7)$ on $\mathbb{O}$. Hence $Spin(7) \subset Cl(\mathbb{R}^7)$ also acts on $\mathbb{O}$, and this identifies $\mathbb{O}$ with the spin representation of $Spin(7)$. This identification is in fact G_2-equivariant (cf. [**9**, page 122]). In particular, the restriction of the spin representation to G_2 is isomorphic to $\mathbb{R}^1 \oplus \mathbb{R}^7$, the direct sum of the trivial and standard representations of G_2.

Hence on a manifold M^7 with a G_2-structure φ the spinor bundle S is isomorphic to $\Lambda^0T^*M \oplus \Lambda^1T^*M$. Under this identification Clifford multiplication by some $\alpha \in \Omega^1(M)$ (which comes from the octonion multiplication) corresponds to

$$\Omega^0(M) \oplus \Omega^1(M) \to \Omega^0(M) \oplus \Omega^1(M),\ \ (\alpha, (f, \beta)) \mapsto (-<\alpha, \beta>,\ f\alpha + *(\alpha \wedge \beta \wedge *\varphi)),$$

and the Dirac operator is identified with

$$\Omega^0(M) \oplus \Omega^1(M) \to \Omega^0(M) \oplus \Omega^1(M), \quad (f, \beta) \mapsto (d^*\beta,\ df + *(d\beta \wedge *\varphi)). \tag{6·2}$$

If M is a G_2-manifold we can see directly from (6·2) that $\eth^2$ is identified with the Hodge Laplacian on $\Omega^0(M) \oplus \Omega^1(M)$ (this could also be proved in the vein of proposition 4·1, using a Weitzenböck formula). This allows us to deduce the index of the Dirac operator from that of the Laplacian (proposition 5·3).

Let $\mathcal{H}_0^S$ be the bounded harmonic spinors on the G_2-manifold M^7, and $\mathcal{H}_\infty^S$ the translation-invariant harmonic spinors on the cylinder $X \times \mathbb{R}$. We can consider $\rho\mathcal{H}_\infty^S$ as a space of spinors on M supported on the cylindrical part, where ρ is a cut-off function for the cylinder.

PROPOSITION 6·4. *Let M be an EAC G_2-manifold with rate δ_0, $k \geqslant 1$ and $0 < \delta < \min\{\epsilon_1, \delta_0\}$.*

$$\eth : C_\delta^{k+1,\alpha}(S) \to C_\delta^{k,\alpha}(S) \tag{6·3a}$$

is injective and its image is the L^2-orthogonal complement of $\mathcal{H}_0^S$, while

$$\eth : C_\delta^{k+1,\alpha}(S) \oplus \rho\mathcal{H}_\infty^S \to C_\delta^{k,\alpha}(S) \tag{6·3b}$$

is surjective with kernel $\mathcal{H}_0^S$.

Proof. The argument is analogous to that on page 328. $\mathcal{H}_\infty^S \cong \mathcal{H}_\infty^0 \oplus \mathcal{H}_\infty^1 \cong 2\mathcal{H}_X^0 \oplus \mathcal{H}_X^1$, so has even dimension (as X is Kähler). Let $2i = \dim \mathcal{H}_\infty^S$. By proposition 5·3

$$\triangle : C_\delta^{k+1,\alpha}(S) \to C_\delta^{k,\alpha}(S)$$

has index $-2i$, so the index of (6·3a) is $-i$. Hence (6·3b) has index $+i$. The kernel of (6·3b) is contained in $\mathcal{H}_0^S$, while by integration by parts the image of (6·3a) is contained in the L^2-orthogonal complement of $\mathcal{H}_0^S$. Hence

$$\begin{aligned} -i &\leqslant -\dim \mathcal{H}_0^S, \\ i &\leqslant \dim \mathcal{H}_0^S. \end{aligned} \tag{6·4}$$

Since equality holds the result follows. □

Remark 6·5. If M has a single end then $\dim \mathcal{H}_\infty^S = 2 + b^1(X)$, while corollary 5·13 implies that $\dim \mathcal{H}_0^S = 1 + b^1(M)$. Hence the equality in (6·4) can also be seen as a consequence of proposition 5·16.

6·4. *The slice*

Fix $k \geqslant 1$, $\delta > 0$, $\alpha \in (0, 1)$ and $\varphi \in \mathcal{X}'_\delta$. We find a direct complement K in $T_\varphi \mathcal{Z}_Q^3$ to the tangent space of the $\mathcal{D}_Q$-orbit. Then we define a submanifold $\mathcal{S} \subseteq \mathcal{Z}_Q^3$ whose tangent space at φ is K. We will use $\mathcal{S}$ as a slice in $\mathcal{Z}_Q^3$ for the $\mathcal{D}_Q$-action at φ.

The fixed torsion-free G_2-structure φ is used to define an EAC metric and a Hodge star. It also defines type decompositions of the exterior bundles and spaces of harmonic forms, as described in subsection 4·1. The relevant decompositions of the exterior powers of the cotangent bundle are

$$\begin{aligned} \Lambda^2 &= \Lambda_7^2 \oplus \Lambda_{14}^2 \\ \Lambda^3 &= \Lambda_1^3 \oplus \Lambda_7^3 \oplus \Lambda_{27}^3 \\ \Lambda^4 &= \Lambda_1^4 \oplus \Lambda_7^4 \oplus \Lambda_{27}^4 \\ \Lambda^5 &= \Lambda_7^5 \oplus \Lambda_{14}^5, \end{aligned}$$

where Λ^m_d is a subbundle of the exterior cotangent bundle Λ^m of rank d. Its sections $\Omega^m_d(M)$ are the 'type d m-forms'. Note that the map $TM \to \Lambda^2_7,\ v \mapsto v \lrcorner \varphi$ is a bundle isomorphism. Therefore Lie derivatives $\mathcal{L}_V\varphi = d(V \lrcorner \varphi)$ are precisely exterior derivatives of 2-forms of type 7.

Having restricted our attention to G_2-structures in $\mathcal{Z}^3_{\mathcal{Q}}$ is convenient since the asymptotic values of elements of $T_\varphi \mathcal{Z}^3_{\mathcal{Q}}$ are harmonic. Recall the notation for harmonic forms from subsection 5·2, in particular that $\mathcal{H}^m_0$ and $\mathcal{H}^m_\infty$ denote the spaces of bounded harmonic forms on M and harmonic translation-invariant forms on $X \times \mathbb{R}$ respectively.

For convenience we identify translation-invariant 3-forms on $X \times \mathbb{R}$ with pairs of 3- and 2-forms on X by $\sigma + dt \wedge \tau \leftrightarrow (\sigma, \tau)$. This identifies the tangent spaces $\mathcal{H}_{SU}$ and $\mathcal{H}_{SU,A}$ of $\mathcal{Q}$ and $\mathcal{Q}_A$ with subspaces $\mathcal{H}^3_{SU}, \mathcal{H}^3_{SU,A} \subseteq \mathcal{H}^3_\infty$. Let

$$\mathcal{Z}^3_{cyl} \subseteq C^{k,\alpha}_\delta(\Lambda^3) \oplus \rho\mathcal{H}^3_{SU,A}$$

be the subspace of closed forms. Clearly $T_\varphi \mathcal{Z}^3_{\mathcal{Q}} \subseteq \mathcal{Z}^3_{cyl}$, and we show below that equality holds. The tangent space to the pre-moduli space of torsion-free G_2-structures at φ will turn out to be the subspace

$$\mathcal{H}^3_{cyl} \subseteq \mathcal{Z}^3_{cyl}$$

of harmonic forms. This is exactly the subspace of elements of $\mathcal{H}^3_0$ which are tangent to cylindrical deformations of the G_2-structure, i.e. whose boundary values lie in $\mathcal{H}^3_{SU}$. The boundary map $B : \mathcal{H}^3_0 \to \mathcal{H}^3_\infty$ maps $\mathcal{H}^3_{cyl}$ precisely onto $\mathcal{H}^3_{SU,A}$. Together with the Hodge decomposition (5·6) it follows that

$$\mathcal{Z}^3_{cyl} = \mathcal{H}^3_{cyl} \oplus C^{k,\alpha}_\delta[d\Lambda^2]. \tag{6·5}$$

Remark 6·6. $dC^{k+1,\alpha}_\delta(\Lambda^m)$ is the space of exterior derivatives of decaying forms, while we use $C^{k,\alpha}_\delta[d\Lambda^m]$ to denote the space of exact decaying forms. $dC^{k+1,\alpha}_\delta(\Lambda^m) \subseteq C^{k,\alpha}_\delta[d\Lambda^m]$ is a closed subspace of finite codimension.

LEMMA 6·7. *$\mathcal{Z}^3_{\mathcal{Q}}$ is a manifold, and $T_\varphi \mathcal{Z}^3_{\mathcal{Q}} = \mathcal{Z}^3_{cyl}$.*

Proof. If ψ is a 3-form asymptotic to an element $(\Omega', \omega') \in \mathcal{Q}_A$ then the condition $[\Omega'] \in A^3$ implies that $d\psi \in dC^{k,\alpha}_\delta(\Lambda^3)$. Therefore

$$d : C^{k,\alpha}_\delta(\Lambda^3) + \rho\mathcal{Q}_A \to dC^{k,\alpha}_\delta(\Lambda^3)$$

is a submersion, and the result follows from the implicit function theorem. □

Let $\mathcal{D}^{k+1,\alpha}_{\mathcal{Q}}$ be the group of diffeomorphisms of M which are isotopic to the identity, and $C^{k+1,\alpha}_\delta$-asymptotic to a cylindrical automorphism of the cylindrical G_2-structure $\Omega + dt \wedge \omega$. The elements of a neighbourhood of the identity in $\mathcal{D}^{k+1,\alpha}_{\mathcal{Q}}$ can be written as $\exp(V + \rho V_\infty)$, where V is a $C^{k+1,\alpha}_\delta$ vector field on M and V_∞ is a translation-invariant vector field on $X \times \mathbb{R}$ with $\mathcal{L}_{V_\infty}(\Omega + dt \wedge \omega) = 0$, i.e. $V_\infty \in (\mathcal{H}^1_\infty)^\sharp$. Therefore if we let

$$D = \rho(\mathcal{H}^1_\infty)^\sharp \lrcorner \varphi \subseteq \Omega^2_7(M)$$

then the tangent space to the $\mathcal{D}^{k+1,\alpha}_{\mathcal{Q}}$-orbit at φ is

$$\{\mathcal{L}_{V+\rho V_\infty}\varphi : V \in C^{k+1,\alpha}_\delta(TM),\ V_\infty \in (\mathcal{H}^1_\infty)^\sharp\} = d\big(C^{k+1,\alpha}_\delta\big(\Lambda^2_7\big) \oplus D\big).$$

We take the tangent space K to the slice at φ to be the kernel of the formal adjoint of $d : \Omega^2_7(M) \to \Omega^3(M)$, i.e.

Definition 6·8. Let K be $\ker \pi_7 d^*$ in $\mathcal{Z}^3_{cyl}$.

LEMMA 6·9. *If $\beta \in \Omega^3_{27}(M)$ then $\pi_7 d\beta = 0$ if and only if $\pi_7 d^*\beta = 0$.*

Proof. An instance of [**5**, Proposition 3]. $\square$

Definition 6·10. Let $W = C^{k,\alpha}_\delta[d\Lambda^2] \cap \Omega^{27}(M)$.

PROPOSITION 6·11. *$K = \mathcal{H}^3_{cyl} \oplus W$, and*

$$\mathcal{Z}^3_{cyl} = d\big(C^{k+1,\alpha}_\delta\big(\Lambda^2_7\big) \oplus D\big) \oplus K.$$

Proof. K obviously contains $\mathcal{H}^3_{cyl}$, and it also contains W by lemma 6·9. By integration by parts K is L^2-orthogonal to $d\big(C^{k+1,\alpha}_\delta\big(\Lambda^2_7\big) \oplus D\big)$. Hence by (6·5) it suffices to show

$$C^{k,\alpha}_\delta[d\Lambda^2] = d\big(C^{k+1,\alpha}_\delta\big(\Lambda^2_7\big) \oplus D\big) \oplus W. \tag{6·6}$$

We can identify the spinor bundle S with $\Lambda^0 \oplus \Lambda^2_7$ and with $\Lambda^3_{1\oplus 7}$ (shorthand for $\Lambda^3_1 \oplus \Lambda^3_7$) so that the Dirac operator $\eth : \Gamma(S) \to \Gamma(S)$ is identified with

$$\Omega^0(M) \oplus \Omega^2_7(M) \to \Omega^3_{1\oplus 7}(M), \quad (f, \eta) \mapsto \pi_{1\oplus 7} d\eta + *(df \wedge \varphi).$$

If $\beta \in C^{k,\alpha}_\delta[d\Lambda^2]$ then by surjectivity of the Dirac operator map (6·3*b*)

$$\pi_{1\oplus 7}\beta = \pi_{1\oplus 7} d\eta + *(df \wedge \varphi)$$

for some $\eta \in C^{k+1,\alpha}_\delta\big(\Lambda^2_7\big) \oplus D$, $f \in C^{k+1,\alpha}_\delta(\Lambda^0) \oplus \rho\mathcal{H}^0_\infty$. By integration by parts $f = 0$. Hence $\beta - d\eta$ is exact and of type 27, i.e. $\beta - d\eta \in W$. $\square$

We want to take as our slice for the $\mathcal{D}_\delta$-action at φ a submanifold $\mathcal{S} \subseteq \mathcal{Z}^3_{\mathcal{Q}}$ with $T_\varphi \mathcal{S} = K$. To this end we pick a map

$$\exp : U \to \mathcal{Z}^3_{\mathcal{Q}} \tag{6·7}$$

on a neighbourhood U of 0 in $\mathcal{Z}^3_{cyl} = T_\varphi \mathcal{Z}^3_{\mathcal{Q}}$, such that $D\exp_0 = id$. We also insist that exp maps decaying forms to decaying forms, and smooth forms to smooth forms. We can do this since by (6·5) the decaying forms have a finite-dimensional complement of smooth forms in $\mathcal{Z}^3_{cyl}$. We then choose

$$\mathcal{S} = \exp(K \cap U). \tag{6·8}$$

6·5. *The pre-moduli space is a manifold*

Let $\mathcal{R}_\delta \subseteq \mathcal{S}$ be the subset of $C^{k,\alpha}_\delta$ torsion-free G_2-structures. $\mathcal{R}_\delta$ is the pre-moduli space of torsion-free G_2-structures near φ. In order to show that $\mathcal{R}_\delta$ is a submanifold we will exhibit it as the zero set of a function F with surjective derivative, and apply the implicit function theorem.

Recall that by theorem 2·5 a G_2-structure ψ is torsion-free if and only if $d\psi = 0$ and $d^*_\psi \psi = 0$. To emphasise the non-linearity of the second condition we define

Definition 6·12. For a G_2-structure ψ on M let $\Theta(\psi) = *_\psi \psi$.

So with this notation

$$\mathcal{R}_\delta = \{\psi \in \mathcal{S} : d\Theta(\psi) = 0\}.$$

If $\psi \in \mathcal{Z}^3_{\mathcal{Q}}$ then ψ is asymptotic to a torsion-free cylindrical G_2-structure, so $d\Theta(\psi)$ decays. Moreover, by definition elements of $\mathcal{Z}^3_{\mathcal{Q}}$ are asymptotic to elements of $\mathcal{Q}_A \subseteq \mathcal{Q}$. Therefore

$\Theta(\psi)$ is asymptotic to $\frac{1}{2}\omega'^2 - dt \wedge \hat{\Omega}'$, with $[\omega'^2] \in A^4$ (cf. (6·1) and remark 2·12). This implies that $d\Theta(\psi) \in dC^{k+1,\alpha}_\delta(\Lambda^4)$.

$\Theta : \Lambda^3_+ T^*M \to \Lambda^4 T^*M$ is point-wise smooth, so by the chain rule

$$\mathcal{Z}^3_Q \to dC^{k+1,\alpha}_\delta(\Lambda^4), \quad \psi \to d\Theta(\psi)$$

is a smooth function. We need to adjust this map to obtain a function with surjective derivative. If β is a 3-form such that $d^*\beta \in d^*C^{k+1,\alpha}_\delta(\Lambda^3)$ then by Hodge decomposition (5·6) there is a unique $\beta_E \in C^{k+1,\alpha}_\delta[d\Lambda^2]$ such that $d^*\beta = d^*\beta_E$. We can think of β_E as the exact part of β. The image of β_E under the projection $C^{k+1,\alpha}_\delta[d\Lambda^2] \to W$ in the direct sum decomposition (6·6) is the unique $P(\beta) \in W$ such that

$$d^*\beta - d^*P(\beta) \in d^*d\big(C^{k+1,\alpha}_\delta\big(\Lambda^2_7\big) \oplus D\big).$$

Definition 6·13. For ψ close to φ in $\mathcal{Z}^3_Q$ let

$$F(\psi) = P(*\Theta(\psi)). \tag{6·9}$$

$F(\psi)$ is essentially a component of the exact part of $*\Theta(\psi)$, so $d\Theta(\psi) = 0 \Rightarrow F(\psi) = 0$. In order to show that the converse also holds, so that we do not 'lose any information' by considering zeros of F instead of $\psi \mapsto d\Theta(\psi)$, we need a simple algebraic lemma.

LEMMA 6·14. *Let ψ a G_2-structure on M with $d\psi = 0$. Then for any vector field V*

$$d\Theta(\psi) \wedge (V \lrcorner \psi) = 0.$$

Proof. By [**24**, Lemma 11.5] there is for any G_2-structure ψ a linear relation between $\pi_7 d\psi$ and $\pi_7 d^*\psi$ (where the type component and codifferential are defined by ψ). This implies the result. □

PROPOSITION 6·15. *For $\psi \in \mathcal{Z}^3_Q$ sufficiently close to φ, ψ is torsion-free if and only if $F(\psi) = 0$.*

Proof. By definition $F(\psi) = 0$ implies that

$$*d\Theta(\psi) = d^*d\eta$$

for some $\eta \in C^{k+1,\alpha}_\delta\big(\Lambda^2_7\big) \oplus D$. We need to show that $d^*d\eta = 0$. By integration by parts it would suffice to show that $\pi_7 d^*d\eta = 0$, but unfortunately there is no *a priori* reason why that should be the case. Nevertheless, if ψ is close to φ then $\|\pi_7 d^*d\eta\|$ is small relative to $\|d^*d\eta\|$, for if $V \lrcorner \varphi \in \Lambda^2_7$ then

$$<d^*d\eta, V \lrcorner \varphi> = <d^*d\eta, V \lrcorner (\varphi - \psi)> \leqslant \|d^*d\eta\|\|\psi - \varphi\|\|V\| = \tfrac{1}{3}\|d^*d\eta\|\|\psi - \varphi\|\|V \lrcorner \varphi\|$$

by lemma 6·14. Hence point-wise

$$\|\pi_7 d^*d\eta\| \leqslant \tfrac{1}{3}\|\psi - \varphi\|\|d^*d\eta\|. \tag{6·10}$$

On the other hand we can find a reverse inequality for weighted L^2-norms: if we pick $0 < \delta' < \delta$ then there is a constant $C > 0$ such that

$$\|\pi_7 d^*d\eta\|_{L^2_{\delta'}} \geqslant C\|d^*d\eta\|_{L^2_{\delta'}} \tag{6·11}$$

for any $\eta \in L^2_{2,\delta'}\big(\Lambda^2_7\big) \oplus D$. To prove this inequality we use weighted Sobolev $L^2_{2,\delta'}$ norms, defined analogously to the weighted Hölder norms in definition 5·1. $C^{k+1,\alpha}_{\delta} \hookrightarrow L^2_{2,\delta'}$ continuously for $k \geqslant 1$. Since $\pi_7 d^* d : \Omega^2_7(M) \to \Omega^2_7(M)$ is elliptic [**18**, Theorem 6.2] (discussed on page 326) ensures that

$$\pi_7 d^* d : L^2_{2,\delta'}\big(\Lambda^2_7\big) \oplus D \to L^2_{\delta'}\big(\Lambda^2_7\big)$$

is Fredholm. Therefore for a suitable choice of C the inequality (6·11) holds for η in a complement of the kernel of $\pi_7 d^* d$. By integration by parts the elements of the kernel are closed, so (6·11) holds for all $\eta \in L^2_{2,\delta'}\big(\Lambda^2_7\big) \oplus D$. Combining the inequalities (6·10) and (6·11) gives that if $\|\psi - \varphi\|_{C^0} < 3C$ then $d^* d\eta = 0$. Hence

$$F(\psi) = 0 \Rightarrow d\Theta(\psi) = 0. \qquad \square$$

Next we show that $F : \mathcal{S} \to W$ satisfies the hypotheses of the implicit function theorem.

PROPOSITION 6·16. *$F : \mathcal{Z}^3_{\mathcal{Q}} \to W$ is smooth with derivative*

$$DF_{\varphi} : \mathcal{Z}^3_{cyl} \to W, \;\; \beta \mapsto P\big(\tfrac{4}{3}\pi_1\beta + \pi_7\beta - \pi_{27}\beta\big). \tag{6·12}$$

Proof. According to [**12**, Lemma 3.1.1] the derivative at φ_0 of the point-wise model $\Theta : \Lambda^3_+(\mathbb{R}^7)^* \to \Lambda^4(\mathbb{R}^7)^*$ is

$$D\Theta_{\varphi_0} : \Lambda^3(\mathbb{R}^7)^* \to \Lambda^4(\mathbb{R}^7)^*, \;\; \beta \mapsto *\tfrac{4}{3}\pi_1\beta + *\pi_7\beta - *\pi_{27}\beta, \tag{6·13}$$

and the result follows by the chain rule. $\square$

PROPOSITION 6·17. *$DF_{\varphi} : \mathcal{Z}^3_{cyl} \to W$ is 0 on $d\big(C^{k+1,\alpha}_{\delta}\big(\Lambda^2_7\big) \oplus D\big) \oplus \mathcal{H}^3_{cyl}$ and $-id$ on W.*

Proof. For any $V \lrcorner \varphi \in C^{k+1,\alpha}_{\delta}\big(\Lambda^2_7\big) \oplus D$

$$d^* * D\Theta_{\varphi}(d(V \lrcorner \varphi)) = *d(D\Theta_{\varphi})(\mathcal{L}_V \varphi) = *\mathcal{L}_V(d\Theta(\varphi)) = 0$$

so $DF_{\varphi}(d(V \lrcorner \varphi)) = 0$. The type components of harmonic forms are harmonic, so for any $\chi \in \mathcal{H}^3_{cyl}$ we have $d^*(\frac{4}{3}\pi_1\chi + \pi_7\chi - \pi_{27}\chi) = 0$, and hence $DF_{\varphi}(\chi) = 0$. If $\kappa \in W$ then $DF_{\varphi}(\kappa) = P(-\kappa) = -\kappa$, as P is a projection to W. $\square$

We have taken the pre-moduli space $\mathcal{R}_{\delta}$ near φ to consist of the torsion-free EAC G_2-structures in the slice $\mathcal{S}$. We can, however, only prove that it has the properties we want close to φ. We will therefore repeatedly replace $\mathcal{S}$ by a neighbourhood of φ in $\mathcal{S}$ in order to ensure that $\mathcal{R}_{\delta} \subseteq S$ has the desired properties. The first step is to ensure that $\mathcal{R}_{\delta}$ is a manifold. Proposition 6·15 shows that if $\mathcal{S}$ is sufficiently small then $\mathcal{R}_{\delta}$ is the zero set of F in $\mathcal{S}$. Applying the implicit function theorem to $F : \mathcal{S} \to W$ we deduce

PROPOSITION 6·18. *If the slice $\mathcal{S}$ near φ is shrunk sufficiently then the space $\mathcal{R}_{\delta}$ of torsion-free EAC G_2-structures in $\mathcal{S}$ is a manifold. Its tangent space at φ is $\mathcal{H}^3_{cyl}$.*

6·6. *Regularity*

In order to use the pre-moduli space $\mathcal{R}_{\delta}$ to construct a moduli space of *smooth* G_2-structures we require two regularity results. We show that if the slice $\mathcal{S}$ is sufficiently small then the elements of the pre-moduli space $\mathcal{R}_{\delta} \subseteq \mathcal{S}$, which *a priori* are merely $C^{k,\alpha}_{\delta}$, are smooth and EAC. We also need to show that isometries of EAC metrics are EAC. To prove the regularity of elements of $\mathcal{R}_{\delta}$ we use a regularity result about solutions of elliptic operators which are asymptotically translation-invariant in a weighted Hölder sense.

Definition 6·19. A differential operator A on M is $C^{l,\alpha}_\delta$*-asymptotic* to the translation-invariant operator A_∞ on the cylinder $X \times \mathbb{R}$ if the difference between the coefficients of A and A_∞ (in the sense of (5·1)) is $C^{l,\alpha}_\delta$.

The theorem below is similar to e.g. [**19**, Theorem 6.3], and can be obtained from interior estimates like those of Morrey [**22**, Theorems 6.2.5 and 6.2.6].

THEOREM 6·20. *Let M^n be an asymptotically cylindrical manifold, E a vector bundle on M associated to TM, and A a linear elliptic order r differential operator on the sections of E that is $C^{l,\alpha}_\delta$-asymptotic to a translation-invariant operator with $C^{l,\alpha}$ coefficients. If $s \in C^{r,\alpha}_\delta(E)$ and $As \in C^{l,\alpha}_\delta(E)$ then $s \in C^{l+r,\alpha}_\delta(E)$, and there is a constant $C > 0$ independent of s such that*

$$\|s\|_{C^{l+r,\alpha}_\delta} < C\big(\|As\|_{C^{l,\alpha}_\delta} + \|s\|_{C^0_\delta}\big).$$

Now consider again a G_2-manifold M with a torsion-free G_2-structure φ, and the pre-moduli space $\mathcal{R}_\delta$ of torsion-free G_2-structures in the slice $\mathcal{S} = \exp(K \cap U) \subseteq \mathcal{Z}^3_{\mathcal{Q}}$ for the $\mathcal{D}_{\mathcal{Q}}$-action at φ. We use theorem 6·20 in a boot-strapping argument to show that the elements of $\mathcal{R}_\delta$ are EAC. A priori they are $C^{k,\alpha}_\delta$-asymptotic to elements of $\mathcal{Q}_A$. Like in proposition 6·18 we can only work close to φ, and must replace $\mathcal{S}$ by a neighbourhood of φ in $\mathcal{S}$.

PROPOSITION 6·21. *If the slice $\mathcal{S}$ near φ is shrunk sufficiently then the pre-moduli space $\mathcal{R}_\delta \subseteq \mathcal{S}$ consists of smooth exponentially asymptotically translation-invariant forms.*

Proof. We want to show that if $\psi \in \mathcal{S}$ is sufficiently close to φ and $d\Theta(\psi) = 0$ then ψ is smooth and exponentially asymptotically translation-invariant. Write $\psi = \varphi + \beta$.

$$D(d{*}d\Theta)_\varphi = -dd^* \circ \big(\tfrac{4}{3}\pi_1 + \pi_7 - \pi_{27}\big),$$

so we can write

$$d{*}d\Theta(\varphi+\beta) = -dd^*\big(\tfrac{4}{3}\pi_1\beta + \pi_7\beta - \pi_{27}\beta\big) + P(\beta, \nabla\beta, \nabla^2\beta) + R(\beta, \nabla\beta),$$

where P consists of the quadratic terms of $d{*}d\Theta(\varphi+\beta)$ that involve $\nabla^2\beta$, and R consists of the remaining quadratic terms. P and R depend only point-wise on their arguments, and P is linear in $\nabla^2\beta$.

By the definition of the map exp (6·7) we can write $\beta = \kappa + \gamma$, with $\kappa \in W$, and γ smooth and exponentially asymptotic to some element of $\mathcal{Q}_A$. As κ is closed of type 27

$$-dd^*\big(\tfrac{4}{3}\pi_1\kappa + \pi_7\kappa - \pi_{27}\kappa\big) = \triangle\kappa.$$

Considering β and $\nabla\beta$ as fixed we can define a second-order linear differential operator $A : \zeta \mapsto P(\beta, \nabla\beta, \nabla^2\zeta)$. Then the condition $d{*}d\Theta(\psi) = 0$ is equivalent to

$$(\triangle + A)\kappa = -R + dd^*\big(\tfrac{4}{3}\pi_1\gamma + \pi_7\gamma - \pi_{27}\gamma\big) - A\gamma. \tag{6·14}$$

If $\beta = 0$ then $A = 0$, so $\triangle + A$ is elliptic. Ellipticity is an open condition, so $\triangle + A$ is in fact elliptic for any β sufficiently small in the uniform norm.

Now suppose κ is $C^{l+1,\alpha}_\delta$ and is a solution of (6·14). R and the coefficients of A depend smoothly on κ and $\nabla\kappa$. Therefore $\triangle + A$ and the RHS of (6·14) are $C^{l,\alpha}_\delta$-asymptotically translation-invariant. Since the RHS of (6·14) is decaying *a priori* it is $C^{l,\alpha}_\delta$. If β is sufficiently small that $\triangle + A$ is elliptic then by theorem 6·20 κ is $C^{l+2,\alpha}_\delta$. Since κ is $C^{1,\alpha}_\delta$ it is $C^{l,\alpha}_\delta$ for all l by induction.

Hence $\psi = \varphi + \kappa + \gamma$ is smooth and exponentially asymptotically translation-invariant.

Myers and Steenrod [**23**] show that any isometry of smooth Riemannian manifolds is smooth. We wish to generalise this, and show that isometries of EAC metrics are EAC (in the sense of definition 2·19).

We can think of the choice of diffeomorphism $M_\infty \to X \times \mathbb{R}^+$ in definition 2·13 as defining a 'cylindrical-end structure', and of two such structures as being 'δ-equivalent' if they differ by a rate δ EAC diffeomorphism – then they define equivalent notions of exponential translation-invariance etc. The next proposition can be interpreted as stating that the cylindrical-end structure of an EAC manifold can be recovered from the metric.

PROPOSITION 6·22. *Any isometry of EAC (rate $\delta > 0$) manifolds is C^∞ and EAC with rate δ.*

Sketch proof. By [**23**, Theorem 8] the isometries are C^∞, so we just need to prove that they are also EAC.

Let M be a manifold with a Riemannian metric g. We need to show that if for $i = 1, 2$ $M_{i,\infty} \subseteq M$ have compact complements, and $\Psi_i : M_{i,\infty} \to X_i \times \mathbb{R}^+$ are diffeomorphisms defining cylindrical-end structures with respect to which g is EAC, then $\Psi_1 \circ \Psi_2^{-1}$ is EAC.

Consider the space R of half-lines in M, i.e. equivalence classes of unit speed globally distance-minimising geodesic rays $\gamma : [0, \infty) \to M$, where two rays are equivalent if one is a subset of the other. We can define a distance function on R by

$$d([\gamma], [\sigma]) = \lim_{u\to\infty} \inf_{v} d(\gamma(u), \sigma(v)). \qquad (6·15)$$

g is pushed forward to an EAC metric on $X_i \times \mathbb{R}^+$ by Ψ_i. It is straight-forward to solve the geodesic equation in local coordinates to show that for each $x \in X_i$ there is a unique half-line $[\gamma]$ such that the X-component of $\gamma(u)$ approaches x as $u \to \infty$. Ψ_i therefore induce isometries $\Xi_i : R \to X_i$. Then $\Xi_1 \circ \Xi_2^{-1}$ is smooth by [**23**, Theorem 8]. If t_i is the $\mathbb{R}^+$-coordinate on $X_i \times \mathbb{R}^+$ then $grad(t_i) - \frac{d\gamma}{du}$ decays exponentially as $u \to \infty$ for each half-line $[\gamma]$, so $\Psi_1 \circ \Psi_2^{-1}$ is exponentially asymptotic to $(x, t) \mapsto (\Xi_1 \circ \Xi_2^{-1}(x), t + h)$ for some $h \in \mathbb{R}$. $\square$

6·7. *Constructing the moduli space*

For each $\varphi \in \mathcal{X}'_\delta$ we have constructed a pre-moduli space $\mathcal{R}_\delta$. $\mathcal{R}_\delta$ is a manifold, its elements are smooth and EAC, and its tangent space at φ is $\mathcal{H}^3_{cyl}$. We now want to use slice arguments to show that we can take the pre-moduli spaces $\mathcal{R}_\delta$ as coordinate charts to define a smooth structure on $\mathcal{M}_\delta$. In [**7**] Ebin gives a very detailed account of a slice construction on a compact manifold. While the basic idea of the slice is the same, it is not so convenient for our purposes to attempt to extend Ebin's arguments to the EAC setting. It is much easier to study the charts for $\mathcal{M}_+$ in terms of the projection to the de Rham cohomology which appears in the statement of the main theorem 3·2.

$$\pi_{\mathcal{M}} : \mathcal{X}_+ \to H^3(M) \times H^2(X), \quad \varphi \mapsto ([\varphi], [B_e(\varphi)]),$$

where we use $B_a(\beta) + dt \wedge B_e(\beta)$ to denote the asymptotic limit of an asymptotically translation-invariant form β.

We first check that $\pi_{\mathcal{M}}$ is an embedding on $\mathcal{R}_\delta$. If we allow ourselves to shrink $\mathcal{R}_\delta$ this amounts to showing that the derivative of $\pi_{\mathcal{M}}$ at φ is injective, and from the definition the derivative is plainly

$$(\pi_H, \pi_{H,e}) : \mathcal{H}^3_{cyl} \to H^3(M) \times H^2(X), \quad \beta \mapsto ([\beta], [B_e(\beta)]).$$

The kernel consists of harmonic, exact, decaying forms, so it is trivial.

Recall from subsection 6·1 that we chose a pre-moduli space $\mathcal{Q}$ near the Calabi-Yau structure (Ω, ω) on X defined by the asymptotic limit of φ, and that $\mathcal{X}_{\mathcal{Q}} \subseteq \mathcal{X}'_\delta$ is the subset of torsion-free G_2-structures whose asymptotic limits lie in $\mathcal{Q}$. $\mathcal{D}_{\mathcal{Q}} \subseteq \mathcal{D}_\delta$ is the subgroup of smooth EAC diffeomorphisms of M whose asymptotic limits lie in the automorphism group of (Ω, ω). $\mathcal{D}_{\mathcal{Q}}$ acts on $\mathcal{X}_{\mathcal{Q}}$ by proposition 4·5, and as an intermediate step for our slice result we prove that $\mathcal{R}_\delta$ is a coordinate chart for $\mathcal{X}_{\mathcal{Q}}/\mathcal{D}_{\mathcal{Q}}$.

PROPOSITION 6·23. *The natural map $\mathcal{R}_\delta \to \mathcal{X}_{\mathcal{Q}}/\mathcal{D}_{\mathcal{Q}}$ is a homeomorphism onto a neighbourhood of $\varphi\mathcal{D}_{\mathcal{Q}}$.*

Proof. We chose $\mathcal{S}$ in consideration of proposition 6·11, ensuring that the derivative at (φ, id) of

$$\mathcal{S} \times \mathcal{D}^{k+1,\alpha}_{\mathcal{Q}} \to \mathcal{Z}^3_{\mathcal{Q}}, \quad (\beta, \phi) \mapsto \phi^*\beta$$

is surjective. By the submersion theorem it is an open map on a neighbourhood of (φ, id).

Thus if $U \subseteq \mathcal{R}_\delta$ is a small neighbourhood of φ then $U\mathcal{D}^{k+1,\alpha}_{\mathcal{Q}}$ is open in the closure of $\mathcal{X}_{\mathcal{Q}}$ in $\mathcal{Z}^3_{\mathcal{Q}}$. If $\psi \in \mathcal{R}_\delta$, $\phi \in \mathcal{D}^{k+1,\alpha}_{\mathcal{Q}}$ and $\phi^*\psi$ is smooth and EAC then ϕ is also smooth and EAC by proposition 6·22. Therefore $U\mathcal{D}^{k+1,\alpha}_{\mathcal{Q}} \cap \mathcal{X}_{\mathcal{Q}} = U\mathcal{D}_{\mathcal{Q}}$, and $U\mathcal{D}_{\mathcal{Q}}$ is open in $\mathcal{X}_{\mathcal{Q}}$.

Hence $\mathcal{R}_\delta \to \mathcal{X}_{\mathcal{Q}}/\mathcal{D}_{\mathcal{Q}}$ is an open map. It is also injective, since $\pi_{\mathcal{M}}$ is $\mathcal{D}_\delta$-invariant and injective on $\mathcal{R}_\delta$. $\square$

For our argument to work we may need to shrink $\mathcal{Q}$ by replacing it with a neighbourhood of (Ω, ω) in $\mathcal{Q}$.

LEMMA 6·24. *If the pre-moduli space $\mathcal{Q}$ of Calabi-Yau structures is shrunk sufficiently then $\mathcal{X}_{\mathcal{Q}}/\mathcal{D}_{\mathcal{Q}}$ is homeomorphic to a neighbourhood of $\varphi\mathcal{D}_\delta$ in $\mathcal{X}'_\delta/\mathcal{D}_\delta$.*

Proof. The natural map $f : \mathcal{X}_{\mathcal{Q}}/\mathcal{D}_{\mathcal{Q}} \to \mathcal{X}'_\delta/\mathcal{D}_\delta$ is injective by proposition 4·5.

Let $\mathcal{Y}$ be the space of Calabi-Yau structures on X. The construction of the moduli space of Calabi-Yau structures on X uses slice results similar to proposition 6·23. This allows us to define on a neighbourhood U of (Ω, ω) in $\mathcal{Y}$ a continuous map

$$P : U \to C^\infty(TX) \times \mathcal{Q}, \quad (\beta, \gamma) \mapsto (V, \Omega', \omega')$$

such that $(\beta, \gamma) = (\exp V)^*(\Omega', \omega')$ for any $(\beta, \gamma) \in U$.

Let $\mathcal{X}_U = \{\psi \in \mathcal{X}'_\delta : B(\psi) \in U\}$. If $\psi \in \mathcal{X}_U$ let $V = P(B(\psi))$, $\phi = \exp \rho V \in \mathcal{D}_\delta$ and $g(\psi) = \phi^*\psi$. Then $B(g(\psi)) \in \mathcal{Q}$, so $\psi \in \mathcal{X}_{\mathcal{Q}}$. Obviously $f(g(\psi)\mathcal{D}_{\mathcal{Q}})) = \psi\mathcal{D}_\delta$. Since f is injective g induces a well-defined map $\mathcal{X}_U\mathcal{D}_\delta \to \mathcal{X}_{\mathcal{Q}}/\mathcal{D}_{\mathcal{Q}}$. g is an inverse for f on a neighbourhood of $\varphi\mathcal{D}_\delta$ in $\mathcal{X}'_\delta/\mathcal{D}_\delta$, so the result follows. $\square$

THEOREM 6·25. *$\mathcal{M}_\delta$ has a unique smooth structure such that*

$$\pi_{\mathcal{M}} : \mathcal{M}_\delta \to H^3(M) \times H^2(X)$$

is an immersion.

Proof. Proposition 6·23 and lemma 6·24 show that for any $\varphi \in \mathcal{X}'_\delta$ the pre-moduli space $\mathcal{R}_\delta$ is homeomorphic to a neighbourhood U of $\varphi\mathcal{D}_\delta$ in $\mathcal{M}_\delta$, and

$$\pi_{\mathcal{M}} : U \to H^3(M) \times H^2(X)$$

is a homeomorphism onto an embedded manifold. To prove that we can use the maps $\mathcal{R}_\delta \to U$ as coordinate charts for $\mathcal{M}_\delta$ we need to check that the transition functions are smooth. But on an overlap $U_1 \cap U_2$ both charts define the unique smooth structure for which $\pi_{\mathcal{M}} : U_1 \cap U_2 \to H^3(M) \times H^2(X)$ is an embedding, so we are done. □

If $\delta_1 > \delta_2 > 0$ then $\mathcal{M}_{\delta_1} \to \mathcal{M}_{\delta_2}$ is injective by proposition 6·22, and $\mathcal{M}_{\delta_1}$ must be an open submanifold of $\mathcal{M}_{\delta_2}$ since $\pi_{\mathcal{M}}$ is an immersion on both spaces. Similarly $\mathcal{M}_\delta \to \mathcal{M}_+$ is injective for any $\delta > 0$, so

$$\mathcal{M}_+ = \bigcup_{\delta>0} \mathcal{M}_\delta.$$

To finish the proof of the main theorem 3·2 it remains only to observe

LEMMA 6·26. *For any $\delta > 0$ the natural map $\mathcal{M}_\delta \to \mathcal{M}_+$ is a homeomorphism to an open subset.*

Proof. We need to show that $\mathcal{M}_\delta \to \mathcal{M}_+$ is open, i.e. that if $U \subseteq \mathcal{X}'_\delta$ with $U\mathcal{D}_\delta$ open in $\mathcal{X}_\delta$ then $U\mathcal{D}_+$ is open in $\mathcal{X}_+$. By the definition of the topology on $\mathcal{X}_+$ this means that $U\mathcal{D}_+ \cap \mathcal{X}_{\delta'}$ is open in $\mathcal{X}_{\delta'}$ for any $\delta' > \delta$. But proposition 6·22 implies that $U\mathcal{D}_+ \cap \mathcal{X}_{\delta'} = U\mathcal{D}_{\delta'}$, which is open in $\mathcal{X}_{\delta'}$ since $\mathcal{M}_\delta \to \mathcal{M}_{\delta'}$ is a local diffeomorphism. □

This concludes the proof of theorem 3·2.

6·8. *Properties of the moduli space*

We look at some local properties of the moduli space $\mathcal{M}_+$ on an EAC G_2-manifold M, which follow from the existence of a pre-moduli space $\mathcal{R}$ with tangent space $\mathcal{H}^3_{cyl}$.

Firstly, the boundary map B maps $\mathcal{H}^3_{cyl}$ onto $\mathcal{H}^3_{SU,A}$, so proposition 6·2 implies that $B : \mathcal{R} \to \mathcal{Q}_A$ is a submersion. As $\mathcal{Q}_A$ is a homeomorphic to an open set in $\mathcal{N}_A$ it follows that $B : \mathcal{M}_+ \to \mathcal{N}_A$ is a submersion onto its image, and we have proved theorem 3·6.

We can now deduce corollary 3·7. It suffices to show that the fibres of $B : \mathcal{M}_+ \to \mathcal{N}_A$ are locally diffeomorphic to the compactly supported subspace $H^3_0(M) \subseteq H^3(M)$.

LEMMA 6·27. *Let φ be an EAC torsion-free G_2-structure on M, $\mathcal{R}$ the pre-moduli space of EAC torsion-free G_2-structures near φ and $\mathcal{Q}$ the pre-moduli space of Calabi-Yau structures near $B(\varphi)$. The map*

$$\mathcal{R} \to H^3(M), \quad \psi \mapsto [\psi - \varphi]$$

maps a neighbourhood of the fibre of $B : \mathcal{R} \to \mathcal{Q}_A$ containing φ diffeomorphically to an open subset of $H^3_0(M)$.

Proof. If ψ is in the same fibre as φ then $\psi - \varphi$ is exponentially decaying, so $[\psi - \varphi] \in H^3_0(M)$. The tangent space to the fibre at φ is the kernel of the derivative of the submersion B, i.e. the subspace $\mathcal{H}^3_+$ of decaying forms in $\mathcal{H}^3_{cyl} = T_\varphi\mathcal{R}$. By theorem 5·10 $\mathcal{H}^3_+ \cong H^3_0(M)$, and the result follows. □

Finally, to confirm the formula for the dimension in proposition 3·5 we just have to compute the dimension of $\mathcal{H}^3_{cyl}$. Recall from subsection 5·2 that A^m is the image of $j^* : H^m(M) \to H^m(X)$, that $H^m(X) = A^m \oplus E^m$ is an orthogonal direct sum, and that the harmonic representatives of the summands are denoted by $\mathcal{A}^m$ and $\mathcal{E}^m$ respectively.

LEMMA 6·28. *Let M^{4k+3} be an oriented EAC manifold with cross-section X. Then $A^{2k+1} \subseteq H^{2k+1}(X)$ has dimension $\frac{1}{2}b^{2k+1}(X)$.*

Proof. $H^{2k+1}(X)$ is a symplectic vector space under the Poincaré pairing. In particular $b^{2k+1}(X)$ is even. $* : H^{2k+1}(X) \to H^{2k+1}(X)$ maps A^{2k+1} isomorphically to its orthogonal complement E^{2k+1}. The Poincaré pairing on $H^{2k+1}(X)$ can be expressed as $< \cdot, *\cdot >$, so $A^{2k+1} \subseteq H^{2k+1}(X)$ is a Lagrangian subspace. $\square$

In particular for any EAC G_2-manifold M with cross-section X the long exact sequence (2·4) for relative cohomology gives

$$\dim H^3_0(M) = b^3(M) - \tfrac{1}{2}b^3(X). \tag{6·16}$$

LEMMA 6·29. $\dim \mathcal{H}^3_{cyl} = b^4(M) + \frac{1}{2}b^3(X) - b^1(M) - 1.$

Proof. As before we let $E^2_8 = E^2 \cap H^2_8(X)$ etc. As a consequence of corollary 6·3 and theorem 5·9 we find that $\pi_H : \mathcal{H}^3_{cyl} \to H^3(M)$ is surjective, and the kernel is mapped bijectively to E^2_8 by $\pi_{H,e} : \mathcal{H}^3_{cyl} \to H^2(X)$. Hence $\dim \mathcal{H}^3_{cyl} = b^3(M) + \dim E^2_8$.

The dimension of E^2 can be computed from the long exact sequence (2·4) for relative cohomology together with (6·16).

$$\begin{aligned}\dim E^2 &= \dim \ker(e : H^3_{cpt}(M) \to H^3(M)) \\ &= b^4(M) - \dim H^3_0(M) = b^4(M) - b^3(M) + \tfrac{1}{2}b^3(X)\end{aligned}$$

By propositions 5·15 and 5·17

$$\dim E^2_6 = \dim A^1 = b^1(M), \quad \dim E^2_1 = \dim A^0 = 1.$$

Hence

$$\dim E^2_8 = b^4(M) - b^3(M) + \tfrac{1}{2}b^3(X) - b^1(M) - 1.$$

$\square$

7. *A topological criterion for $Hol = G_2$*

In this section we prove theorem 3·8, which gives a topological criterion for when the holonomy group of an EAC G_2-manifold M^7 is precisely G_2 and not a proper subgroup. As stated in corollary 2·6 the holonomy group of a metric defined by a torsion-free G_2-structure is always a subgroup of G_2. For compact G_2-manifolds there is a known necessary and sufficient condition for the holonomy group to be exactly G_2.

THEOREM 7·1 ([**13**, Proposition 10.2.2]). *Let M^7 be a compact G_2-manifold. Then $Hol(M) = G_2$ if and only if the fundamental group $\pi_1(M)$ is finite.*

We summarise the proof, and then generalise the result to the EAC case. Note that all covering spaces will be presumed to be equipped with the Riemannian metric pulled back by the covering map. In particular all covering maps will be local isometries, and all covering transformations are isometries.

It is a consequence of the Cheeger-Gromoll line splitting theorem that any compact Ricci-flat Riemannian manifold M has a finite cover isometric to a Riemannian product $T^k \times N$, where T^k is a flat torus (of dimension k possibly 0) and N is compact and simply-connected (see [**3**, Corollary 6.67]). So for a G_2-manifold M let $\tilde{M}$ be a cover of that form.

If $\pi_1(M)$ is infinite then $\tilde{M} = T^k \times N$ with $k > 0$, so $Hol(\tilde{M}) \subseteq SU(3)$. $Hol(\tilde{M})$ is a finite quotient of $Hol(M)$, so $Hol(M)$ cannot be G_2.

If $\pi_1(M)$ is finite then $\tilde{M}$ is the universal cover of M. By the classification of Riemannian holonomy groups ('Berger's list', see e.g. [**2**, Theorem 3] or [**13**, Theorem 3.4.1]) the only proper subgroups of G_2 that can be the holonomy group of a simply-connected Riemannian manifold are 1, $SU(2)$ and $SU(3)$ (up to conjugacy). Thus if $Hol(\tilde{M})$ is not G_2 then it fixes at least one vector in its action on $\mathbb{R}^7$. By proposition 2·2 this implies that there exists a parallel 1-form ϕ on $\tilde{M}$. Since $\tilde{M}$ is compact and Ricci-flat ϕ is harmonic. But Hodge theory gives an isomorphism $\mathcal{H}^1 \to H^1(\tilde{M})$ between harmonic forms and de Rham cohomology. $\tilde{M}$ is simply-connected, so $b^1(\tilde{M}) = 0$ and there can be no harmonic 1-forms on $\tilde{M}$. Hence $Hol(M) = Hol(\tilde{M}) = G_2$.

To generalise theorem 7·1 to EAC G_2-manifolds M we use that by proposition 5·17 the space of parallel 1-forms on M is exactly $\mathcal{H}^1_0$, and that by corollary 5·13 the natural map $\mathcal{H}^1_0 \to H^1(M)$ from bounded harmonic forms to de Rham cohomology is an isomorphism when M has a single end. Recall from theorem 2·21 that a Ricci-flat EAC manifold either has a single end or is isometric to a product cylinder. We also need the following lemma.

LEMMA 7·2. *Let M be a Ricci-flat EAC manifold.*

(i) *if M has a finite normal cover homeomorphic to a cylinder then M or a double cover of M is homeomorphic to a cylinder,*
(ii) *if $\pi_1(M)$ is infinite then M has a finite cover $\tilde{M}$ with $b^1(\tilde{M}) > 0$.*

Proof.

(i) If $\tilde{M}$ is a finite normal cover of M homeomorphic to a cylinder then it is isometric to a product cylinder $Y \times \mathbb{R}$. M is a quotient of $Y \times \mathbb{R}$ by a finite group A of isometries. The isometries are products of isometries of Y and of $\mathbb{R}$ (since they preserve the set of globally distance minimising geodesics $\{\{y\} \times \mathbb{R} : y \in Y\}$). The elements of A have finite order, so they must act on the $\mathbb{R}$ factor as either the identity or as reflections. The subgroup $B \subseteq A$ which acts as the identity on $\mathbb{R}$ is either all of A, in which case M is the cylinder $(Y/B) \times \mathbb{R}$, or a normal subgroup of index 2, in which case $(Y/B) \times \mathbb{R}$ is a cylindrical double cover of M.
(ii) M is homotopy equivalent to a compact manifold with boundary, so $\pi_1(M)$ is finitely generated. Since M is complete and has non-negative Ricci curvature volume comparison arguments show that the volume of balls in the universal cover of M grows polynomially with the radius, and this can be used to deduce that $\pi_1(M)$ has 'polynomial growth' (see Milnor [**21**] for details). A result of Gromov [**8**, Main theorem] states that any finitely generated group with polynomial growth has a nilpotent subgroup of finite index.
So let $G_0 \subseteq \pi_1(M)$ be a nilpotent subgroup of finite index. G_0 is soluble, so the derived series $G_{i+1} = [G_i, G_i]$ reaches 1. Therefore there is a largest i such that $G_i \subseteq \pi_1(M)$ has finite index. Let $\tilde{M}$ be the cover of M corresponding to $G_i \subseteq \pi_1(M)$. G_i/G_{i+1} is an infinite Abelian group, so has non-zero rank. Hence

$$b^1(\tilde{M}) = rk(\pi_1(\tilde{M})/[\pi_1(\tilde{M}), \pi_1(\tilde{M})]) = rk(G_i/G_{i+1}) > 0.$$

The lemma implies that if M is an EAC G_2-manifold then one of 4 possible cases holds:

(i) $\pi_1(M)$ is finite and M is homeomorphic to a cylinder. Then M is isometric to $Y \times \mathbb{R}$ for some compact Calabi-Yau manifold Y^6. The same arguments as in the proof of theorem 7·1 show that the holonomy of Y cannot be a proper subgroup of $SU(3)$. Thus $Hol(M) = SU(3)$.

(ii) $\pi_1(M)$ is finite, M has a single end, and has a double cover homeomorphic to a cylinder. Then the double cover has holonomy $SU(3)$, so $Hol(M) \neq G_2$.

(iii) $\pi_1(M)$ is infinite. Then M has a finite cover $\tilde{M}$ with $b^1(M) > 0$. By theorem 5·9 together with proposition 5·17 there is a parallel 1-form on $\tilde{M}$, so $Hol(\tilde{M}) \subseteq SU(3)$, and $Hol(M) \neq G_2$.

(iv) $\pi_1(M)$ is finite and neither M nor any double cover of M is homeomorphic to a cylinder. Then the universal cover $\tilde{M}$ of M is an EAC G_2-manifold with a single end. The only proper subgroups of G_2 that can be the holonomy group of a complete simply-connected manifold are 1, $SU(2)$ and $SU(3)$, so if $Hol(\tilde{M})$ is not G_2 then there is a parallel vector field on $\tilde{M}$. But $b^1(\tilde{M}) = 0$, so by corollary 5·13 and proposition 5·17 there are no parallel 1-forms on $\tilde{M}$. Hence $Hol(M) = G_2$.

$Hol(M)$ is exactly G_2 only in case (iv), so we have proved theorem 3·8. Examples of the cases (ii) and (iii) are provided below. The author expects that EAC manifolds with holonomy exactly G_2 can be constructed by methods similar to those used by Joyce to construct compact examples in [**13**], and that both $SU(3)$ and proper subgroups such as $\mathbb{Z}_2 \ltimes SU(2)$ can occur as the holonomy of the cross-section of such manifolds. In particular the converse to the following corollary of theorem 3·8 is probably false.

COROLLARY 7·3. *Let M be an EAC G_2-manifold with cross-section X, and suppose that M is not finitely covered by a cylinder. If $Hol(X) = SU(3)$ then $Hol(M) = G_2$.*

Proof. Suppose that $Hol(M)$ is a proper subgroup of G_2. Then $\pi_1(M)$ is infinite, so M has a finite cover $\tilde{M}$ with $b^1(\tilde{M}) > 0$. Let $\tilde{X}$ be the cross-section of $\tilde{M}$. By proposition 5·16 $b^1(\tilde{X}) = 2b^1(\tilde{M}) > 0$, so $Hol(\tilde{X})$ is a proper subgroup of $SU(3)$. $\tilde{X}$ is a finite cover of X, so it follows that $Hol(X)$ is a proper subgroup of $SU(3)$. □

Example 7·4. There exist EAC manifolds W^6 with holonomy precisely $SU(3)$ (see Kovalev [**14**, Theorem 2.7]). Then we can define a torsion-free G_2-structure on the product $W \times S^1$ as in proposition 2·11. Of course $Hol(W \times S^1)$ is not all of G_2, but just $SU(3)$. Furthermore $b^1(W \times S^1) > 0$, so by theorem 3·8 *no* EAC G_2-structure on $W \times S^1$ can have holonomy exactly G_2.

Example 7·5. Let $Y \subset \mathbb{C}P^5$ be the complex projective variety defined by the equations $\sum X_i^2 = 0$, $\sum X_i^4 = 0$. Y is a complete intersection of hypersurfaces, so is a smooth complex 3-fold. As described in [**13**, page 40] the adjunction formula can be used to show that the first Chern class $c_1(Y)$ vanishes. The Lefschetz hyperplane theorem, stated in the form [**4**, Theorem I], can be applied to show that $\pi_1(Y) = 1$.

Since the polynomials defining Y are real the complex conjugation map on $\mathbb{C}P^5$ restricts to an involution $a : Y \to Y$. a is anti-holomorphic, and since the defining polynomials have no roots over $\mathbb{R}$ the involution has no fixed points.

Let ω_{FS} be the restriction of the Fubini-Study Kähler form to Y. $a^*\omega_{FS} = -\omega_{FS}$. Since $c_1(Y) = 0$ Yau's solution to the Calabi conjecture [**28**] implies that there is a unique Kähler form ω in the cohomology class of ω_{FS} such that the corresponding metric is Ricci-flat, making Y into a Calabi-Yau manifold. The cohomology class of ω_{FS} is preserved by $-a^*$ and $-a^*\omega$ is a Kähler form defining a Ricci-flat metric, so the uniqueness part of the assertion implies that $-a^*\omega = \omega$ (cf. [**13**, Proposition 15.2.2]).

Pick a global holomorphic non-vanishing 3-form ϕ on Y. $\overline{a^*\phi}$ is also holomorphic, so equals $\lambda^2\phi$ for some $\lambda \in \mathbb{C}$. Replacing ϕ with $\lambda\phi$ we can assume without loss of generality that $\lambda = 1$. Then $\Omega = \mathrm{re}\ \phi$ is preserved by a^*. We can rescale Ω to ensure that (Ω, ω) is a Calabi-Yau structure in the sense of definition 2·9.

Now define a G_2-structure on $Y \times \mathbb{R}$ by $\varphi = \Omega + dt \wedge \omega$. By proposition 2·11 φ is torsion-free. Let M be the quotient of $Y \times \mathbb{R}$ by $a \times (-1)$. M has a single end and $\pi_1(M)$ has order 2. $(a \times (-1))^*\varphi = a^*\Omega + (-dt) \wedge a^*\omega = \varphi$, so φ induces a well-defined torsion-free G_2-structure on M.

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